Introduction to interfacial waves

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Lecture - 10
The non-linear pendulum

We were looking at elliptic functions and we looked at some identities concerning elliptic functions.

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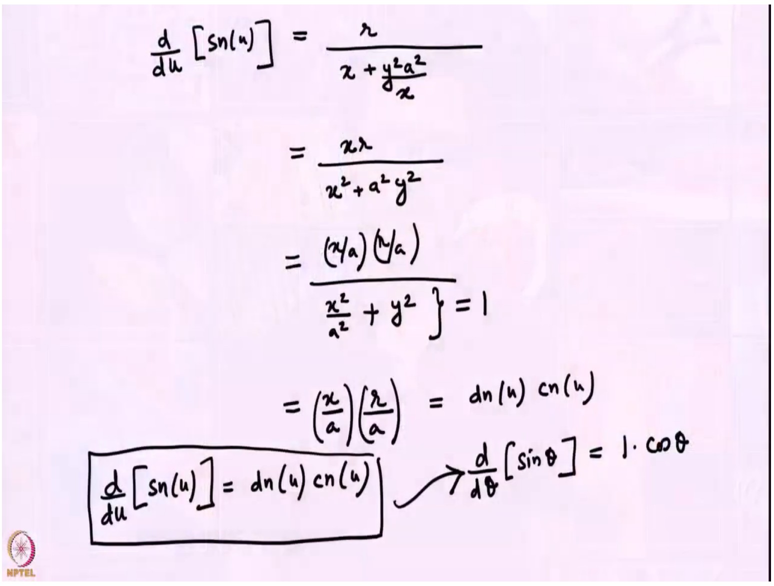

And then we learnt how to take the derivative of the first elliptic function which is d by d u of sn u and we found that it generalizes what we know for circular functions d by d theta of sin theta is equal to cos theta. Let us do one more derivative, let us find out how to take the derivative of c and u.

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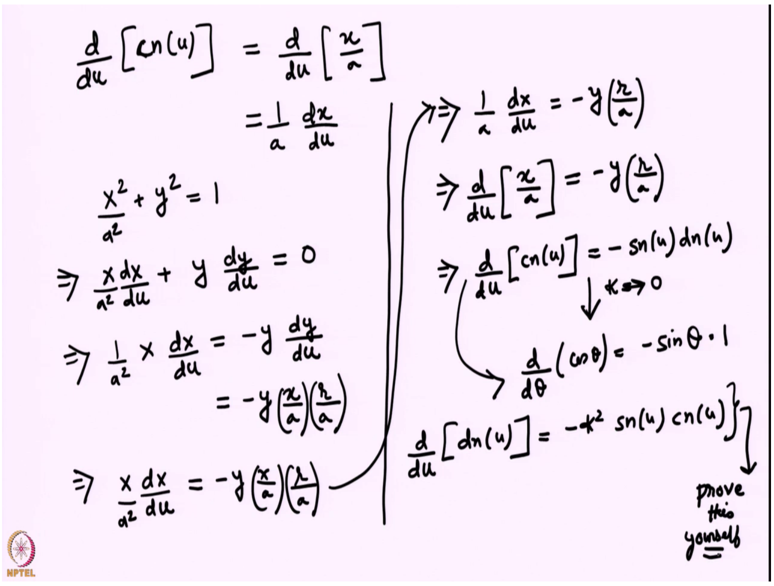

So, we employ a very similar procedure. So, we want d by d u of cn u. Once again remember that k is being held constant. And so, I am replacing the partial derivative with the ordinary derivative. So, d by d u of cn u is basically d by d u of x by a that is the definition of cn u now recall that a is related to k.

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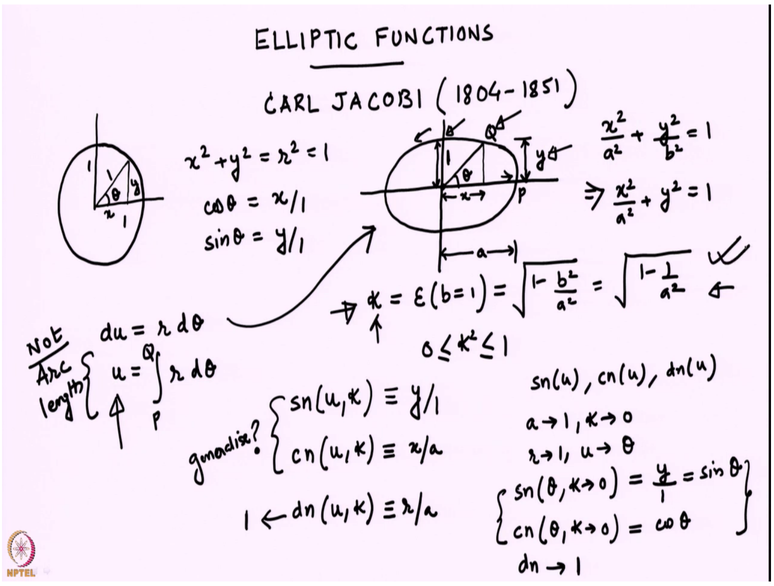

