Introduction to interfacial waves

Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture - 13
Introduction to perturbation methods

In the last video we had looked at the exact solution to the non-linear pendulum. Recall that we had written down the exact non-linear equation and we had solved this equation using elliptic functions. And we had expressed it in this manner.

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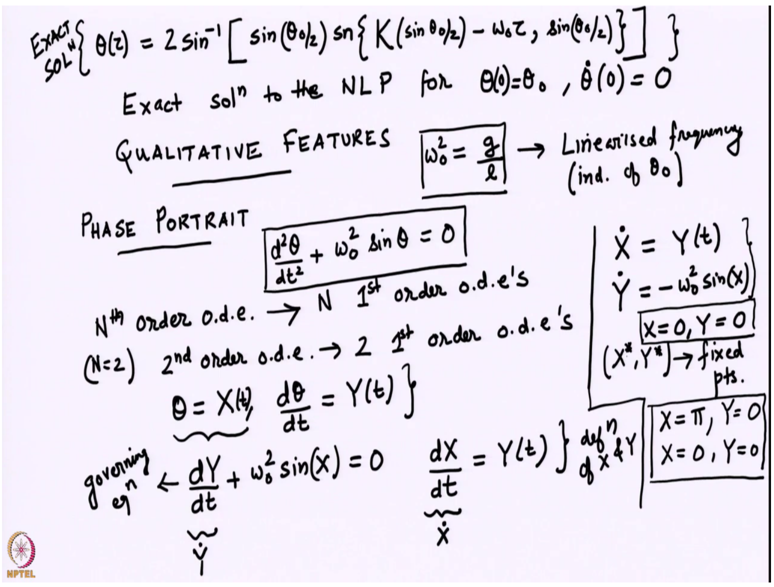

And we had found that it is a rather complicated looking function involved involving an sn inverse and within the square brackets there is the elliptic sn. Now, in order to understand the qualitative features of this of this non-linear oscillator, we had also plotted the phase portraits of the system, ok.

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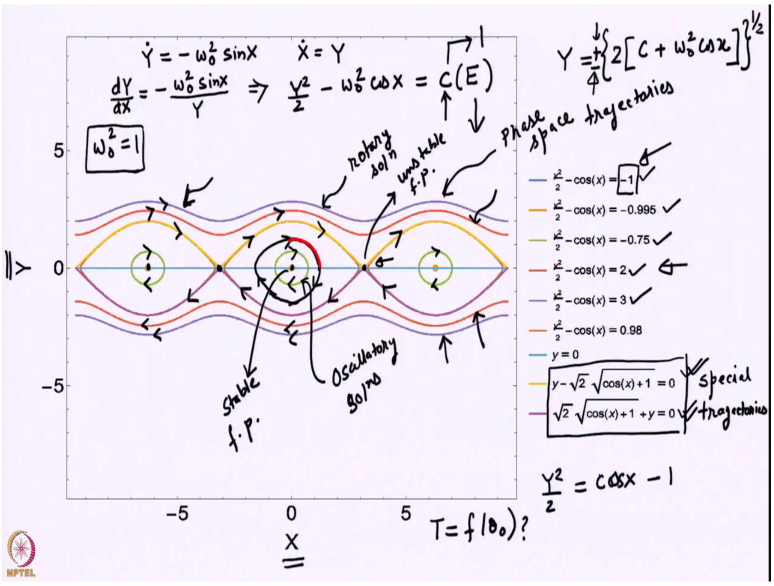

So, recall that we had seen two kind of trajectories, one was a rotary trajectory and another was an oscillatory trajectory. The rotary trajectories are open curves, oscillatory trajectories are closed curves. And there is in general a special trajectory which separates between these two. The oscillatory solutions are ones which we had physically explained before and the rotary solutions are one where the energy is so high that it actually executes a rotary motion about its center.

Now let us continue further.

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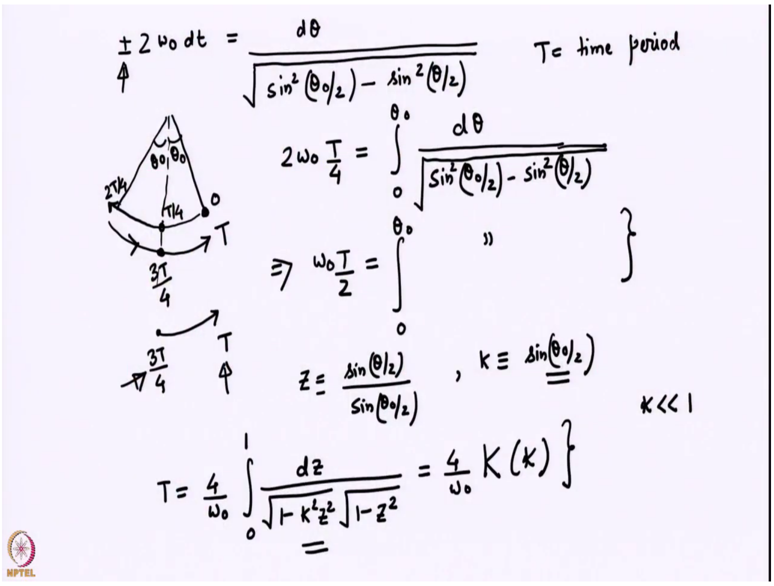

We had also seen that we could make an approximation for the time period of the pendulum by writing it as 4 times the time taken to execute one-fourth of the oscillation, and then we had got another elliptic integral.

