Introduction to interfacial waves
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Lecture - 12
Time period of the non-linear pendulium

We were looking at the phase portrait and we had said that the phase portrait will help us understand the qualitative features of the non-linear pendulum. Let us write the equation of motion of a non-linear pendulum once again.

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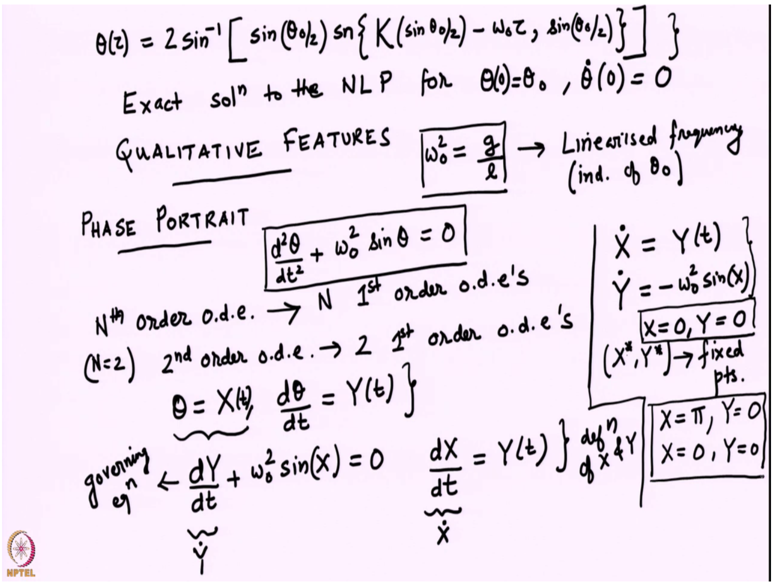

The basic idea of drawing a portrait is the following. If we have an nth order o d e, the idea of a phase portrait is to convert it into N first order ode's. So, if we have an N-th order ode it need not be linear it can be linear or non-linear that does not matter. The idea is to first convert it into N 1st order ode's.

In our case we have a 2nd order ode. So, N is equal to 2 for us. So, we will convert a 2nd order ode to 2 1st order ode's. This ode's will typically be non-linear and will also be coupled to each other. So, let us see how we do this.

So, the first thing is to rewrite theta as another variable X, then the derivative of theta and call it another variable Y; obviously, X and Y are functions of time just as theta and d theta by dt are both functions of time. Now, we go back to our governing equation and we see how can we write it as a first order equation.

You can see that if I think of this the first term as dY by dt then it is a first order equation in Y. So, let me write it as dY by dt plus omega 0 square sin X is equal to 0; that is my first equation for the evolution of the variable Y. What about the evolution of the equation variable X?

We have to write one more, so I need one more, dX by dt. You can immediately see, that if you take this equation theta is equal to X of t this is a definition. If you take this and differentiate it you would get dX by dt is equal to d theta by dt, but we have defined d theta by dt to be Y. So, dX by dt is just Y. This follows from the definition.

So, this is the governing equation rewritten as a first order evolution equation for the variable Y, this is from the definition of X and Y.

So, have we written it as a set of 2 1st order ordinary differential equations; you can see that we have, let me use the symbol X dot. So, this is X dot. So, X dot is equal to Y of t. Y dot which is this is equal to minus omega 0 square sin of X. Those two are our equations which govern how X and Y evolve in space.

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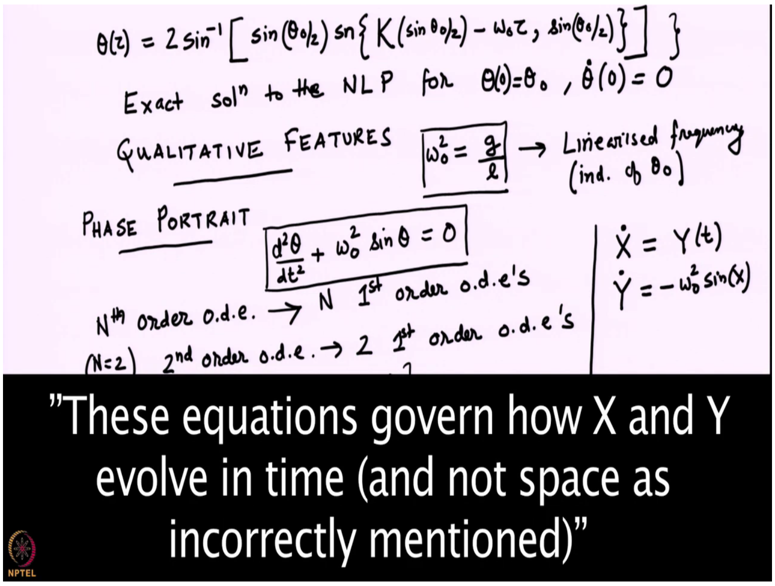

