Introduction to interfacial waves
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Lecture – 01
Introduction

Welcome to this course Introduction to interfacial waves. Now, this course will have many this is a very rich course and it will have applications in chemical engineering. But as we go along the course you will also find that it has applications to other areas of engineering such as mechanical and aerospace engineering.

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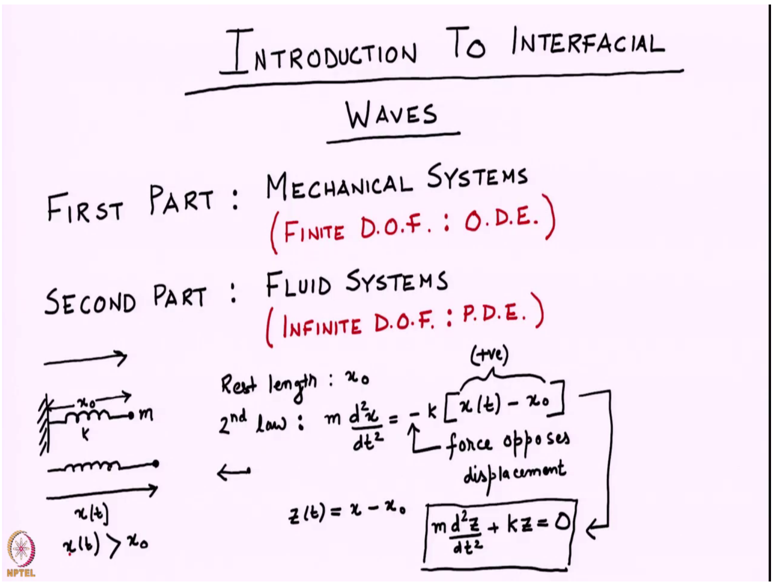

Now, the way the course will be structured, there will be two parts to the course. In the first part, we will be looking at vibrations in mechanical systems. The idea is to introduce the mathematical techniques, which will be necessary to study interfacial waves in the second part of the course.

In the first part, we are going to look at mechanical systems with typically finite number of degrees of freedom, point mass systems connected through springs and so on. In this case, it will be governed by the mathematics will be governed by ordinary differential equations.

We will learn the basic mathematical techniques in the context of finite degrees of freedom. And in the second part, we will go on to study interfacial waves in fluid systems. In this case, there will be infinite degrees of freedom and the systems will be governed by partial differential equations ok.

So, with that let us start with one of the most elementary examples of a vibrating system a spring mass system. So, let us say I have a spring connected to a point mass, the rest length of the spring is x naught let us say, which may essentially means that at when the spring is of length x naught it does not exert any force on the mass m. The spring constant is k. Now, I disturb the spring and the mass and extend it.

Now, its position is some x. And if I leave it, x will vary as a function of time. We know that it is going to oscillate. Let us write down Newton second law of motion applied to this mass. So, the rest length is x naught. Newton 2nd law implies, the minus sign indicates that the force opposes displacement.

The way I have drawn it x t is greater than x 0, and so the force will be towards the left. Note that the positive sign. So, every quantity, which is from left to right is positive and vice versa. So, this quantity the way I have drawn it is positive because x of t is greater than x 0. So, the force acts in the opposite direction.

So, if you draw the mass like this the in this configuration, the force opposes the displacement. This is also true when x t is less than x 0. In that case, x t minus x 0 would be negative. And this additional negative sign outside would make the force positive, which means the force would be again opposing the displacement. Now, this is something which we have studied, which we have all studied earlier.

Now, how do you solve this equation? We can see that it is there is x t minus x 0 on the right hand side, whereas only x t on the left hand side. So, we make a simple transformation. We define a new variable Z of t is defined as x minus x 0. And in terms of Z of t, we can immediately see that this equation becomes m d square Z by d t square plus K Z is equal to 0.

So, now let us try to. So, this is our governing equation, let me put it in a box. Now, I will introduce a method called method of normal modes to solve this equation. We are all of you I think; know how to solve this equation. This is a linear constant coefficient ordinary differential equation you know how to solve it. But I will introduce in particular a method called method of normal modes. And the advantage is that this method will generalize to more complicated systems where we will have more than one mass and more than one spring.