We have defined k in terms of a when we obtain the elliptic functions when we define k and k is related to a in this manner. So, if I am holding k constant this implies I am holding a constant. So, this implies that in this derivative a is a constant and so, can be pulled out. a is a parameter of the ellipse I am on the same ellipse I am not changing the ellipse.

So, a can be ruled out and so, this is just d x by d u. Now, this can be easily evaluated from the equation of the ellipse x square by a square plus y square is equal to 1. If I take the derivative of this with respect to u then this is x d x by d u divided by a square plus y d y by d u is equal to 0 and d x by d u can be easily expressed in terms of d y by d u.

So, you can see that this is 1 by a square x d x by d u is equal to minus y d y by d u and recall that y was just sn u and so, we already know what is d y by d u.

So, this is minus y it is cn u into d n u; cn u is x by a d n u is r by a and so, we have this thing which is 1 x d x by d u let me keep it a square here is equal to minus y x by a r by a let us continue.

So, I can cancel out an a an x by a on both sides, I am left with 1 by a d x by d u is equal to minus y r by a and so, d by d u of x by a s minus y r by a and this is what we want, this is d by d u of cn u is equal to minus sn u, y is basically y by 1 which is sn u and r by a is basically d n u.

We do expect a minus sign here because this we expect it to generalize the identity d by d theta of cos theta is equal to minus sin theta. So, that is where the minus sign comes from you can see immediately that this in the limit of small k going to 0, d by d u of cn u goes to d by d theta of cos theta d n goes to 1, sn u goes to sin theta and the minus remains intact. So, this is again a generalization of something we already know for circular functions. Similarly, you can get more derivatives.

So, for example, you can find out that d by d u of d n u is minus k square sn u c n u. Again you can see that this is consistent in the limit of k going to 0 because d n u just goes to 1 a constant. So, d by d theta of a constant is just 0 and the right hand side also goes to 0 because k goes to 0. So, you can try to prove this on your own ok.

So, we have defined elliptic functions we have found a few identities which will relate them. Using those identities we have used some of those identities along with the equation of the ellipse to obtain relations of how to differentiate them now let us find out if we can find ways of expressing elliptic functions or inverse of elliptic functions as integrals and this is what we will actually need when we are solving the full non-linear pendulum. So, once again let us go back to circular functions to understand the basic idea.

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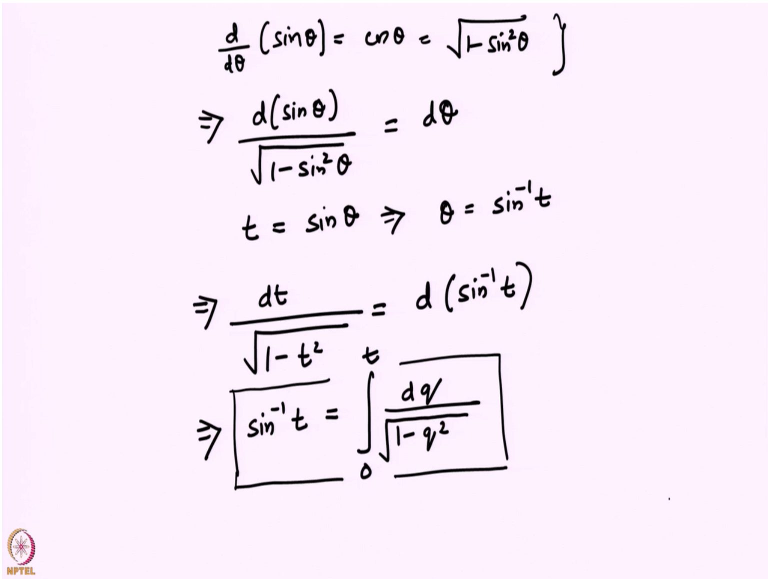

So, we know that d by d theta of sin theta is cos theta which can be written as 1 minus sin square theta. Now, using this I can actually obtain an integral representation for sine inverse. So, this is d of sin theta into square root 1 minus sin square theta is equal to d theta. If I take t is equal to sin theta this is a definition this implies theta is equal to sin inverse t.