Let us now plot X and Y and identify some important features of this equation. Notice that, this these two set of equations say that if you choose X is equal to 0 and Y equal to 0. So, this tells you this equation is a non-linear equation, they are coupled to each other, it is non-linear because there is a sin X term they are coupled to each other. And these tell you that if I start from a certain value of X and a certain value of Y at X equal to at time equal to 0; how will X change in time and how will Y change in time.

So, I can draw an X Y plane and I can start at a certain point on that plane; that would be equivalent to setting some initial conditions. I can and I can draw trajectories on that plane that trajectories will be governed by these two equations. In particular note that there are some special points in the plane such that if I use those if I start with those points I will not go anywhere. By definition those points will be solutions where the right hand side should go to 0.

If with the right hand side of both the equations, if there are such special some special points X star and Y star such that add X star and Y star X dot is equal to 0 and Y dot equal to 0, then you can immediately see that the point is if you start at that point you remain at that point you do not go anywhere.

Those points are called equilibrium points in this context this is the base state or they are also called in the language of phase portraits they are also sometimes called fixed points. You can see immediately see that X is equal to 0 Y equal to 0 is a fixed point. What does that physically imply?

It implies that if you take a pendulum hang it vertically downward at its equilibrium position do not give it a initial velocity; X is equal to 0 is theta equal to 0, Y equal to 0 is theta dot equal to 0. So, if I start my pendulum at the lower most position, do not give it any initial velocity; the pendulum is going to stay there. This is just a mathematical way of saying this simple fact.

You also see that there is a non trivial fixed point of these equations, X is equal to pi Y is equal to 0. This is telling us again one more physical equilibrium situation that if I take the pendulum. So, let us say I have this pen and this is like a pendulum is pivoted here and if the point is here and if I do not give it an initial velocity this is a fixed point of the system, but I can also take it to the topmost point the angle is now pi.

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So, X is equal to pi I do not give it any initial velocity we know that this is also a fixed point, if you leave the pendulum here it stays there. The reason why it falls down is because this is an unstable point, whenever we try to put it here we always introduce some perturbations and the perturbations causes it to fall. But if you in principle if you could do a very careful experiment where you do not inject any perturbation and bring it exactly at the vertical position it will stay there.

So, these are the two fixed points, of course mathematically you can put more. You can put X is equal to m pi where m is plus minus 1 plus minus 2 plus minus 3. Those points do not give more physical solutions they are just repetitions of the same two physical points the 0 the lowermost point and the topmost point.

So, this so, X is equal to 0 Y equal to 0 and X is equal to pi Y equal to 0 are the two fixed points of this system. We also know that 0 0 is stable pi comma 0 is unstable, we also intuitively know this from experience. Now let us draw the phase portrait of a pendulum of this non-linear pendulum and let us examine the trajectories on the phase portrait. This plot of X and Y is called the phase portrait. Let us draw the phase portrait for this pendulum.

