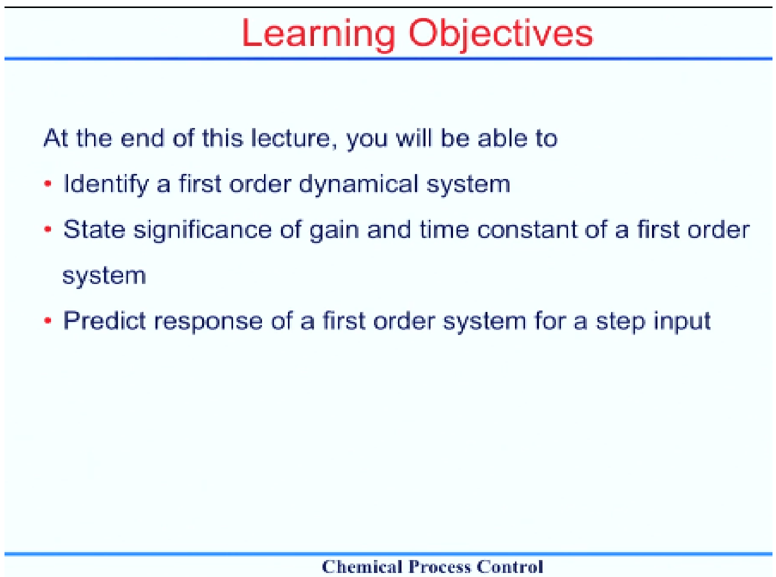
Chemical Process Control

Prof. Sujit S. Jogwar
Department of Chemical Engineering
Indian Institute of Technology-Bombay

Lecture - 09
Introduction to First Order Dynamical Systems

Hello students. Welcome to the second week of this course. In this week, we will be discussing the dynamics of one of the simplest type of dynamical system which is known as first order systems. So here are the objectives for this particular week.



At the end of this week, you would be able to identify a first order system. You would be able to state the significance of two main parameters of a first-order system which are gain and time constant and lastly you would be able to predict how a first order system responds to different types of disturbances. So let us get started.

A first order system is a dynamical system whose dynamics are given by a first order differential equation, which will be typically of the form

$$a_1 \frac{dy}{dt} + a_0\, y = b\, f(t)$$

where we can say *y* is the output of the system, *f(t)* is known as the forcing function or input and we have three parameters a_1, a_0, and b.

Now out of these constants, we have to ensure that a_1 is not equal to 0, because if $a_1 = 0$ then automatically this derivative goes away and you do not have any dynamical system. All you have is an algebraic system. So it automatically violates the definition of first order dynamic system. For any first order dynamic system you have to ensure that the coefficient of the first derivative is always nonzero. These other coefficients can be zero. Let us look at how we can simplify the representation. There are two types of first order systems depending on the value of this other coefficient a_0.

Case 1: - $If\ a_0 \neq 0$

$$\frac{a_1}{a_0}\frac{dy}{dt} + y = \frac{b}{a_0}f(t)$$

$$\tau\frac{dy}{dt} + y = k_p\, f(t)$$

This is the standard form in which a first order dynamical system is written and we represent it by two parameters, one is τ which is known as a time constant, and the other parameter is k_p which is known as process gain. We will see the significance of these as we move along into this lecture. So any first order system where a_0 is not equal to zero, we have two parameters which describe the system. One is time constant and other is the process gain. Now let us see what would happen if a_0 is indeed equal to 0.

Case 2: - $If\ a_0 = 0$

$$a_1\frac{dy}{dt} = b\, f(t)$$

$$\frac{dy}{dt} = \frac{b}{a_1}\, f(t)$$

$$\frac{dy}{dt} = k_p\, f(t)$$

This type of system we call it as a purely capacitive process. So we have only one parameter which is gain.

In the previous lecture I told you that irrespective of the way in which we develop the dynamic model the analysis we typically try to cover in Laplace domain. So in both the cases, we will try to take the Laplace transform of the equation and then try to find out how the relationship between input and output is represented as a transfer function.

Case 1: - $If\ a_0 \neq 0$

$$\tau \frac{dy}{dt} + y = k_p\, f(t)$$

Applying Laplace transform,

$$L\left[\frac{dy}{dt}\right] = s\, y(s) - \ y(0)$$

Typically when we write the dynamic equation or dynamic model of a process we generally write it in a deviation form which means the variable ỹ(t) represents deviation of output from the steady state. So as we write the dynamic model as a deviation from a steady state and we typically assume that at t = 0 or anytime which is at or before 0, we are at steady state. So this assumption automatically helps us to get ỹ (0) = 0 and this when plugged in here we have removed this variable from the equation. This is the whole motivation of writing the equation in a deviation form since this initial value gets canceled or removed from the Laplace transform.

