Introduction to interfacial waves
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Lecture - 11
The non-linear pendulum (continued.)

We were looking at the non-linear pendulum, we had written down the equation of motion for the pendulum.

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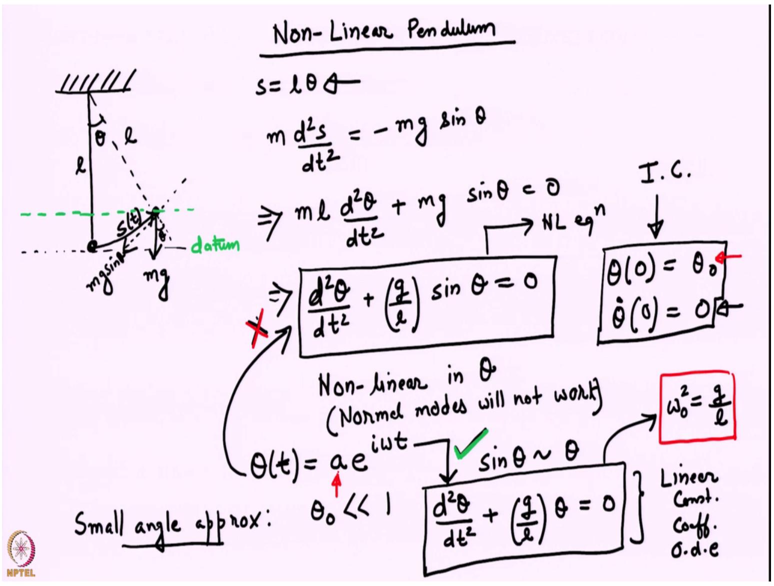

It was found to be d square theta by dt square plus g by l sin theta equal to 0. We wanted to solve this equation subject to the initial conditions that I give the pendulum an initial angular displacement and 0 angular velocity.

So, theta of 0 is theta 0 and theta dot of 0 is 0. Now the first thing that we noticed already was that the normal mode approximation will not be appropriate, because this is a non-linear equation. So, if you substitute theta is equal to some number a into e to the power i omega t, this non-linear equation we will not be able to satisfy this and do it the way we did it in the case of linear equations.

Now, we also discussed that this equation is typically solved in the small angle approximation where sin theta is approximated as theta, this is typically expected to be valid when the initial displacement theta naught is small. So, when theta naught is sufficiently small, then you can expand sin theta and replace it by this first term in the Taylor approximation as theta.

And then the resultant equation becomes a linear equation in theta; this equation is amenable to normal mode analysis, here the equilibrium state is 1 where the pendulum is horizontal. So, that is the base state. So, when we write theta is equal to a e to the power i omega t on the linearized equation it works.

And leads to a frequency relation which tells us that omega 0 square is equal to g by l. Once again like before the amplitude a is has to be determined from initial conditions and it is in general a complex number. Now, before we go further discuss our solution of the non-linear pendulum in terms of elliptic functions, notice that the frequency of the linearized pendulum is purely a function of the system parameters.

The acceleration due to gravity and the length of the pendulum l, the length of the string which we have assumed to be inextensible here. In particular notice that omega 0, the natural frequency is independent of theta naught that the angular displacement with which we leave the pendulum at time t equal to 0.

Let us look at the solution to the exact problem without doing this linearization and let us learn about the features of the non-linear problem.

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Energy of the pendulum (at any time t) is:

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 + mgl(1-\cos\theta) = mgl(1-\cos\theta_0)$$

$$\Rightarrow \frac{1}{2}\cancel{m}l^{\cancel{2}}\left(\frac{d\theta}{dt}\right)^2 + \cancel{m}g\cancel{l}(1-\cos\theta) = \cancel{m}g\cancel{l}(1-\cos\theta_0)$$

$$\Rightarrow \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{g}{l}\right)\left[\cos\theta - \cos\theta_0\right]$$

$$\Rightarrow \left(\frac{d\theta}{dt}\right)^2 = 2\omega_0^2\left[\cos\theta - \cos\theta_0\right] \qquad \omega_0^2 \equiv g/l$$

$$\Rightarrow \left(\frac{d\theta}{dt}\right)^2 = 2\omega_0^2\left[1 - 2\sin^2(\theta/2) - 1 + 2\sin^2(\theta_0/2)\right]$$

$$\Rightarrow \left(\frac{d\theta}{dt}\right)^2 = 4\omega_0^2\left[\sin^2(\theta_0/2) - \sin^2(\theta/2)\right]$$

So, we started by looking at the energy of the pendulum and we wrote an equation of this form, we equated instantaneous energy to the initial energy, in this case the initial energy is purely potential energy.