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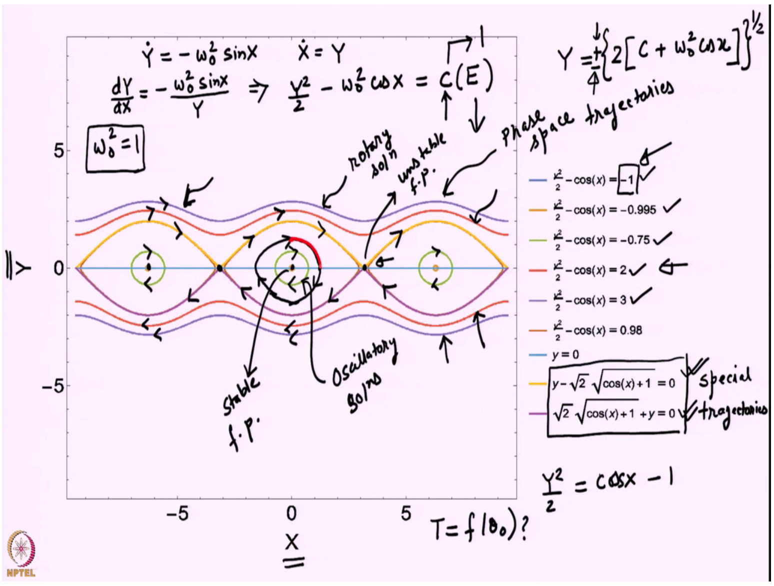

I have already drawn it here you can see that this is X and this is Y. Now let me explain how I have plotted these trajectories. Our equation was so, I will just write it here; our equation was Y dot is equal to minus omega 0 sin X and X dot is equal to Y. If I divide both the equations then I get the dt cancels out dY by dX is equal to minus omega 0 square sin X divided by Y.

This is an equation which can be readily integrated. If I integrate it then I get Y square over 2 minus omega 0 square cos X is equal to constant integral sin X is equal to minus cos X. Now you can see that on the X Y plane I can choose different values of the constant and I can plot curves in the implicit form Y and X, this is what I have drawn here.

The X axis is the horizontal axis the Y axis is the vertical axis. I have used you can see that I can write it in this form if you want to go from the implicit to the explicit form I can write this as there is a half and there is plus minus. So, for any value of c you will get a curve and for every value of c you will get two such curves; one on the upper half plane and one on the lower half plane.

I have plotted it for different-different values of c. So, these are what are known as phase space trajectories. For each value of c we get a phase space trajectory. What is the physical meaning of c? You can see that c is related to the energy of the system, c is not exactly energy it is a function of the energy.

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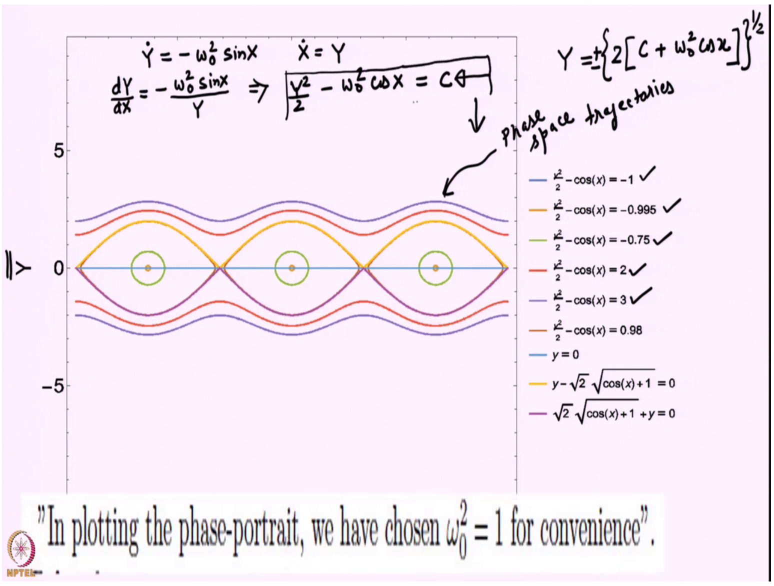

In fact, this equation is nothing but another version of an equation we have already written before.

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Energy of the pendulum (at any time t) is:

$$\frac{1}{2} m \left(\frac{ds}{dt}\right)^2 + mgl(1-\cos\theta) = mgl(1-\cos\theta_0)$$

$$\Rightarrow \frac{1}{2} \cancel{m} l^{\cancel{2}} \left(\frac{d\theta}{dt}\right)^2 + \cancel{m}g\cancel{l}(1-\cos\theta) = \cancel{m}g\cancel{l}(1-\cos\theta_0)$$

$$\Rightarrow \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{g}{l}\right)\left[\cos\theta - \cos\theta_0\right]$$

$$\Rightarrow \left(\frac{d\theta}{dt}\right)^2 = 2\omega_0^2\left[\cos\theta - \cos\theta_0\right] \qquad \omega_0^2 \equiv g/l$$

$$\Rightarrow \left(\frac{d\theta}{dt}\right)^2 = 2\omega_0^2\left[1 - 2\sin^2(\theta/2) - 1 + 2\sin^2(\theta_0/2)\right]$$

$$\Rightarrow \left(\frac{d\theta}{dt}\right)^2 = 4\omega_0^2\left[\sin^2(\theta_0/2) - \sin^2(\theta/2)\right]$$