If I substitute it here then this becomes d t divided by square root 1 minus t square is equal to d of sin inverse t. I can integrate on both sides and if I write the left hand side on the right and vice versa then if I integrate it if I integrate both sides then I get sin inverse t is equal to 0 to some t and I am using a dummy integration dummy variable for integration now. So, I will use maybe q. So, I am replacing all the t s by q and I am putting the limit of integration as t.

So, that will give me a function of t because all the q s will get replaced after integration with t. So, this is 1 minus q square. So, this is something we know that integral 0 to t d q by square root 1 minus t q square is sin inverse t from 0 to t ok. Now if this suggests that if we know the analogous relation for how to differentiate elliptic functions sn u and cn u, I can get integrals which represent sn inverse u and cn inverse u let us do it for sn inverse.

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$$\frac{d}{du}\left[sn(u)\right] = cn(u)\,dn(u) = \sqrt{1-sn^2(u)}\,\sqrt{1-k^2 sn^2(u)}$$

$$\Rightarrow \frac{d\left[sn(u)\right]}{\sqrt{1-sn^2(u)}\,\sqrt{1-k^2 sn^2(u)}} = du$$

$$\text{Let } t = sn(u) \Rightarrow u = sn^{-1}(t)$$

$$\Rightarrow \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}} = d\left[sn^{-1}(t)\right]$$

$$\Rightarrow sn^{-1}(t) = \int_0^t \frac{dq}{\quad}$$

So, we will use the same strategy d by d u of sn u we have just found is cn u d n u we also know relations between them. So, I will express the right hand side completely in terms of sn u. So, cn u is I am taking the positive square root and d n u with the relation that I have just written is 1 minus k square sn square u. Again a generalization of what we have written d by d theta of sin theta is cos theta if k goes to 0 then just this part survives the other square root goes to unity.

So, now I can do the same trick that I did before I can write this is d. Let t be equal to sn u this implies u is equal to sn inverse of t ok. So, this just becomes d t divided by 1 minus t square 1 minus d of sn inverse t.

Again integrating both sides and using a dummy variable in place of t from 0 to t, we will get sn inverse t is equal to 0 to t d q. I am replacing t by the dummy variable q and I am putting the upper limit of integration as t so, that the whole thing will become a function of t because I have a function of t on the left hand side.

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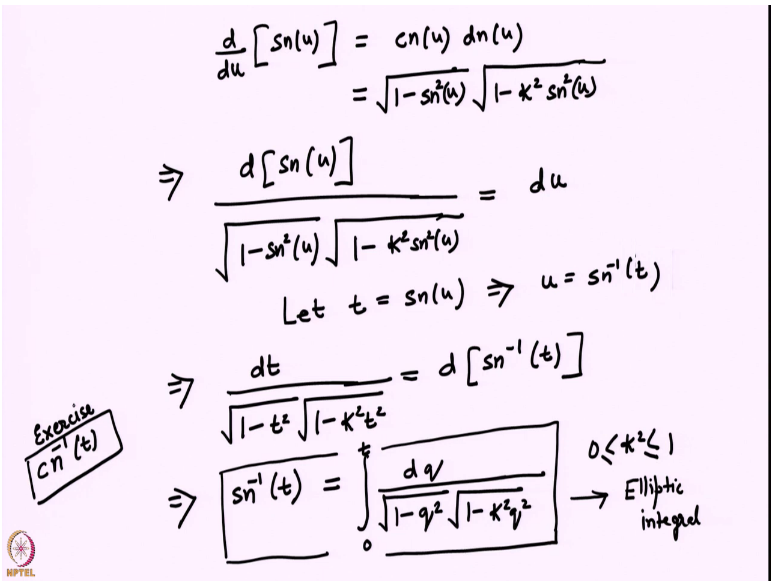

0 to t this is d whatever I used d q and you have to remember that k square between 0 and 1. This is something that we are going to need while solving the non-linear pendulum exactly. You can see that sn u is a known function told you there is a function which has been defined

it is an oscillatory function, it is different from sin theta. It generalizes sin theta and sn u is known.