And it finally led us after some algebraic manipulations, it led us to an equation which was of this form. Notice that this is already a first order equation and that is because we did not start with the equation of motion, but we started with the first integral which is the energy. Now let us continue further.

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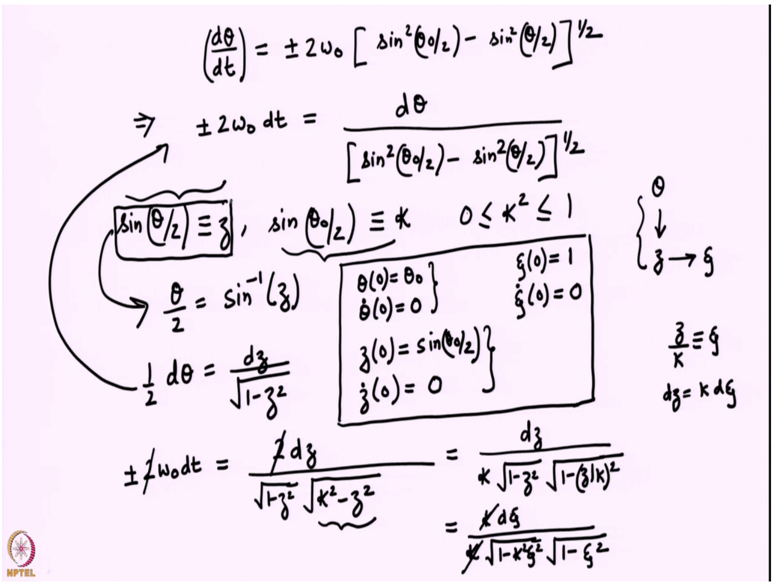

So, we found that d theta by dt is equal to plus minus 2 omega naught sin square theta naught by 2 minus sin square theta by 2 to the power half. Now, our next task will be to integrate this equation and I will show you that we can using a simple set of manipulations, we can convert this into elliptic integrals.

Once we convert this into an elliptic integral, we have already discussed the elliptic integral which can be expressed in terms of the inverse of the elliptic functions and so, that will allow us to express theta as a function of time. So, let us continue. So, I will write this as plus minus 2 omega naught dt is equal to.

This is just a simple algebraic rearrangement and then let us substitute sin theta by 2 to be equal to z sin theta naught by 2 equal to K. So, you can see that this is the same K that we had introduced earlier in the case of elliptic functions this was a measure of the shape of the ellipse.

So, K square was between 0 and 1; you can see that if you are defined K as sin theta by 2 then K square is between 0 and 1 and it is never negative. So, now, we have to go from theta to z. So, let us express. So, this is our relation which is allowing us to do this. So, I can write this as theta by 2 is equal to sin inverse z.

And so, half d theta is equal to 1 by square root 1 minus z square dz. If I substitute it into this equation, I would write to replace theta by z. So, the equation becomes twice omega 0 dt is equal to twice dz divided by square root 1 minus z square into square root K square minus z square.

Now, we can do one more round of substitutions, but before we do that let me pull out. So, I can cancel the 2 and the 2 here and then I have dz by I will pull out that K square from this part and so, the K square will come out as K. And this is z by K whole square and now, let me substitute z by K.

I will define it as some zeta. So, once again dz is equal to K d zeta and if you express it in terms of zeta then this just becomes K d zeta divided by K into 1 minus K square zeta square into 1 minus z by K zeta sorry, so zeta square.

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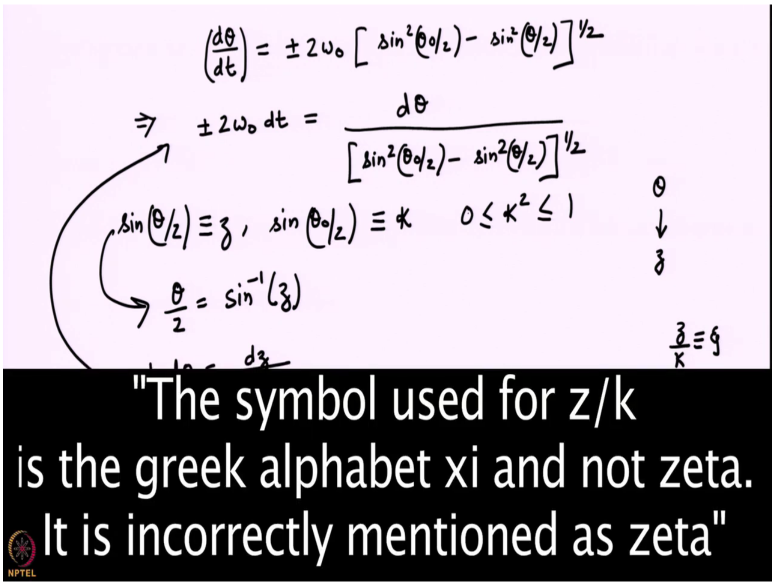