So, if you if we go back to our old equations then you can see that the equation that we wrote down here is actually just being written down there you know on the face portrait in terms of X and Y; d theta by dt is X Y square. So, if you bring the 2 to the left hand side this is half Y square minus omega 0 square cos X is equal to minus omega 0 square cos theta 0.

So, you can see that the constant that we have in our phase portrait is actually related to the energy, it is not quite the energy because something has been subtracted from energy by the time we arrive at this step. So, it is a function of the energy; is actually a function of the energy e or the initial energy this is a conservative system. So, the energy stays the same at all times.

Now by choosing different-different values of this constant, you are essentially initializing the system by giving it different amounts of energy. Note that these each of these trajectories each trajectory corresponds to a certain amount of energy. So, if I take my pendulum and give it leave it at 20 degree angle I am injecting it certain amount of gravitational potential energy with that energy it will keep oscillating at twenty degrees.

If I leave it at a larger angle it will execute it has it starts with a bigger energy. So, it will execute a more energetic motion, but it will still be periodic. You can see that each of these trajectories have a sense. So, you can work this out that each of these trajectories goes in the clockwise sense.

In particular there is one such trajectory which corresponds to these two. Notice that we had drawn these trajectories for the special case of omega 0 square by assuming that omega 0 square is equal to 1. Now, there are 2 kinds of very special trajectories which occurs when c is equal to 1.

You can see that if c is equal to plus 1 then I get these two trajectories. What is special about them? These trajectories separate two kinds of solutions; we have one kind of solutions which are represented by this, these are oscillatory solutions whereas, these are rotary solutions.

So, physically what does this mean? It means that if you give it an initial energy such that if you leave it here then this is an oscillatory solution. But if I start from this angle let us say and I give it so, and I give it a kick which is so hard that it actually goes over the top and comes back. Then you can see that this pendulum is going to have a rotary solution. So, these kind of solutions are rotary solutions.

So, these special trajectories separate two kinds; outside them we have rotary solutions, inside them we have oscillatory solutions. It is now possible to understand why did we choose a negative sign in our earlier exercise.

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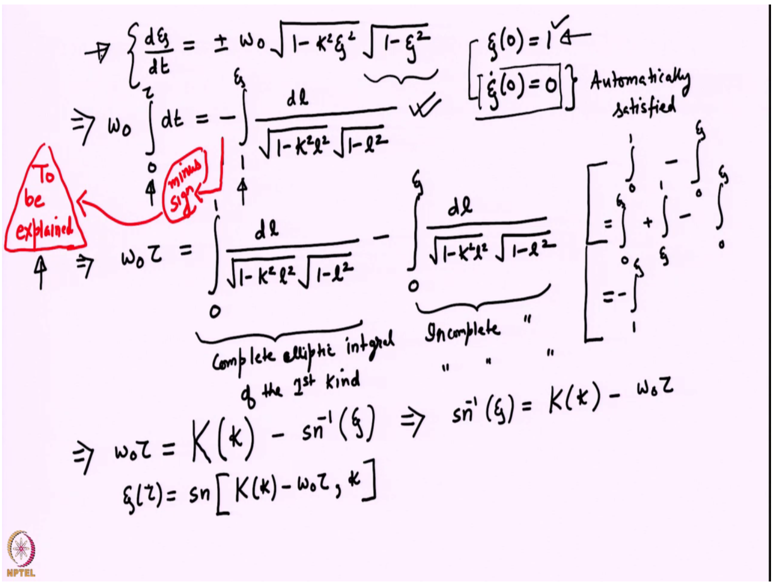

So, here recall that we have chosen a negative sign. You can see that this is an expression for d zeta by dt. I am starting the pendulum at some initial angle theta naught that corresponds to zeta of 0 is equal to 1. So, on the phase portrait, I am starting somewhere on the X axis and there will be a trajectory which will be going like this which will be passing through that point in that sense.