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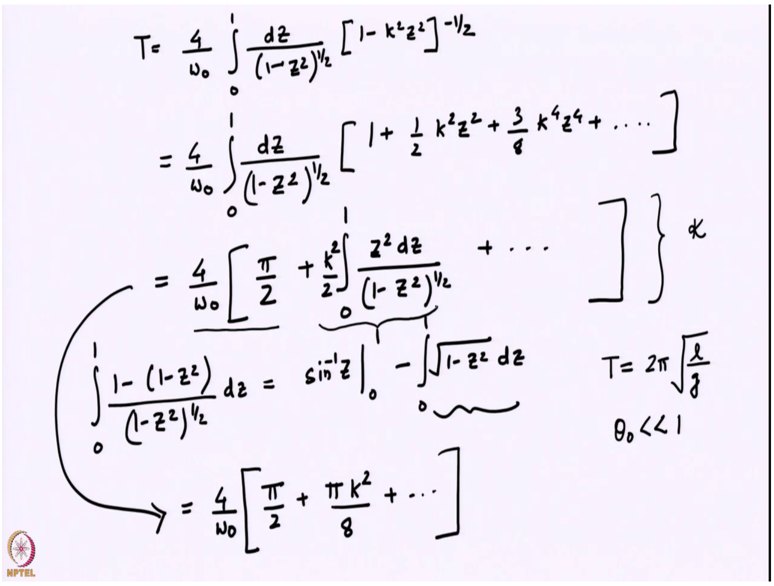

And we had solved this elliptic integral by doing a Taylor series expansion for small k; the modulus of the elliptic function. So, small k would imply that sin theta naught by 2 is small and so, we had obtained an expression for the time period of the pendulum.

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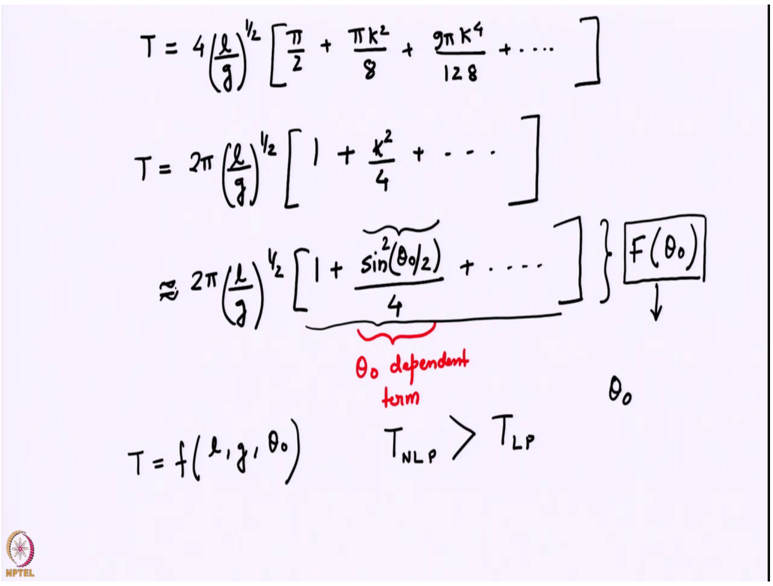

Now, even without this it is possible to understand you can see that the time period of the pendulum is a function of k. It is only at a linear approximation that we only retain the first term in the infinite series and then there is no k in the resultant expression.

Consequently in the linear limit we recover our familiar thing which is T is equal to 2 pi root l by g. But one has to understand that this is an approximation, this is the first term in an expansion, if your initial theta naught is much much small than 1; let us say we are writing theta naught in radians. If it is much much less than 1 then you can do a Taylor series expansion of sin and then you will recover this result.

However, as your theta naught increases you will actually need to take into account these corrections. So, the first correction is this term and you can see that this is this has a coefficient k square, this is why we had calculated the first term and you can see that there is a theta naught dependent term. So, this tells us an important thing which is generic feature of non-linear oscillators that the time period of motion actually depends on the initial perturbation amplitude, ok.

So, T is not just a function of l and g as linear theory would tell us, but it is also a function of theta naught. First theta naught sufficiently small this, this can be expressed this function. So, this infinite series actually represents a function some function of let us call it theta naught you will get this series. So, this is essentially what we have computed, ok. So, so you can think of this as a infinite series which represents this function, it is, ok.