So, if the function is known its inverse is also known. So, sn inverse of t is also known. So, this it is expressible is this integral and we will find that when we try to solve the non-linear pendulum exactly without replacing sin theta by theta in the first term in the Taylor series expansion if you do not do that, then we will be able to integrate the equation and obtain an integral like this.

So, the we have two choices one we can do the integration numerically or we can express it in terms of an inverse sn function which is what we are going to do and we will find that the final answer for a non-linear pendulum can be represented in terms of sn the Jacobean elliptic function. This integral is called an elliptic integral and it just generalizes what we know for circular functions.

When k goes to 0 the second square root in the integration just goes to 1 and this just goes over to sin inverse of t is 0 to t d q by square root 1 minus q square that is a formula we all know. So, this just generalizes that. So, this is about. So, similarly you can find I encourage you to find integral representation of cn inverse t, you can follow the same procedure start from d by d u of cn u you know the what is the formula it is minus sn u d n u then express the entire right hand side in terms of cn u bring all the cn u s to the left hand side keep the u on the right hand side and then follow the same procedure you will get another elliptic integral ok.

So, we will not do that I leave it to you as an exercise, it is important to know these integrals because frequently these allow it is useful to write these when we encounter these kind of integrals while solving non-linear ordinary differential equations.

One I can replace this integral by a function which is known and whose behavior is known. So, that is the useful thing ok. So, this is as much as we will require in order to solve the non-linear pendulum ok.

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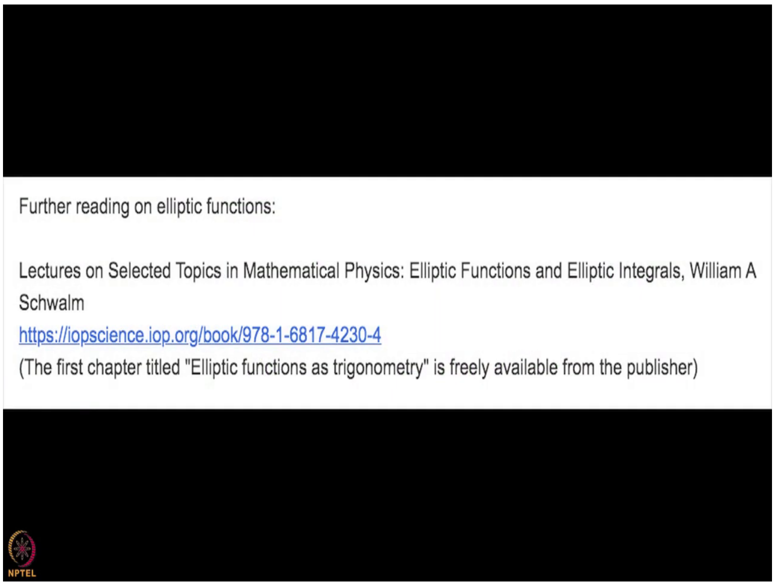

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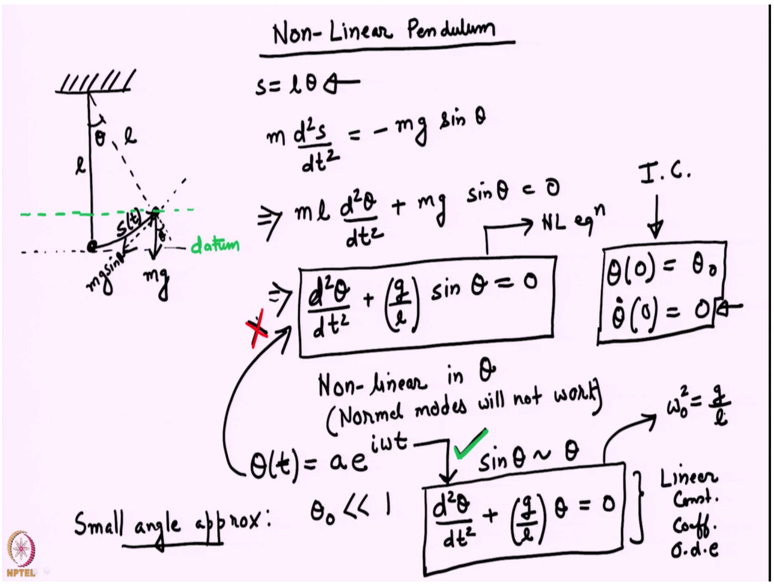

So, with that let us move over to the next topic which is the non-linear pendulum. So, you see we are until now we have looked at linear systems, we have looked at linear we started with a point mass connected to a single spring that was a single degree of freedom system connected to a linear spring it gave us a linear ordinary differential equation then we were looked at couple systems we went up to analyze them when we they were an arbitrary number of masses in the coupled system.