So, you can immediately see that I will be going into the negative lower half of the plane, ok. So, the that the lower part of the curve is identified by this minus sign, ok, so that is why we chose the negative sign. You can also see that the fixed points on this is this, this, this and so on.

This is the stable fixed point. That point is the unstable fixed point. You can see that there are two trajectories which are coming into the fixed point and there are two trajectories which are going out of the fixed point and all of these are this sense, ok. So, this fixed point is the same as that fixed point.

Now the question is, if you make the if you keep going so, you can see that the lower value there is a lower value of c there is no upper value of c you can give it as much energy as you want, but there is a minimum value of c and that is minus 1.

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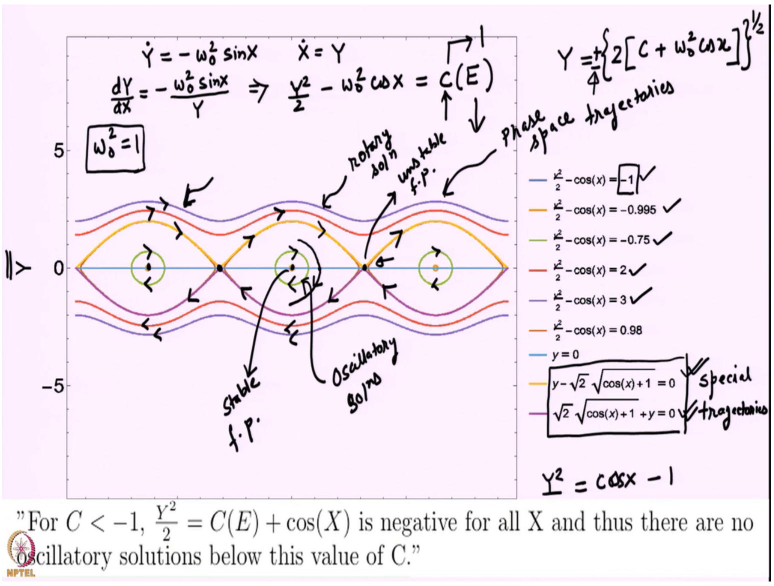

So, this is the minimum value that we can give to c. However, there is no upper limit on what is the maximum value. So, for example, if we give 2 then we recover this trajectory and the negative of that is this trajectory. So, we just take the plus and the minus sign there and give it c is equal to plus 2. If we give it 3, then our trajectory goes even further. So, these are rotary solutions which move faster and faster.

Now let us ask that for the oscillatory solutions. So, I am looking at these kind of solutions, where it goes like this. As I go from smaller and smaller amplitude to larger and larger amplitude what happens to the time period. Looking at this phase portrait, it is not obvious that the time period is independent of the amplitude. In this case is T a function of theta naught that is the question that we are asking. So, let us answer that question in an approximate manner.

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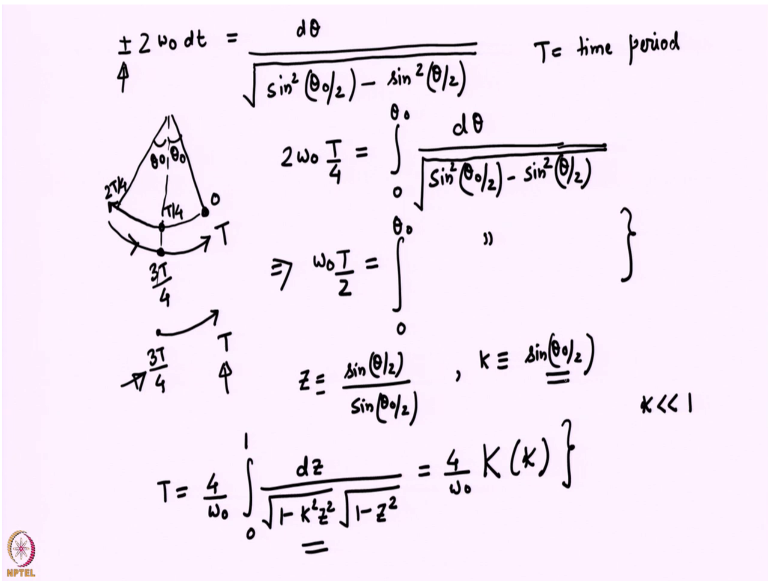

So, we have seen that; now suppose we leave the pendulum at this theta some value of theta naught we are interested in the oscillatory solutions, not the rotary solution and we are trying to compute the time period of the oscillatory solution. If I leave it at a positive value of theta naught, I expect the pendulum to go to the other side at the same angle theta naught and then come back.