So, now with this theta naught dependent term we can understand that the time period of a linear oscillator, as you can see this term is positive because it is sin square theta naught by 2. And so, the time period of a non-linear oscillator or a non-linear pendulum is actually greater than the time period of a linear pendulum, ok. The inverse would hold true for frequency and you can see it from here. Now how much greater is this T NLP compared to T LP depends on theta naught.

Now, remember that this is an approximation, this time period is an approximation and for larger values of theta naught we may take into account we may have to take into account other higher terms in the expansion. So, now, let us look at some; what I will show you next is some numerically solved plots of the exact solution which we had derived earlier. So, we had derived this was the exact solution. And I have solved this, I have just plotted this for different-different values of theta naught and I am comparing this with the linear solution.

For reference I have also plotted the numerical solution to the full non-linear ordinary differential equation. You will see that the exact solution and the numerical solution lie on top of each other. For small values of theta naught the linear solution also lies on top of the other two, but as we increase theta naught this exact solution will start differing from the linearised corresponding solution. So, let us look at that.

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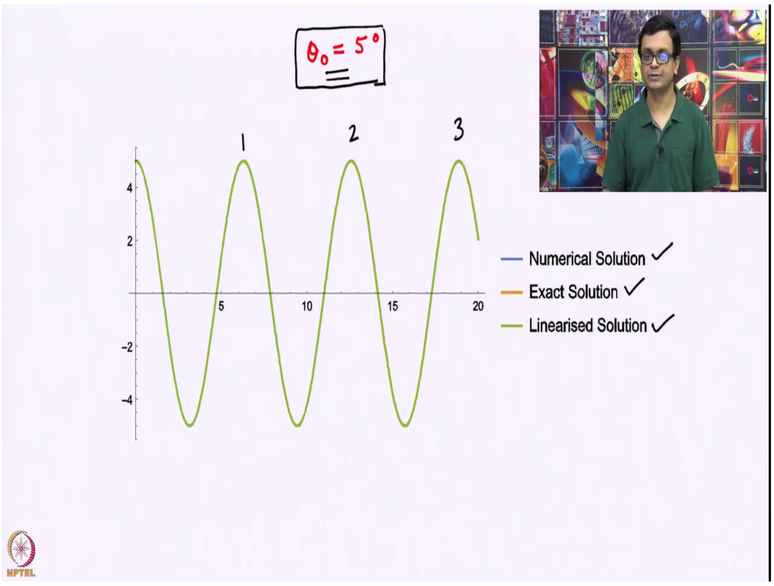

So, this is a plot where I am showing all three solutions. Now because the numerical solution; the numerical solution implies that it is a solution to the full equation solved numerically for the same initial conditions. So, the numerical solution and the exact solution will be indistinguishable from each other.

Now for this value of theta naught this is a reasonably small value of theta naught if you express this in radians, then this is a small value of theta naught. And for this value, you can see that all the three of them are indistinguishable from each other. So, you can only see the green color, the blue and the orange lines are actually on top of the green line and hence you cannot distinguish it ok.

So, you can see that for these kind of angles the linear approximation is an excellent approximation; at least for the first. So, this is 1 time period, this is 2 time period and this is about 3 time periods. So, at least as far as I have plotted for the first 3 time periods if you are interested in predicting the displacement of the pendulum as a function of time; linear theory does an excellent task, non-linear the non-linear solution is not necessary.

Let us increase this theta naught slightly more.

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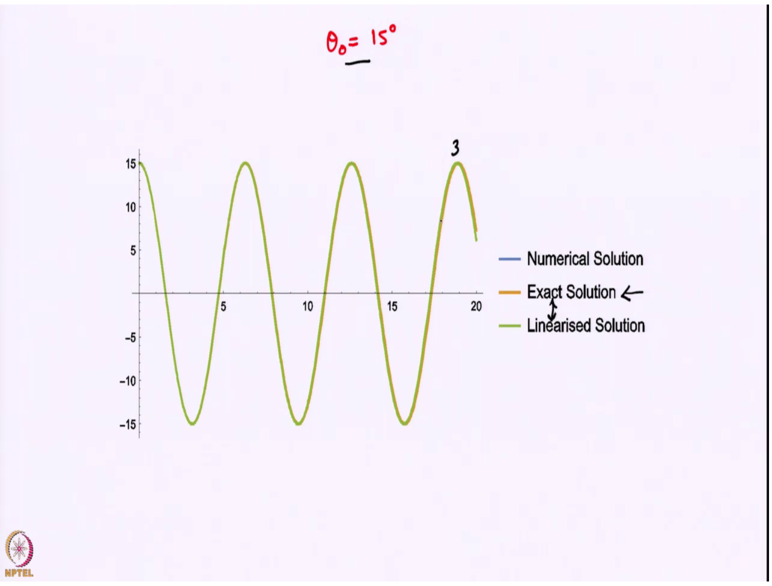

So now, we have made theta naught to be 15 degree. So, now you can see, you can still see that they are almost on top of each other. However, you will see that the exact solution; you can now see the exact solution on the linearised resolution, but towards the end of the third time period which is here you can see that the green and the orange curve these two have a small mismatch for this angle this initial angle.