So when we try to write the Laplace transform of that original equation what we get is,

$$\tau[s\, y(s) - \cancelto{0}{y(0)}] + y(s) = k_p\, f(s)$$

We can simplify this and write down the ratio,

$$\frac{y(s)}{f(s)} = \frac{k_p}{\tau s + 1}$$

This is the transfer function for first order process which again has the two parameters which we had earlier listed, the gain and the time constant. So this transfer function gives the relationship between the input and the output.

$$y(s) = \frac{k_p}{\tau s + 1} f(s)$$

So depending on what input we have whether it is a step change or a sinusoidal input we can take a Laplace transform of the input multiply it by this transfer function and we will get a Laplace of output which eventually have to be inverted to get the time domain response of the output.

Case 2: - $\boldsymbol{If\ a_0 = 0}$

$$\frac{dy}{dt} = k_p\, f(t)$$

Applying Laplace transformation,

$$s\, y(s) - \overset{0}{\cancel{y(0)}} = k_p\, f(s)$$

$$\frac{y(s)}{f(s)} = \frac{k_p}{s}$$

This is the transfer function for a purely capacitive process.

Now let us take different examples of first order systems. So the first example we would consider is again a familiar example, a liquid surge tank.

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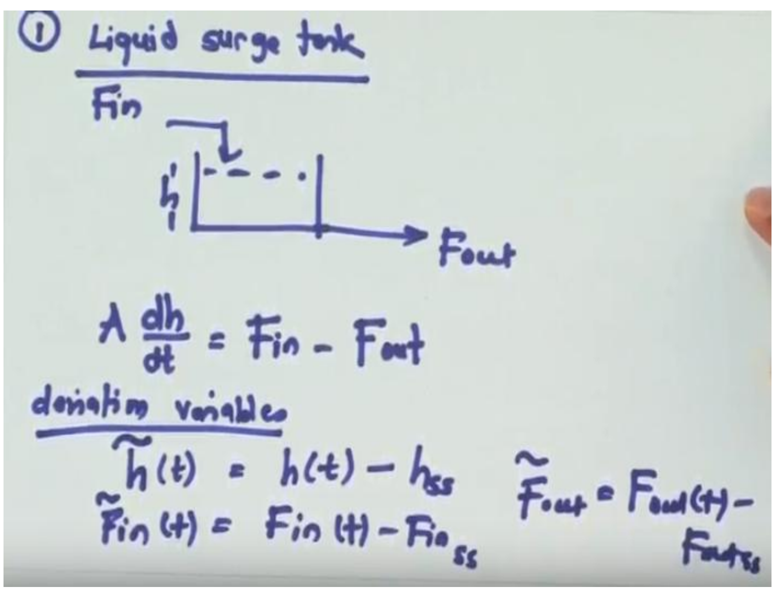
1. Liquid surge tank



Let us consider a liquid surge tank where we have some fluid coming in and some fluid going out. And then the height inside the tank is the variable of interest. We have already written down the dynamic model for this system in the previous lecture which was,

$$A\frac{dh}{dt} = F_{in} - F_{out}$$

So now when we write this system into a deviation form, we need to define deviation variables. So we will be defining deviation variables as departure from steady state, and we represent them by a sign tilde on top. So deviation in height will be,

$$\check{h}(t) = h(t) - h_{ss}$$

similarly, deviation in input flow will be F_{in} - $F_{in,ss}$, and F_{out} will be F_{out} (t) – $F_{out,ss}$.

Applying the above to the dynamic model,

$$A\frac{dh}{dt} = F_{in} - F_{out} \quad (1)$$

$$A\frac{dh_{ss}}{dt} = F_{in,ss} - F_{out,ss} \quad (2)$$

(1) - (2) ⟶

$$A\frac{d\tilde{h}}{dt} = \widetilde{F_{in}} - \widetilde{F_{out}}$$

Taking Laplace transform,

$$A\left[s\,\tilde{h}(s) - \tilde{h}(0)\right] = \widetilde{F_{in}}(s) - \widetilde{F_{out}}(s)$$