So, in this part if I split these trajectories and it will come back up to here. So, if I split this trajectory into four parts then it goes if the total time is T, then when it is here so this is time T equal to 0 this is T by 4 this is 2T by 4. Then when it again comes back this is 3T by 4 and then this is T. By symmetry I know that each of these parts takes equal amount of time.

So, I will just T is the time period, and so I am just going to integrate this equation in the last part when it goes from here to there. So, my time is varying from 3T by 4 to T. In the phase portrait, what does that correspond to? In the phase portrait it corresponds to this part of the motion.

It just corresponds to this part of the trajectory; this one fourth of the trajectory. This entire thing which looks like an ellipse, but really is not an ellipse can be split up into four parts its very symmetric. So, it can be split up into four parts. I am just looking at the time taken to go in the last part and I will multiply that by 4; that will give me the time period of the total motion.

So, I say and you can immediately see that now I have to take the plus sign because I am integrating in the upper half of the phase space portrait. So, if I integrate this it will look like T omega 0 T by 4 0 to theta naught. I am going from angle the angle is 0 when T is equal to 3T by 4 and the angle is again theta naught when t is equal to capital T. So, 3T by T minus 3T by 4 is T by 4 and I get this integral. So, we can write this as; now it is essentially the same thing.

Now, I can immediately do the same exercise that we did earlier, convert this into an elliptic integral. However, I am going to do an approximation on the elliptic integral and not use the elliptic functions per se. So, let us see how we do that. So, I define Z is equal to sin theta by 2 divided by sin theta naught by 2 and K is sin theta naught by 2. So, Z is basically sin theta by 2 divided by K.

If we do this then it is it can be shown with a little bit of algebra that this equation just converts to 4 by omega 0, 0 to 1. You can see that when the upper limit is theta naught then Z is 1, so it is 0 to 1 dZ square root 1 minus K square Z square. Let me write Z that manner into square root 1 minus Z square and we have seen that this is just 4 by omega 0 into big K of small k, the complete elliptic integral of the first kind.

However, we can do an approximation on this integral. Suppose I do not know what this function looks like. Let me do an approximation on this integral for small k; k remember is sin theta by 2. If my angle is small, but not very small I can do a Taylor series approximation and retain more and more terms in the Taylor series approximation. And the hope is that, that will give me a glimpse into what is the first correction due to non-linearity. So, let us do that.

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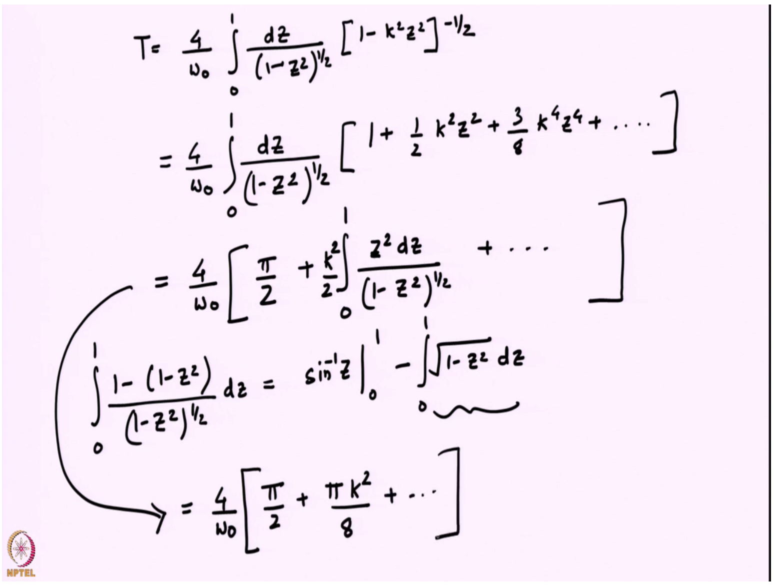

So, you can see that T is equal to 4 by omega 0 0 to 1 dZ by 1 minus Z square to the power half and I will take the other term because that contains K. So, this is 1 minus K square Z square to the power minus half. And now I can use binomial theorem for small k and expand this.