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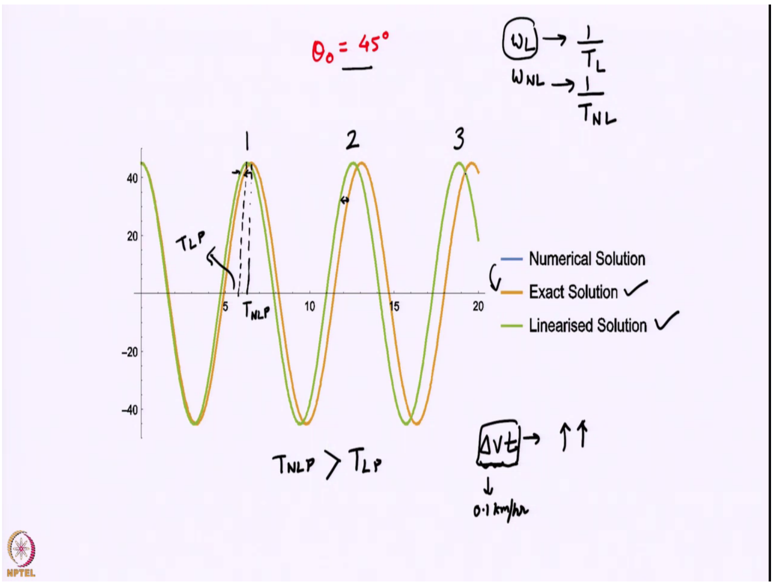

Now, let us go further to larger angles. Now I have taken a much larger angle pi by 4. So, this and now you can see visible differences between the exact solution and the linearised solution.

The numerical solution still lies on the on top of the exact solution and so you cannot really see; because the orange curve is completely overwhelming the blue line ok. So, now, you can see that after one oscillation there is some difference, after 2 oscillation the difference increases, after 3 oscillations the difference becomes very pronounced.

So, for example, this predicts that the time that it takes to take one or to complete 1 oscillation for the non-linear case; this is the time when the non-linear or the exact solution the non-linear pendulum will complete 1 time period this is slightly more than 6; whereas, the time that the linearised solution would take is slightly less. So, this is T NLP and this is T LP and we have seen that T NLP is greater than T LP. So, this is consistent with what we had seen earlier from our approximate expression.

Now, you can easily understand why is this difference becoming large, you can think I will give you a very simple analogy here and this analogy is not exact, but you can think of it like this. That suppose we have 2 cars, here there are two frequencies there is a linear frequency and there is a non-linear frequency. We have seen that the linear frequency and the non-linear frequency are different from each other, ok.

So, corresponding to this there is a linear time period it is inversely proportional to the linear frequency and corresponding to this there is a non-linear time period. Now think of the frequency, frequency we know is like angular speed.

So, now, think of this example where we have 2 cars starting at the same location with slightly different speeds. So now, if the car has one of them has a speed of 1 kilometer per hour, the other car has a speed of 1.1 kilometer per hour; a small difference of only 0.1 kilometers per hour.

Now, if I look at them after let us say 1 hour there will be a certain distance apart you can calculate, ok. So, let us say they are at a certain distance apart at the end of 1 hour. However, if you look at them after at the end of 2 hours you will find that they are a greater distance apart, ok.

So, the distance that they will be apart will be equal to the delta v into t. So, even if delta v is very small which in the our case is just 0.1 kilometer per hour as t go becomes larger and larger the product which represents the distance between the 2 cars becomes larger and larger. The same thing happens here.

So, here thing is a slightly more complicated situation because the speeds are actually angular speeds. An angle is not something which just grows in an unbounded manner because it is a periodic variable. So, after 2 pi it again repeats itself. So, there is only a maximum separation that is possible between the 2, after which it will again reduce and again increase, ok. So, you can see that at the end of. So, this gap is because the initial.

So, we have waited for approximately 6 units of time to see this gap. If you wait for about 12 units of time you will see a larger gap, if you wait for about 18 units of time you will see an even larger gap. I encourage you to think that what will happen if you keep plotting this for larger time.