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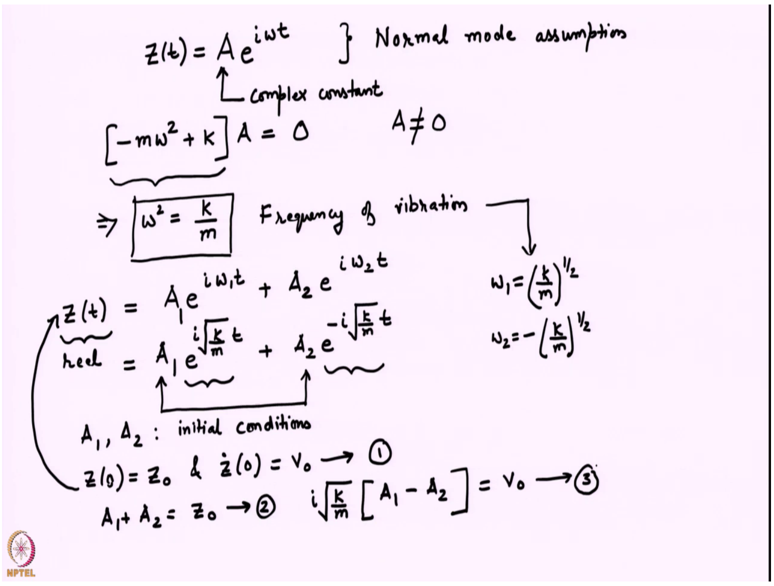

So, let us learn about this method. We set Z of t to be equal to some constant e to the power i omega t. Now, this is the normal mode assumption. Note that we are using complex notation. So, e to the power i omega t has a real part and an imaginary part. And we expect an oscillatory solution with a frequency omega.

Our task is now to determine omega, and how does omega depend on the physical properties of the system. Also note that this A is a constant, but is it is in general a complex constant which means it will have a magnitude as well as a phase. So, if we substitute that into our governing equation, which we have written earlier in the box, then we can immediately see that we have minus m omega square plus K into A is equal to 0.

It our ordinary differential equation becomes an algebraic equation. Now, clearly we are not interested in the solution A is equal to 0. So, we would rather have A not equal to 0 because

A equal to 0 is a trivial solution, where the mass does not oscillate at all. So, then the only other option is to set this part to 0.

And this tells us what is the frequency of oscillation; in terms of the physical parameters of the system the spring constant K, and the mass m. So, this is the frequency of vibration of the system. So, now how do we write the solution to the equation? Using the same complex notation, we will write Z of t is equal to A e to the power i omega 1 t plus or rather A 1 and A 2 e to the power i omega 2 t.

Why omega 1 and omega 2 because from this frequency relation we know that omega 1 is equal to plus square root K by m, and omega 2 is equal to minus square root K by m. They are just negative of each other. So, we can write this as A 1 e to the power square root plus A 2 e to the power minus i square root K by m t.

Now, notice that this part and this part – the exponentials are just the complex conjugates of each other. And Z of t is a displacement. So, it has to be a real quantity. So, we must find that A 1 and A 2 should be complex conjugates of each other. I have told you earlier that A is a complex constant. So, A 1 and A 2 are complex constants. And in order to make Z real, A 1 and A 2 should be complex conjugates of each other ok.

So, now we determine A 1 and A 2 from initial conditions. There are two constants of integration. The equation is a second order ordinary differential equation. So, we have two constants, which are to be determined from the two initial conditions. So, let us say that Z of 0 is some initial displacement that I have given the mass.

And Z dot of 0 Z, a dot indicates derivative with respect to time at time t equal to 0 we have given it also an initial velocity. So, I will call this condition 1. These are my initial conditions, the initial displacement and the initial velocity. And if I go back and substitute this into that relation, then I get two equations in A 1 and A 2.

Those are A 1 plus A 2 is equal to Z 0. Let me call this equation 2. And I also have i square root K by m A 1 minus A 2 is equal to V 0. And I will call this equation 3. Now, you can see that 2 and 3 are two algebraic equations for A 1 and A 2. We can very easily solve these two to determine A 1 and A 2 in terms of Z 0 and V 0. If we do this, we obtain the following expressions.

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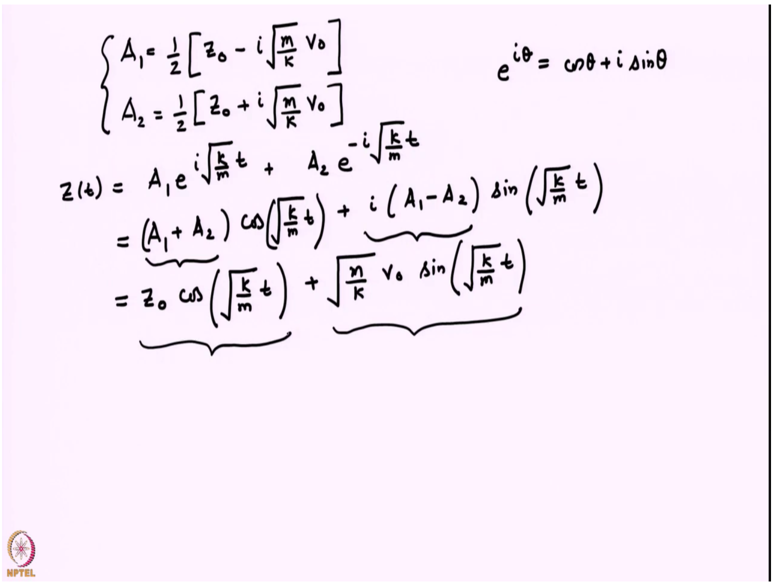