And if I just use binomial theorem I get; now I can integrate term by term. So, the first term is just sin inverse Z when I impose the limits that gives me pi over 2. The second term is this integral 0 to 1; I will put a K square here by 2 Z square d Z divided by 1 minus Z square to the power half plus dot dot dot let us not worry about the other terms.

Now, this term this integral can be easily done using elementary methods. So, you can for example, write it as 1 minus or rather 1 minus 1 minus Z square divided by 1 minus Z square to the power half dZ and then the first integral is just sin inverse Z from 0 to 1 minus square root 1 minus Z square 0 to 1 dZ. And this can be easily solved with the trigonometric substitution.

If you do all of these things you will just find that the second integral is just pi by 8 or rather pi by 4. So, I am just going to write that this is equal to 4 by omega naught into pi by 2 plus pi K square by 8 plus dot dot dot.

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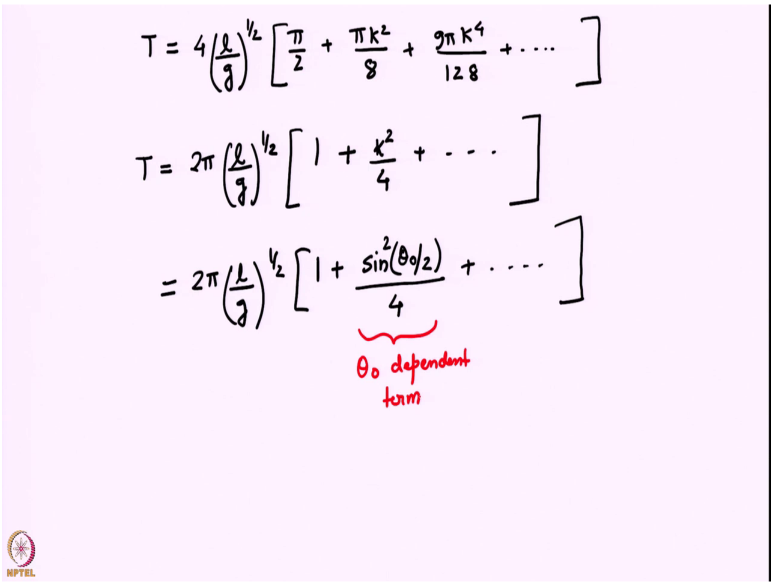

So, our expression for T looks like T is equal to 4 omega naught is l by g to the power. So, it was 1 by omega naught. So, it becomes l by g to the power half, and then we have a pi by 2 plus pi K square by 8 plus dot dot dot. Incidentally the next term can also be calculated with a little bit effort and then that term is 9 pi K 4 by 128. It just requires doing one more integral.

And you can write this as T is equal to; if I pull out a pi by 2 out of the bracket then I have 2 pi square root l by g the familiar time period of a simple pendulum 1 plus. If I pull out a pi by 2 outside then this becomes K square by 4 plus dot dot dot. And this is telling us something very very interesting.

1 plus sin theta naught over 2 sin square divided by 4 plus dot dot dot dot. This is telling us that the time period of a pendulum of a non-linear pendulum is not independent of theta naught. This is an approximation to the time period notice that we have ignored higher order terms in the dot dot dot expressions.

So, this is a theta naught dependent term. This is a very important conclusion that we will see again when we meet interfacial waves; non-linear interfacial waves particularly. We will see that linear interfacial waves are governed by a dispersion relation where the amplitude of the wave does not figure in the dispersion relation.

Here also if you take a linearized pendulum the time period of the pendulum or the frequency of the pendulum does not depend on the amplitude of initial perturbation that is given to the pendulum. However, that is not the complete story the; an actual pendulum this is there is an actual dependence of the time period.

And if you take 2 pendulums and leave one at 1 degrees and the other at 45 degrees their time periods will be slightly different. This will have an effect on their motion at long times. We will see that when we do perturbation methods.