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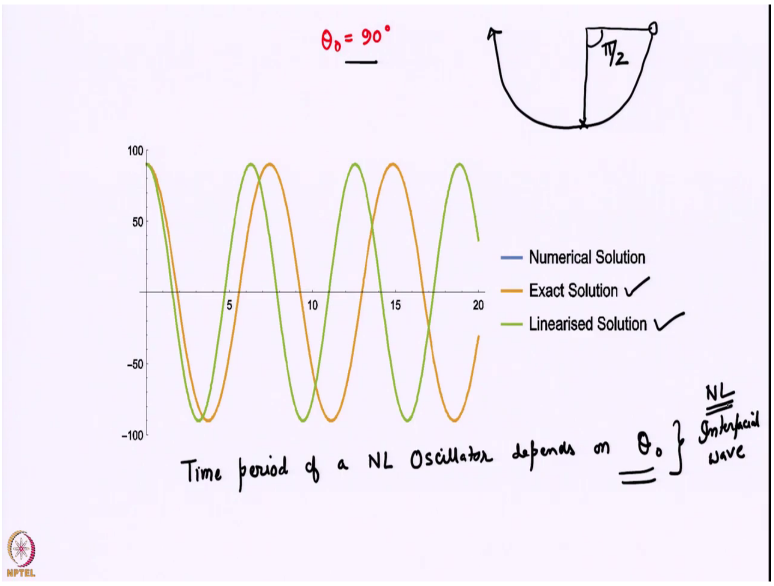

Just as a example I have also plotted this for 90 degree this is a large angle, so now I am leaving my pendulum horizontally. So, this is my vertical position and now I am leaving it at pi by 2. And so this is a large amplitude oscillation that it is going to it is going if it is a non-dissipative pendulum it will just go climb to minus pi by 2 on the other side and keep doing this.

And so, you can clearly see that even the first time period of oscillation is visibly different for the exact problem as compared to the linearised problem, ok. So, the main thing that we learned here is that the time period of a non-linear oscillator depends on the amplitude of perturbation, ok. So, it is a this is a very important conclusion; non-linear oscillator depends on theta naught or the initial perturbation amplitude.

We will see this feature also later in when we study interfacial waves. We will see that when we understand interfacial waves using linearised equations of motion, we will obtain analogous frequency relation, we will call it the dispersion relation. And you will see that for a linearised interfacial wave the dispersion relation does not depend on the amplitude of the wave.

However, the moment you go to higher orders of non-linearity we will find that the dispersion relation includes a amplitude correction. And that amplitude correction is obtained only one when go allows for non-linear contributions to the equations of motion. So, we will see more of this when we encounter interfacial waves. So, this is a qualitative similarity between a non-linear pendulum and a non-linear wave that we are going to study later in this course; so, non-linear interfacial wave.

Now I will come to these comparisons once again when we as we start doing introduction to perturbation methods. As I told you before the ability to solve non-linear equations exactly is a limited ability, we will not be always be able to solve non-linear equations exactly the way we could do it for the non-linear pendulum.

So, one requires approximations which allow us to take into account contributions from non-linearity. So, one such class of techniques which allow us to do this is called perturbation methods. So, we are going to start with introduction to perturbation methods now

Once again we will introduce the basic techniques in perturbation using first algebraic equations and then ordinary differential equations. However, whatever we will learn, the various techniques that we will learned as a part of perturbation methods we will find immediate applications when we start doing interfacial waves in the second part of the course.

So, let us move on to perturbation methods. So, what is perturbation method? Perturbation method allows us to solve or to make systematic approximations systematically better approximations to problems which typically cannot be solved exactly.

This does not imply that perturbation does not work for problems which can be solved exactly. In fact, the first set of examples that we will take up we will take up problems where we will be able to solve the problem exactly and we will compare that what does perturbation tell us about the answer; compared to the exact answer. And this will allow us to understand what is the perturbation method really doing.

Now, in any perturbative problem there are a couple of steps, the first step is usually non-dimensionalization this introduces a small parameter in the problem. One typically requires a small parameter in any kind of perturbating perturbative technique, because it allows us to expand typically in some power of the small parameter. Now in the example that I am doing right now, we will skip the non-dimensionalisation step and I will come to non-dimensionalisation shortly afterwards.

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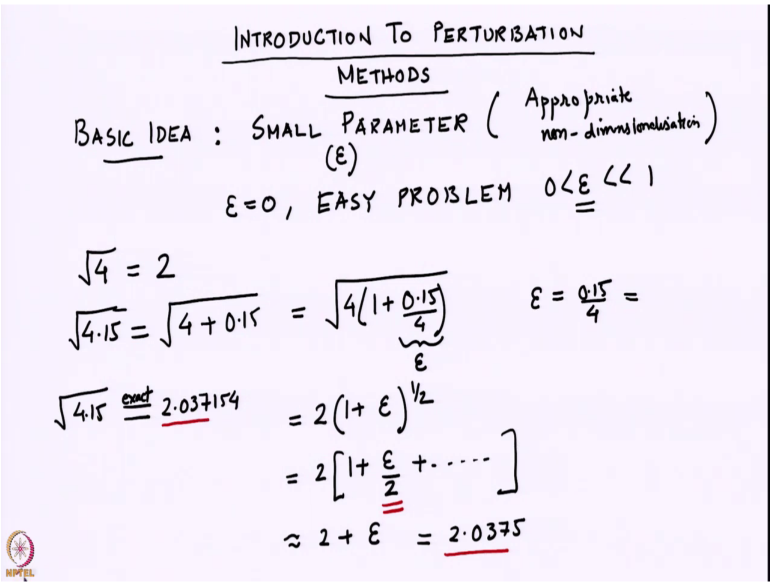

Now, let us see a very simple example the basic idea of perturbation is that there should be a small parameter in the problem. And this small parameter usually comes in through some kind of appropriate non-dimensionalisation; we will come to this shortly. But let us understand what does a perturbative technique involve.