So, what we have done, you can see why we are doing this. So, what we are doing is we are going from theta to z and from z to zeta. You can see why we are doing this, because we want to convert this into a standard form of an elliptic integral ok and you can already see that this is coming in the form of an elliptic integral that we have seen earlier.

So, now we can integrate both sides and then express the right hand side in terms of elliptic integrals, let us do that. Now, before we go further let us keep track of the initial conditions. So, I will put a box here and recall that my initial condition was theta of 0 is equal to theta 0, theta dot of 0 d theta by dt was chosen to be 0.

I went from theta to z and the transformation was this. So, z of 0 so, when theta is 0. So, at t equal to 0 theta is theta 0. So, this is sin theta naught over 2 and z dot of 0 you can see is 0;

that you can see very easily from this equation. If you take the derivative with respect to time on both sides, you will get a dz by dt on the right hand side.

You will have to take the derivative of sin theta by t with theta by 2 with respect to time, that will give you a cos theta by 2 into theta dot. If you apply this equation at t equal to 0 then it is cos theta by 2 which is cos theta naught by 2 into theta dot and theta dot is 0, hence z dot is 0.

So, our initial conditions from theta to z transformed like this and let us see what do they transform on zeta. So, zeta of 0 zeta by definition is z by K, z of 0 is sin theta by 2 K sin theta naught by 2 and K is also defined as recall that K is defined as sin theta naught by 2. So, zeta of 0 is normalized to one. You can also see that zeta dot of 0 is equal to 0 alright. So, these are our initial conditions on the three variables.

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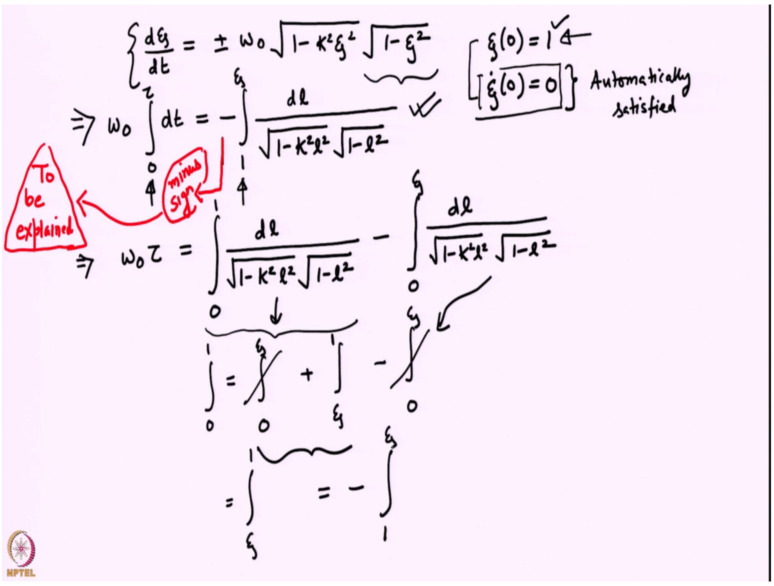

So, now, let us rewrite what we had written in the earlier page. So, we had written t zeta by dt is equal to plus minus omega 0, 1 minus K square zeta square into 1 minus zeta square.

Notice we have also seen that the initial condition is 0, notice that this is redundant; because, in this equation in this equation if you put time t equal to 0. So, then the left hand side becomes zeta dot. So, zeta dot of 0 is equal to the right hand side evaluated at time t equal to 0 and you can see that this term would take the product to 0 because zeta of 0 is 1.

So, the whole thing is automatically 0. So, this condition is automatically satisfied. This is not surprising; the reason is because we have derived all this with taking into account 0 velocity initial conditions. So, the 0 velocity initial condition is built in into the form of this differential equation.

In any case, this is a first order differential equation and we would not have been able to satisfy. They will only once we integrate this there will be only be one constant of integration and we cannot satisfy 2 boundary conditions or 2 initial conditions using 1 constant. So, it is this condition is automatically taken care of and we only need to determine the unknown constant that appears here using this.