Notice that we had inferred earlier that A 1 and A 2 should be complex conjugates of each other. This is indeed so as you can see from these expressions Z 0 and V 0 represent initial displacement and velocity. So, Z 0 and V 0 are real quantities. So, this if it is immediately clear from these expressions that A 1 is the complex conjugate of A 2.

Now, substituting this in the expression for Z of t, recall that Z of t was e to the power i square root K by m t. And if I substitute this then I obtain. Now, I am going to use e to the power i theta is cos theta plus i sin theta. Now, if I substitute that and write it in real notation in terms of cos and sin, then I get cos square root K by m t plus i times A 1 minus A 2 into sin square root K by m t.

Now, as it is clear A 1 plus A 2 is a real number, i times A 1 minus A 2 is again a real number. So, this is going to eventually give us a real expression. And if you just substitute from these two expressions, the value of A 1 plus A 2 and i times A 1 minus A 2 you will finally, obtain this is equal to Z 0 cos square root K by m t plus square root m by K V 0 sin square root K by m t ok.

So, this solves our first problem. The problem of a single point mass attached to a linear spring. And this is the most general solution to the problem. If you give initially if you just give a initial displacement, but do not give any initial velocity, then the second part is 0. If you give it an initial kick without giving it an initial displacement, then the first part is 0.

If you give it both, then this is the answer. And the final answer has been written in real notation ok. So, now let us go on to slightly more complicated problems where our governing systems will still be linear. But we will have more than one point mass and more than one spring.

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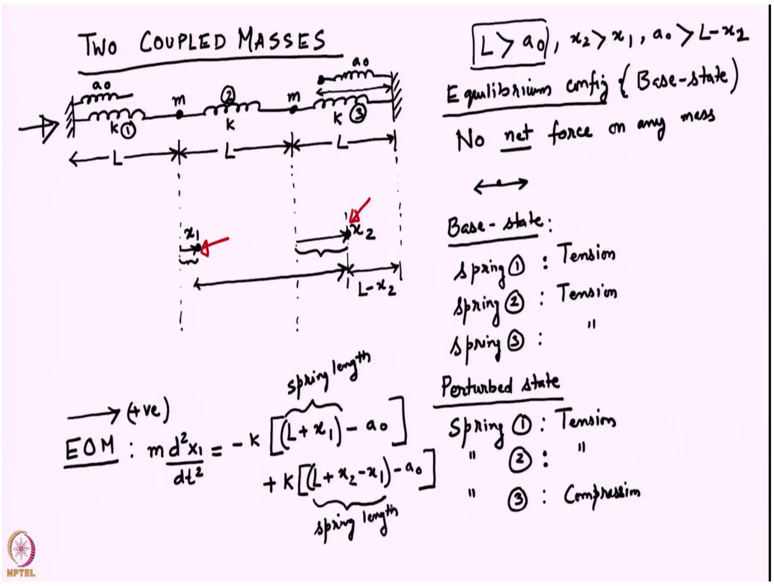

So, two coupled masses ok. So, now, I am going to make a diagram. And you should understand this very clearly because this will help you write down equations of motion for more complicated problems where there may be many masses and many springs. So, I have a spring connected to a point mass, connected to a second mass, and then another spring, which is connected to a wall.

For simplicity, let us assume that both the masses are the same; the spring constants are also the same. In this case, we will assume that ok in this configuration where there is no net force on the mass. This is an equilibrium configuration which essentially means on any no net force on any mass.

So, every mass is acted upon by equal and opposite forces in either directions. So, let us say that the rest length of the springs is slightly less than L. So, I will call the rest length a naught.

So, the same it is the same spring here this also has a rest long length a naught. And the way I have drawn it L is greater than a naught.

So, let us indicate the equilibrium configuration like this. Later on when we do when we do interfacial waves the equilibrium configuration will also be referred to as base state. But let us stick with the word equilibrium configuration here. And now, let us say that we have perturbed. So, again once again my left to right direction is positive, and we are displacing the first mass by a distance x 1. So, only this much is x 1. And we are displacing the; we are displacing the second mass by an amount x 2 ok.