But still linear then we went on to the continuum limit of that still linear, but it we got partial differential equations all of these things systems were analyzable using the method of normal modes. The base state was an equilibrium state and it turned out to be stable that is why we are getting oscillations about the base state and so, we have analyzed we have found out the frequencies of those oscillations using the method of normal modes.

Now, we are going to do a slightly different kind of a problem where we will again reduce our number of degrees of freedom to just 1, but now we are not going to have deal with a linear system we are going to deal with a non-linear system and we are not going to linearize the system and we are going to find out can we solve this system exactly and once we solve the system exactly what are the properties that this system has which is not exhibited by a linear system.

Until now we have looked at what are the properties of a linear system. If I have n degrees of freedom, then there are n normal modes and n frequencies and the general solution is a summation of that. The system may vibrate in the system may move either in a linear superposition of all its states all its eigen modes or in one pure one or more than that depending on what you do initially.

If you set up the system to oscillate in a pure mode, it will continue to remain in that. I also mentioned that in a linear system the energy is not exchanged between modes. The linear equations do not permit any energy exchange between the eigen modes. So, if we inject energy only into one mode, there is no way energy can be produced in the other or transferred to the other modes. So, the only way to excite many modes is to excite them initially as far as a linear system is concerned.

Now, let us go to a non-linear pendulum, here we will first start with having a single degree of freedom and later on we will see if we can introduce more degrees of freedom, but we will learn some qualitatively new things which are not present in the linearized equations that we have studied until now. This is still a ordinary differential equation, but it will turn out to be a non-linear ordinary differential equation.

So, that is my state of rest I am going to make the simplest possible pendulum. So, it is a point mass it does not have a finite size the string is inextensible massless. So, we do not have to worry about moment of inertia and other things.

So, this is it is a point mass and we will just do a force balance or tangential force balance on the point mass which will give us the equation of the well known equation of motion for a pendulum. So, suppose I displace it, it moves in a circle because the string is inextensible.

So, this is a string of length l and this is angle is measured from here, this is some reference state and let us say this s is the arc length it is going along a circle. So, the arc length is a function of time because its moving in a circle s is equal to l of theta s is a function of t l is a constant theta is also a function of t.

Now if the at some instant of time the mass is here, then this is the force on the mass by gravity if I do a tangential force balance the tangent if I draw the tangent to the arc at that point, then the tension in the string does not contribute to the force balance because it is normal to the by definition it is along the radius which is normal to the tangent.

So, the only force which actually acts along the, which has a component along the tangent is the gravitational force. So, if I write down the equation of motion, it is m d square s by d t square is equal to minus m g sin theta. You can see that this angle is theta and so, the component in that direction is m g sin theta.

There is a minus sign because this is a restoring force the force tries to reduce the angle. So, I have to now express s in terms of theta on the left hand side so, as to get my dependent variable in terms of theta. So, this is just m l d square theta by d t square, I have just used this relation plus m g sin theta equal to 0.

This is just the force balance in the regular thing that we do and this leads us to the equation of the pendulum plus g by l sin theta equal to 0. This is the equation that we all learn. Now, the first thing about this equation is, it is a non-linear equation it is non-linear in theta. So, we cannot use the method of normal modes straight away on this equation, you can try that you will see that you will run into difficulties. The way in which this equation is typically solved this is a still a second order equation.

So, you will need initial conditions in this analysis that we are going to do now, we will assume that theta of 0 is theta 0 and theta dot or d theta by dt the initial velocity of the pendulum I will just take it to be 0. You can put some number here and redo the analysis, it will just give you slightly bigger expressions, but the qualitative aspects will already be captured by taking these initial conditions.

So, the method of normal modes will not work for the non-linear equation. So, I cannot say that theta is equal to theta of t is some number a into e to the power i omega t. You will see you can go back and try and see this and if you substitute it you will see that this does not this does not work. So, we will have to find out another way of doing this and that is where the elliptic functions will be useful.