So, that is consistent. So, let us continue further and so, if I integrate this equation then I will write this as omega 0 into some 0 to some time tau d t is equal to minus and I will explain why I choose the minus sign, let me just go ahead and I will remember to explain the minus sign, I will also explain the limit, but let me write down.

So, what I have done here is I have integrated it from time t equal to 0 up to some time tau. On the right hand side, I have replaced the zeta that appears with a dummy variable l and I have put the zeta in the limit of integration this thing comes, because when time is equal to 0 zeta is equal to 1.

So, we are integrating from 1 to some value of zeta which is the value the zeta takes when time is tau. So, you can see that this is going to, we are integrating it up to some time t and if I carry this forward, then you can see that I can write this as so, once again. So, we will explain why we are taking this minus sign.

But let us right now, let us take this forward with the minus sign and in the next exercise I will explain why did we choose the minus sign. So, you can see that this integral minus 1 to zeta with an integrand can be written as 0 to 1. So, what I am doing is you can see how I have written this.

This integral, I am not writing the integrand because it is the same in both the terms, but this integral 0 to 1 can be split up into 0 to some value of xi plus zeta to some value to 1 and then this is 0 to the xi. So, this part can be written as these, this part is just this term the first term and the third term cancels each other.

And what I have left is an integral which is xi to 1, but what my original integral was 1 to xi. So, this integral can just be written as minus of 1 to xi, that is the argument using which I am splitting it like this.

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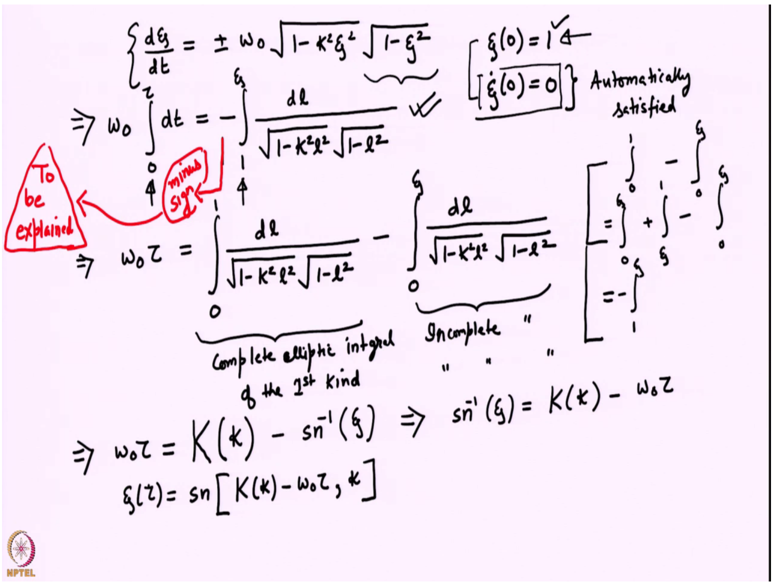

So, the argument that I am using here is, I am writing the first integral. So, the first integral is 0 to 1 and the second integral is 0 to xi and I am writing this as 0 to xi plus xi to 1 minus 0 to xi. And this the first and the third cancelled each other and the second one can be written as minus 1 to xi which is our original integral. So, that is the argument which I am using here ok.

So, this is useful because, you can see immediately that both the integrands are exactly in the form of elliptic integrals. Now, this is what is known as complete elliptic integral of the first kind and this is what is known as incomplete elliptic integral of the first kind. You can see this is a the first one is a definite integral, the second one is an indefinite integral.

And so, this is the first one is called complete and the second one is called incomplete and we have seen earlier that these are related to s n inverse ok. Now, there is a very standard way of writing down the first integral, I will just use that standard notation. The standard notation so, you can see that this, the first integral the complete elliptic integral of the first kind depends on one particular parameter.

l is anyway going to be integrated over and the limits of integration are constants, they are just numbers. So, this integral is just a function of k small k. So, the way to write the complete elliptic integral is the symbol big K and big K is actually a function of small k, small k recall is the modulus of the elliptic function.

It is related to the shape of the ellipse. In this case small k is sin theta naught by 2. So, the magnitude of small k is determined by the initial angular displacement that I am giving to the pendulum, minus this is second one is just sn inverse of zeta. So, you can see that I can now write this as sn inverse of zeta is equal to k minus omega 0 tau.

And we can rewrite that as I can shift the sn to the other side that would give me a zeta of tau is sn and recall now, that I am going to now remember that sn actually depends on 2 arguments the first one is u. So, the value of u is K of k minus omega naught into tau and the second one is small k itself.