So, you can notice that the way I have drawn it, I have intentionally so after displacement the second mass is here the first mass is there. So, you can notice that the way I have drawn it, L is greater than a naught x 2 is greater than x 1. So, x 2 is this displacement. And a naught is greater than L minus x 1, L minus x 2. So, what is L minus x 2? L minus x 2 is this length; L minus x 2 is this length. And the way I have drawn it the rest length of the third spring is greater than L minus x 2 ok.

Now, under these conditions, our base state in our base state spring 1 is in tension, spring 2 is also in tension, and spring 3 is once again in tension that you can see very easily because L is greater than a naught. So, in this state, all the springs are extended to a length which is greater than a naught. So, all of them are in tension.

What about the perturbed state? You can see that in the perturbed state the mass 1 is here, so and mass 2 is there. So, spring 1 is in tension even in the perturbed state. Spring 2 is also in tension that you can see because now the distance between the two the mass 1 and mass 2 is given by this much in the perturbed state.

And this is more than L in the way I have drawn it. Spring 3 the way I have drawn it is in compression because spring 3 has been pushed beyond its rest length in the way I have drawn it alright. So, with this it is now easy to set up the equation of motion of the two masses, because there are two masses we will get two ordinary differential equations by applying Newton second law of motion to each of the masses.

If you do that, then for the first mass and for the second, so there are two forces acting on the first mass – one exerted by the first spring and another exerted by the second spring. The first term on the right hand side represents the force exerted by the first ring, the first spring. This is spring 1; this is spring 2 and this is spring 3.

So, we also have to add the force exerted. Now, note that there is a pattern in the way I am writing things the I have written these forces according to the diagram that I have drawn. So, each of these square brackets is positive. The quantity inside each of the square brackets is positive.

And in the curve bracket whatever I have written represents the spring length in the perturbed state. So, this is the spring length of spring 1. Similarly, this is the spring length of spring 2 in the perturb state. Now, you can write down a similar equation of motion for mass 2.

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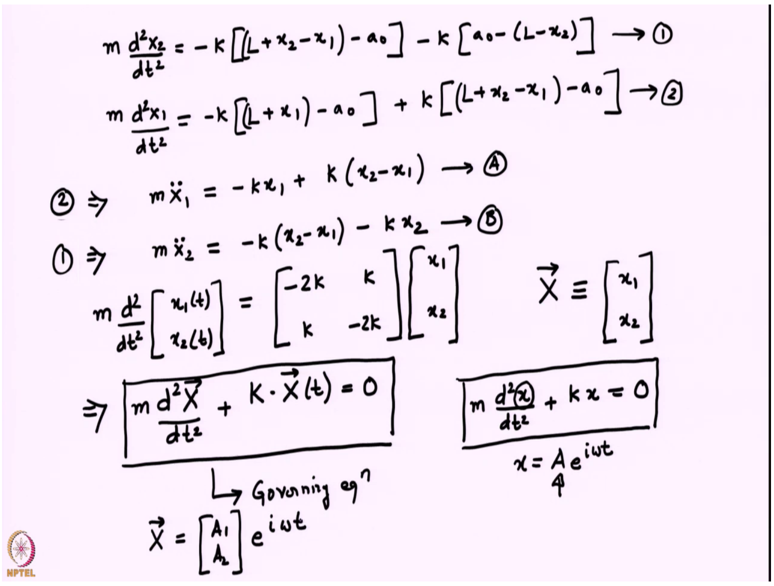

Once again we just follow the same reasoning. And we have and for I am just going to write so that we have both the expressions on the same page alright. So, now we have two equations for the two masses. Notice that these are linear springs; hence the equations of motion are also linear. But unlike before we have now coupled linear equations, and we are going to simplify these equations. And once again solve them by the method of normal modes.

Before doing that, let us simplify the equations. You can see that in both the equation some terms cancel out. And if you do that cancellation, so if I call this 1, and this one 2, then m x 1, I am just shifting to dot notation, so that is easier to write. So, this is the simplified equation let us call this A and B. And now we can write this using matrix notation as a single matrix equation.

So, now, I am going to define a vector X, and that vector is x 1, x 2 a column vector, and this is a function of time. So, I can write this as m d square X by d t square plus some matrix times the vector X is equal to 0. Notice the analogy of this equation with the equation that we had written earlier for a single mass.

This was what we had written in our first example. You can think of this as a column vector with only one entity in it. Since there was only one entity we did not explicitly use matrix notation, but now because we have more than one degree of freedom. Here we have 2 degrees of freedom because there are two masses we now have a governing equation for our system written in terms of using matrix notation.

We are next going to do a normal mode analysis on this by generalizing what we had done. So, here we had done x is equal to A e to the power i omega t. You can think of A as a column vector with only one entity in it. Here I would have to write X is equal to A 1 A 2 e to the power i omega t. So, we will do this next.