Rearranging,

$$\tilde{h}(s) = \frac{1}{As}\widetilde{F_{in}}(s) - \frac{1}{As}\widetilde{F_{out}}(s)$$

$$\tilde{h}(s) = G_1\widetilde{F_{in}}(s) - G_2\widetilde{F_{out}}(s)$$

where,

$$G_1 = \frac{1}{As} \qquad \frac{k_p}{s} \rightarrow k_p = \frac{1}{A}$$

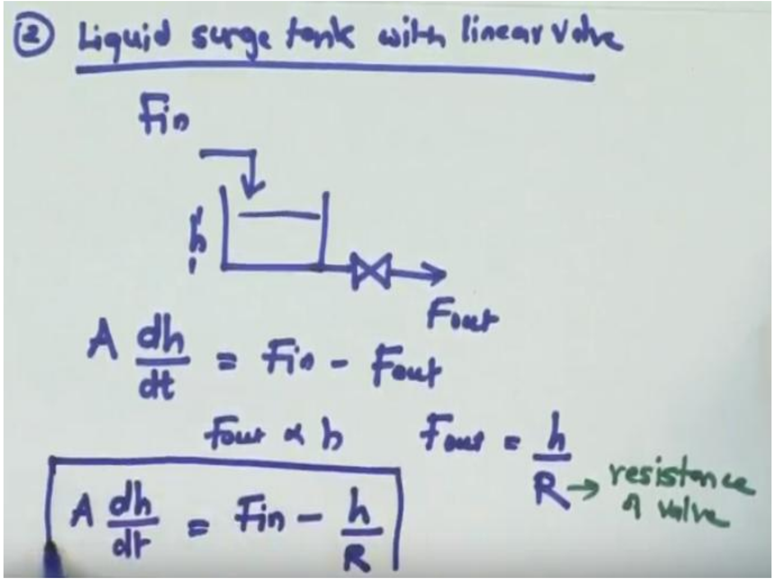
$$G_2 = \frac{1}{As} \qquad \frac{k_p}{s} \rightarrow k_p = -\frac{1}{A}$$

So both these transfer functions are purely capacitive, and it is indeed a first order system which is purely capacitive. The gains of the two inputs are equal but opposite in direction. So it tells me that the rate at which change in input is going to affect the controlled variable is exactly same as the way in which the output change. Only the magnitude will be different. If my input increases height will increase then if F_{out} increases the height will reduce and that automatically comes from the signs of the gain that they are exactly opposite of each other.

So this is the first order system, and we get two purely capacitive first order systems which are parallel to each other, and together they give the change in the height. So let us improve upon this process or try to take more variants of a liquid surge tank.

2. Liquid surge tank with linear valve

In this case input remains the same but instead of a simple output what we have is a valve at the output and we will have certain level in the tank. The dynamic equation remains the same,



$$A\frac{dh}{dt} = F_{in} - F_{out}$$

but now as we have a valve there, F_{out} depends on how much material is there inside the tank. So let us try to say if there was a lot of material inside this tank in that case the hydrostatic head is much higher at the upstream of the valve so there will be more flow through the valve. If the level inside the tank is very low, then there will be less driving force for the liquid to flow and therefore the outlet flow rate will be less. So your outlet flow rate is proportional to height, and we can say that,

$$F_{out} = \frac{h}{R}$$

where R is the resistance of the valve. Substituting in the original equation,

$$A\frac{dh}{dt} = F_{in} - \frac{h}{R} \qquad (1)$$

So we can see that it is a first order dynamic equation. We have still not written it in a deviation form, but even in this current form you can see that it is a first order dynamic equation, so this will be your first order process.

$$\check{h}(t) = h - h_{ss} \; ; \widetilde{F_{in}} = F_{in} - F_{in,ss}$$

$$A\frac{dh_{ss}}{dt} = F_{in,ss} - \frac{h_{ss}}{R} \qquad (2)$$

(1) - (2) ⟶

$$A\frac{d\tilde{h}}{dt} = \widetilde{F_{in}} - \frac{\tilde{h}}{R}$$

$$AR\frac{d\tilde{h}}{dt} + \tilde{h} = R\,\widetilde{F_{in}}$$

$$\tau\frac{d\tilde{h}}{dt} + \tilde{h} = k_p\,\widetilde{F_{in}}$$

So by comparison between these two equations we can say that τ = AR in this case and k_p = R. Now this is the first order dynamic system where we have the time constant τ as equal to A* R and then the process gain is given by the resistance of the valve which is R. Let us now move forward with the same example but moving more closer to reality and which is the case of nonlinear dependence of outlet flow rate on height.

3. Nonlinear dependence of outlet flow rate on height.

In the previous lecture, even though we had assumed a linear relationship between height and flow rate, the actual relationship is square root relationship because the flow is proportional to square root of the pressure and therefore square root of the height and we typically have,

$$F_{out} = \alpha\sqrt{h}$$

So what we can write here is the equation becomes

$$A\frac{dh}{dt} = F_{in} - \alpha\sqrt{h} \quad (1)$$

So this looks like a first order system, but there is a small problem here that we have a nonlinear term. So this is not a linear equation which we can take a Laplace transform of. So the first thing we need to do is to convert this nonlinear system into a linear system and that we do by something known as a linearization. So we will be approximating this system by a linear equivalent, and then we will continue the analysis on that linearized version. So we will take a short break here, and after the break, we will look at how do we linearize this nonlinear system. Thank you.