So, now, I will include generally generically we will indicate our small parameter by the term epsilon; epsilon is usually a non-dimensional number. Now, the basic idea is that that when the small parameter is 0 we should have an easy problem which can be solved exactly. However, the problem becomes harder when epsilon is non-zero and we basically want to ask the question that can we find out what is the answer to the problem as when epsilon is non-zero.

Typically we will be able to solve because epsilon is a small parameter. So, epsilon will be greater than 0, but much much less than 1. Typically we will assume that epsilon is much less than 1 and we will be able to say something about the solution to the problem when epsilon is not 0, but less than 1. Let us look at a very elementary example as a first step.

So, suppose I have to calculate the square root of 4; this is an easy problem we all know the answer and this is 2. I am just taking the positive square root. Now suppose I make the problem slightly more complicated and I ask you; what is the square root of 4.15. We can of course, calculate this by if you know how to take the square root of real numbers. Let us say that we do not we have or we have forgotten how to take the square root of real numbers and we do not have a calculator near our hand. So, let us see if I can find some approximations.

So, this is and then I can write it as 4 into 0.15 by 4. Now I will treat this as my epsilon and so I can write this as square root 4 is 2 and then I can write this as 1 plus epsilon to the power half. We all know the binomial theorem for 1 plus epsilon to the power half and that is an infinite series whose first term is 1 and the second term is epsilon by 2 you can continue further.

Let us find out how good is this approximation 2 plus epsilon, the numerical value of epsilon is 0.15 by 4. And so, this actually turns out to be approximately this ratio turns out to be approximately 2.0375. What is the exact answer? If you use a calculator you can readily find the exact answer to be 2.037154.

So, you can see that just taking the first term in the binomial expansion gives us an answer which is reasonably accurate up to here we have got it right; so up to here we have got it right. If you want more accuracy you could try taking more terms in the expansion. Now this is the basic flavor of perturbative techniques.

Now in order to understand the basic idea of perturbation methods, I am going to introduce some formal symbols in the in this in the next discussion. And I will explain the meaning of that symbol, in particular it is called the order symbol. So, let us understand what is an order symbol.

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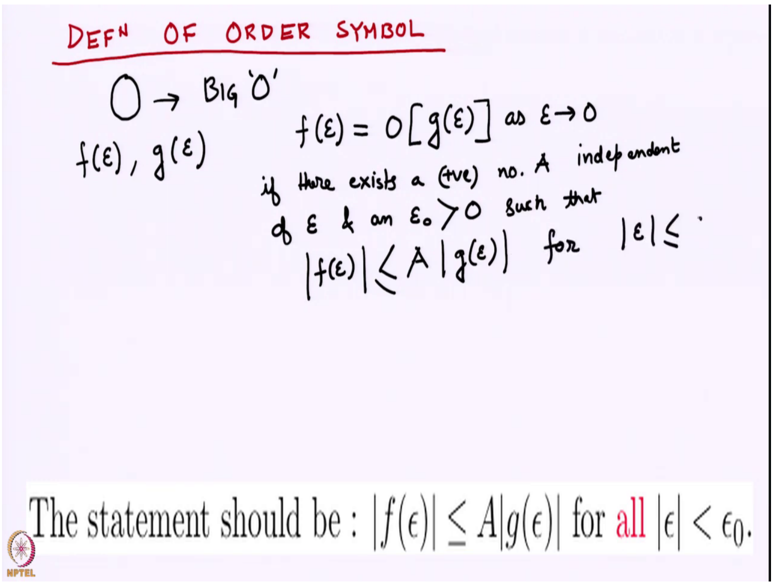

So, definition of order symbol. So, usually indicated by O; big O there is also a small o we will not talk about it right now. And so to differentiate between a small o and a big O, this one is called a big O. Now what does big O mean? So, I am going to write down. So, this order symbol allows us to compare the limiting behavior of functions.

So, let us say we have two functions f of epsilon and g of epsilon. The order symbol says that f of epsilon. So, as ok let me write the order symbol first f of epsilon is of the order g of epsilon as epsilon tends to 0 epsilon is a small parameter; if there exists a positive number A independent of epsilon and an epsilon naught another positive number such that.