Now, if you recall the typical way to solve this is in the small angle approximation when we say that theta naught the initial angle is small. So, theta naught is say small compared to 1, the in terms of radians the maximum theta naught that you can give is pi.

So, theta naught is let us say a small angle and that allows you to approximate sin theta with the first term in the Taylor series which is just theta. Once you do that then there is a qualitative shift in the nature of the equation and this just becomes equal to 0. This is a linear constant coefficient o d e one can apply the method of normal modes to this equation.

So, this works. If you substitute it here you will just get a frequency relation the when the way we had got it for a spring mass system in the first class. There it was k by m, here it will be g by l the frequency will turn out to be g by l. Now, we would like to ask the question that suppose theta naught is not small.

Suppose I take the pendulum and leave it at 45 or even bigger angles, suppose I leave it at 70 degrees, I leave it at 80 degrees is this a good approximation? If it is a bad approximation how bad is the approximation are there better approximations available?

Can this problem be solved exactly? In this particular case it turns out that this problem can be solved exactly with the use of elliptic functions. Later on when we do interfacial waves we will see that most of the times non-linear problems cannot be solved exactly.

So, there we will have to use techniques like perturbative methods in order to find approximate answers to non-linear problems. More importantly we have to understand that what is the qualitative role that non-linearity brings in. So, I will use this example to illustrate many of the things that we will later also see in the context of interfacial waves.

So, now let us take this non-linear equation and let us try to solve it without making any approximations subject to those initial conditions. So, how do we solve this? So, there are two ways of going about it this is a second order o d e it is easy to spot an integrating factor for this o d e. You can immediately see that if I multiply this ordinary differential equation with theta dot on both sides the right hand side is anyway 0 you can integrate this equation once.

You can write the first term as d by d t of half d theta by d t whole square and the second term as d by d t of some scalar multiple into cos theta instead of that. So, that is one way of solving it because that that exercise will reduce this second order equation into a first order equation. There will of course, be an integration constant there because it is equivalent to integrating once.

We will take a slightly more physical approach where we will say that this oscillator this pendulum whether it is approximated by a linear equation or a non-linear equation we know that its energy is always conserved its a conservative system. So, the sum of potential plus kinetic energy is always conserved. Now, the advantage of doing that is that that energy is the first integral of motion.

So, the expression for energy will only depend on the first derivative of theta there will not be a second derivative. So, it is equivalent to already starting by from this equation and being able to integrate it once.

So, let us write down the expression for the energy of this pendulum at any instant of time and then we will take it from there. So, in this configuration you can see that this pendulum is at this level. Now, let us say this is my datum this is my datum of potential energy.

So, if I calculate potential energy the 0 of potential energy to be at the datum, then this is at a slightly higher elevation. The elevation you can see is given by l into 1 minus cos theta and so, we will be able to write down the expression of the potential energy and the kinetic energy.

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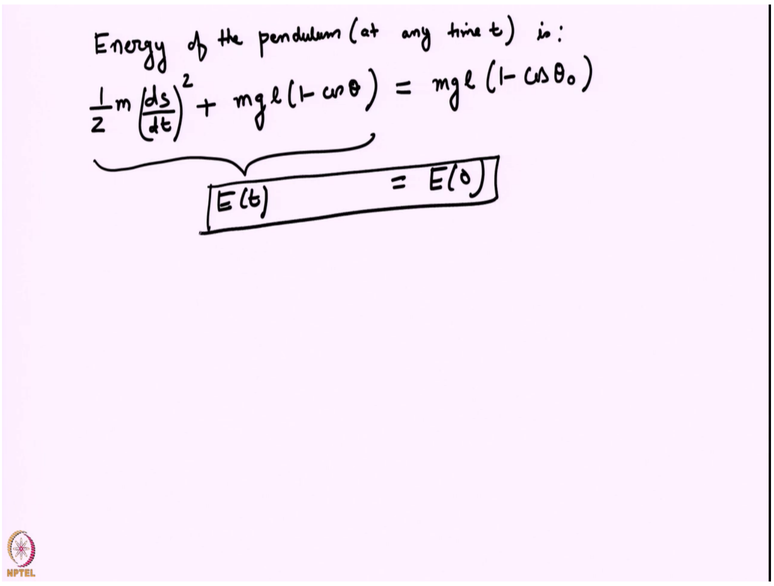

So, let us do that. Energy of the pendulum at any instant of time is half m d s by d t whole square that is its kinetic energy at any instant of time and the way I had drawn it you can see that it is at a distance l into 1 minus cos theta above the data.