So, you can see that we have solved the pendulum in this form, in terms of the sn function. Let us go back to our old variables; we had recall that we had gone from theta to z to zeta.

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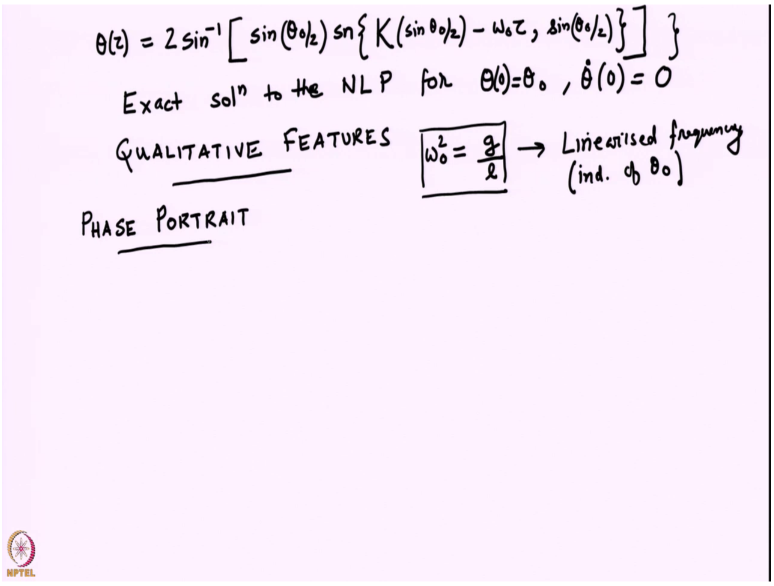

So, so if you do the reverse transformation, you can show that theta as a function of tau is equal to 2 sin inverse sin theta naught by 2.

I am replacing all the small case with sin theta naught by 2, sn has two arguments. This is the exact solution to the non-linear pendulum for theta is equal to theta naught, theta dot is equal to theta of 0 is equal to theta naught theta dot of 0 is equal to 0 or other complicated looking formula.

You can see that the sine inverse comes, because we have we had written our answer in terms of zeta and then when we go back from zeta to z to theta there is a sine transform in this. So, you can see that there is a transform which expresses z as sin theta by 2. So, this is the transform that I am talking about.

So, sin theta by 2 is equal to z. So, when we wrote down our final exact answer in terms of zeta, we have to go back to first z and then we have to go back to theta that will involve a sine inverse; so, the final answer contains a sine inverse. Now, you can see that this is a very rather complicated looking formula.

It is difficult to visualize what does the solution look like, one can anticipate that theta will be an oscillatory function it is an oscillatory function; however, this is telling us that for large angles theta naught much greater than let us say 1, it is not going to be a sinusoidal or a cosine function of time.

We will plot this later, when we do perturbation methods at that time I will show that even if we did not know elliptic functions we can solve this equation in an approximate manner using perturbative techniques, at that time we will compare this exact solution with the perturbative solution.

Now, let us explore this solution a little bit further in order to understand what are the qualitative features, what are the qualitative features of a non-linear oscillator, this represents a non-linear oscillator and so, it is interesting to ask what are the qualitative features of a non-linear oscillator.

So, for that we can either plot this function or we could try to extract some other information from it and see what can we understand qualitatively about a non-linear oscillator. We will definitely plot this as I said earlier, but let us explore a more qualitative way of understanding the important features of a non-linear oscillator which distinguishes it from a linear oscillator.

Recall that we have found that the linear oscillator when sin theta is replaced by theta then omega square was given by g by l, I had also pointed out that the frequency, the linearized frequency or other frequency squared is independent of theta naught.

Now, would this also continue to be true for a non-linear oscillator would the time period of the pendulum remain the same independent of how much I give it any. So, if I take 2 identical pendulums and under identical conditions, if I start one at an angle of 1 degree, it would oscillate at a small angle and if I leave it the other one at an angle of 45 degrees, it would oscillate on either side up to angle 45 degree.

Would the time period of both the pendulums be the same? Linear theory says yes, they would be the same. The question is for initial angles as large as 45 degree, is the linearized approximation a good approximation? So, for that we will explore something called a phase portrait, which will tell us the qualitative features of the non-linear ordinary differential equation that we have just solved exactly.

So, this is a very useful tool of visualization particularly, when there is only 1 or at most 2 degrees of freedom, in this particular case the phase portrait can be drawn quite easily, we will do it in the next video.