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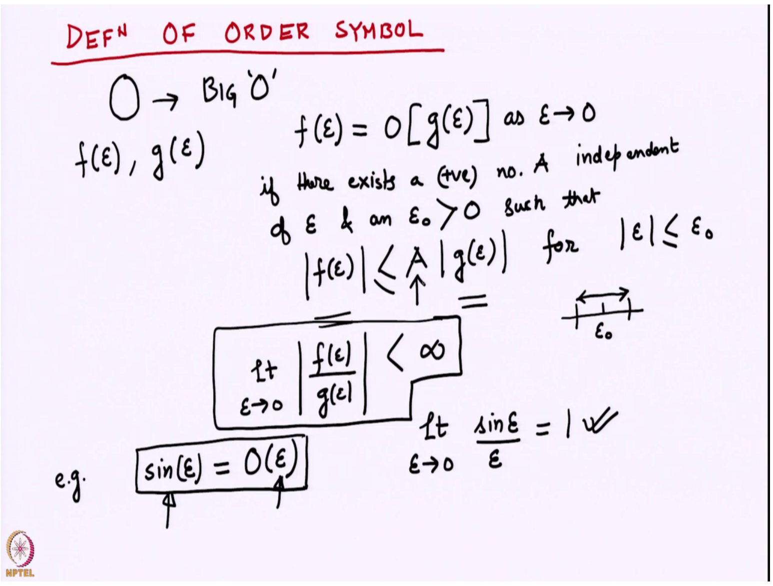

So, this is basically just a formal way of saying; that limit epsilon goes to 0 f of epsilon by g of epsilon is less than infinity. It is a, it is not unbounded that is what it is saying. So now, let us look at some examples let us learn this through examples to get a physical field for this.

So, our first example is sin epsilon as epsilon tends to 0. So, we claim that sin epsilon is of the order epsilon. How do we test this, we take the limit epsilon tends to 0 sin epsilon by epsilon, you can put a mod around it or you need not put a mod around it you can just work out this limit. We know that this limit is 1.

So, this is just checking this formula. Here f of epsilon is sin epsilon g of epsilon is epsilon itself. So, I am taking the limit of f by g and taking epsilon going to 0; you can put a model around it and we are finding that it is a finite value 1. So, sin of epsilon is of the order epsilon.

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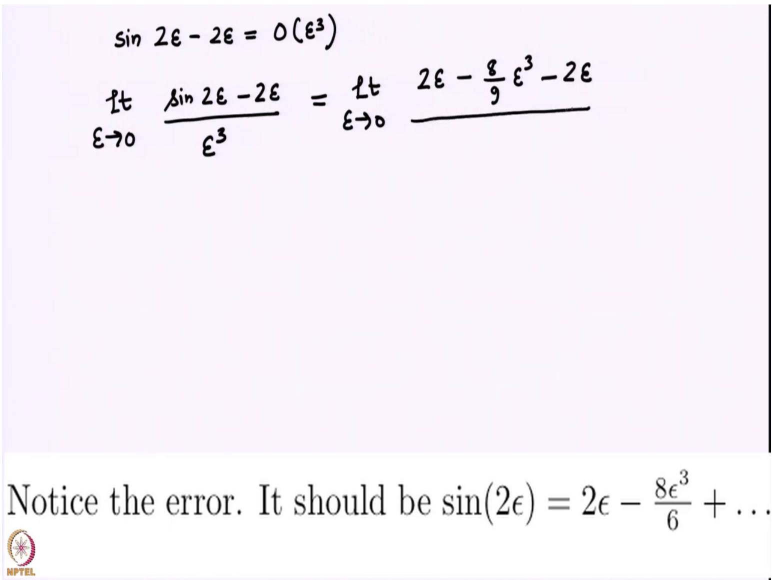

Let us take another example, sin 2 epsilon minus 2 epsilon is of the order epsilon cube. How do we check whether this is true or false? We take the limit epsilon goes to 0 sin 2 epsilon minus 2 epsilon divided by epsilon cube. If I expand sin 2 epsilon in a Taylor series about epsilon equal to 0, then the first term is sin x minus x cube by factorial 3 and so on.

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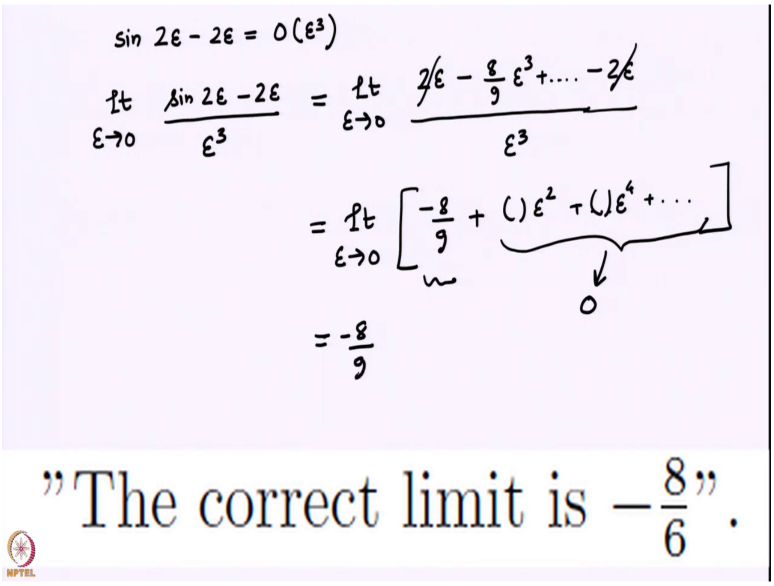