So, m g h; h is l into 1 minus cos theta that is the gravitational potential energy. So, kinetic plus potential is equal to constant how much is the value of the constant that is determined by initial conditions. Remember that I had said that I leave the pendulum at an angle theta naught without any initial velocity.

So, when I leave it at that whatever is its energy, this has to be the same at every instant of time. So, if I write down the energy at time t equal to 0, it would be given by just the potential part because the initial kinetic energy is 0 and the theta would be theta naught. So, I am just taking this expression and applying it to theta equal to theta naught because initially there is no kinetic energy. So, this part is 0 and this part is m g l into 1 minus cos theta naught and so, at any instant.

So, let me rewrite this, this is the energy at any instant of time and this is the energy at 0 and if the system conserves energy they must be the same. So, this is what is my basic idea.

Now, notice that this was for one particular set of initial conditions. If you change the initial conditions the expression on the right has to be changed if you give it a velocity also then you have to add one more term for the kinetic energy that you have imparted it initially ok. So, we will solve it for this particular set of initial conditions and we will find out how.

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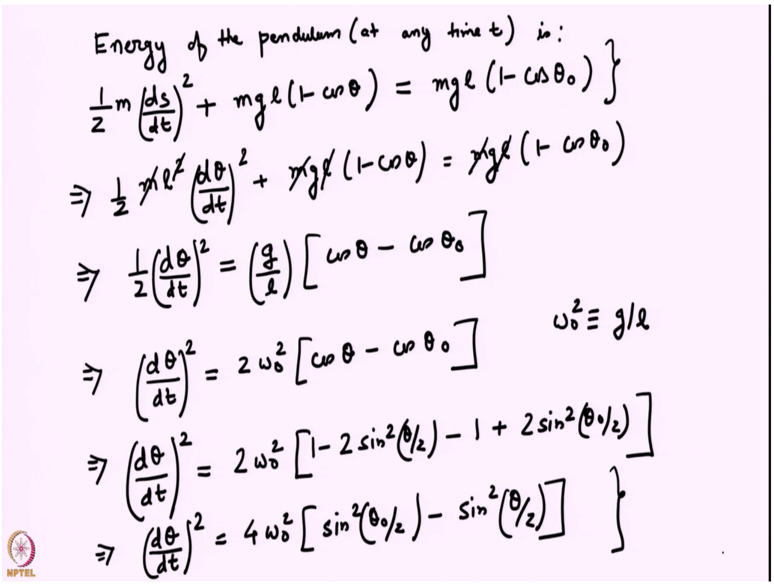

Now, let me express this in terms of theta, s is equal to l theta we have seen that earlier now I can write this as. So, the m g l and the m g l cancels out on both sides and then you will get. So, the m m can be cancelled. So, this and this go I will get rid of a this and then I will get a g by l into cos theta minus cos theta naught and we know that g by l.

If you do have done a linearized if you do the normal mode analysis on this, then you will find that omega square. I will call it omega naught square because it is the linear frequency is just g by l. If you do a e to the power i omega t you will just find that omega square is equal to g by l this just has the units of 1 by time square.

Now so, I will write this as d theta by d d t whole square is equal to 2 omega naught square into cos theta minus cos theta naught omega naught square is defined as g by l. I can make it slightly more compact you can see that I have already reduced the order of the equation by 1.

My original governing equation for motion was a second order equation because I have instead of that equation I have started with the expression for energy. The expression for energy is a first order the derivative which appears is a first order derivative in time there is no second order and so, I have already done one round of integration and this energy initially is the basically is what determines the constant of integration for me.

So, this if I write it like this then what am I doing here, I am basically this is 1 minus 2 sin square theta by 2 I am just using the identity and so, this can be written as square is equal to 4 omega naught square sin square theta naught by 2 minus sin square theta by 2.

We are going to analyze this expression for d theta by d t and we will try to reduce this to an elliptic integral. Once we reduce this to an elliptic integral that we know we can express that integral in in terms of the sn inverse or cn inverse functions. We will do it in the next lecture and we will see that this expression can be exactly solved provided we know how to express the integrals in terms of sn inverse function the elliptic function.