So, the first term is 2 epsilon minus 8 by 9 epsilon cube minus twice epsilon divided by; so again there are I should put a dot dot dot and then a 2 epsilon on that and then an epsilon cube. So, this 2 epsilon and that 2 epsilon canceled each other, the first term in this infinite series in the numerator is proportional to epsilon cube. And so we are left with the first term would just be 8 by 9 if I divide by epsilon cube.

The next term would probably be of the some number into epsilon to the power 5 if I divide by epsilon cube it will be something into epsilon square and so on. So, you will get another number into epsilon maybe 4 plus dot dot. The important point is if you take this limit of epsilon going to 0 everything here will go to 0, except the first term.

So, this limit is minus 8 by 9. The exact value of this limit does not matter. Remember the definition says if you can find some A, A can be a small number or a very large number, it does not matter what is the exact value of A, as long as you find a finite number in the limit f of epsilon is of the order g of epsilon. In this case it is less than 1 this number and if you take the magnitude of it then A is a positive number.

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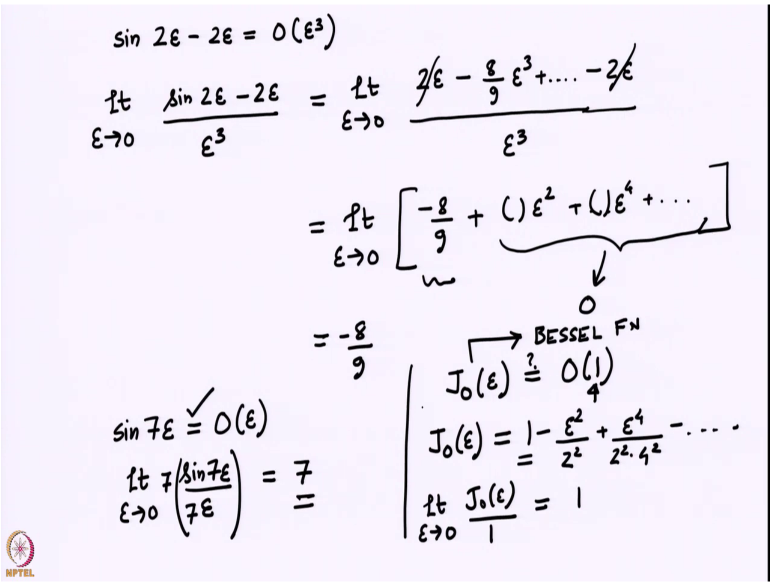

Let us try one more example. Sin 7 epsilon we just saw the sin epsilon is of the order epsilon, now I am saying sin 7 epsilon is also of the order epsilon; would this be true? Yes, because limit epsilon tends to 0 sin 7 epsilon by epsilon you can work this limit out by multiplying and dividing by 7. So, if you put a 7 here and a 7 there then this this part just goes to 1 and so this limit is just 7. Once again the answer is finite, as long as the answer is finite this is true.

One more example. J 0 of epsilon is of the order 1,this is very important. When I say something is of the order 1 then ideally I should be taking J 0 of epsilon in the numerator and 1 in the denominator, ok. The limit should be finite, order 1 does not mean that the limit is has an order of magnitude which is 1; it can be a number which is much much bigger than 1. As long as it is finite we will treat it as an order 1 number.

In this case how do we work this out; we use the series form of we have met this function earlier this is the Bessel function that we have seen earlier when we were looking at vibrations of a circular membrane. So, let us expand J 0 of epsilon in a series; there is a series representation of J 0 you can look it up in any handbook. So, the series representation is 2 square plus epsilon 4 by minus dot dot dot; they are all these other terms are also dependent on various powers of epsilon.

Now if I say, if I have to check this thing then I ask the question limit epsilon goes to 0 what happens to J 0 of epsilon divided by 1, because there is a 1 here. So, what do we get?

This is just 1 because every term after 1 goes to 0 because it is proportional to some power of epsilon. Note that J 0 epsilon is of the order 1, but even if the first term was not 1 if it was let us say 10 which it would be if I take instead of taking J 0 epsilon I take 10 J 0 epsilon or 1000 J 0 epsilon ok. So, 1000 J 0 of epsilon would also be of the order 1, ok.

So, I hope this basic idea is clear. We will look at more such examples in the next video.