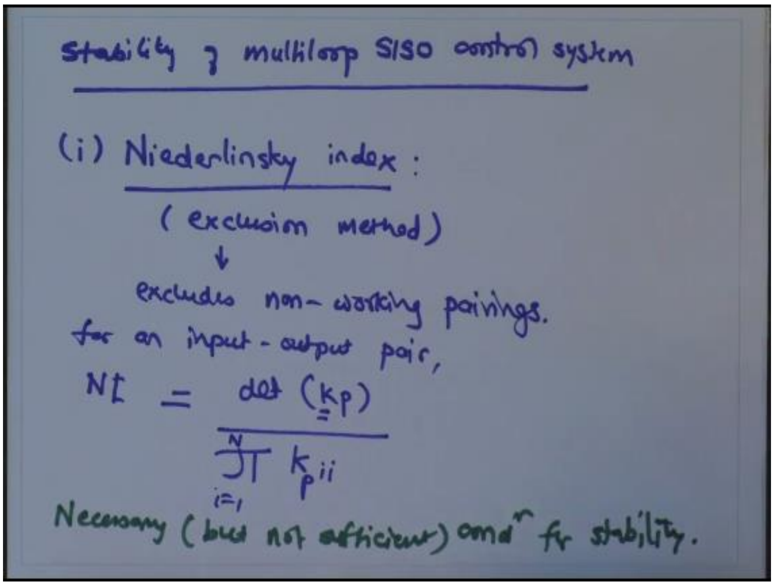
Chemical Process Control
Prof. Sujit S. Jogwar
Department of Chemical Engineering
Indian Institute of Technology – Bombay

Lecture - 44
Input-Output Pairing

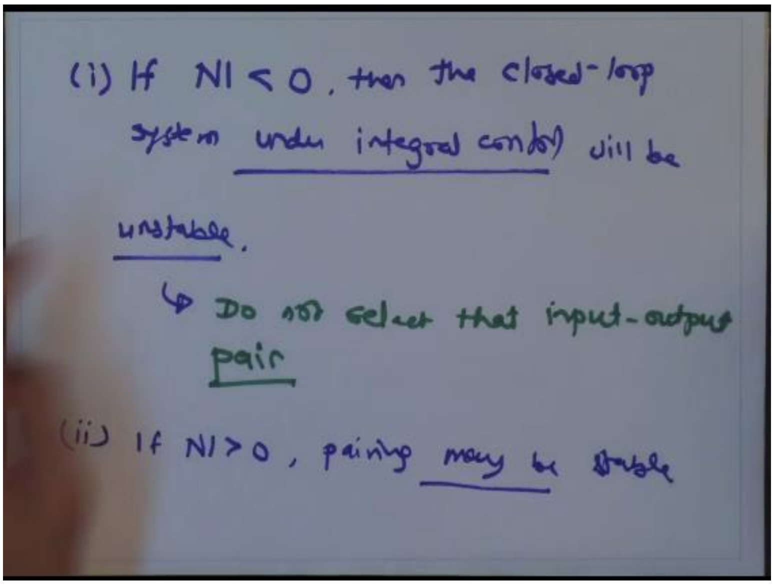
Welcome back. We have been looking at multivariable control and within that, we started looking at one way of implementing multivariable control which is multi-loops SISO controllers. Before the break, we saw that the stability condition of such a system is different than individual SISO controllers because of the interaction term.

Because of these different stability properties for this closed-loop system, the stability of these interacting multi-loop SISO controllers is stringent compared to simple SISO controllers and all the pairings would not always result in stable controllers. Let us now see how do we go about pairing any manipulated input with a controlled variable and there are 2 ways or 2 steps which we would take to really go about doing this pairing.



The first is kind of rejecting any pairing which may result into instability, that is given by an index known as Niederlinski index, the stability of multi-loop SISO controllers. So what we will use is Niederlinski index. This is sort of an exclusion method, it will exclude any non-workable pairing. It excludes non-working pairings. What this Niederlinski index does? It says if I have a pair between and for an input-output pair what you calculate is Niederlinski index and is calculated as a determinant of the gain matrix divided by the product of all the diagonal transfer functions, okay.

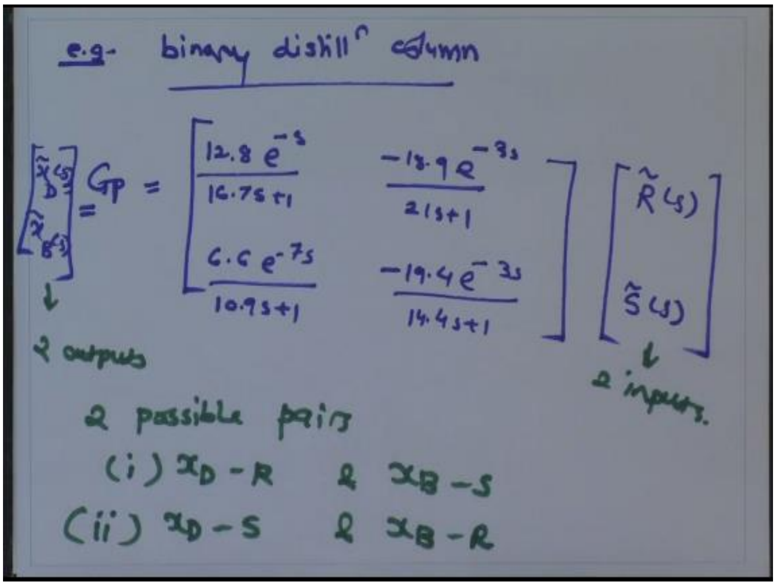
How do we interpret this? This is a necessary condition for stability but not sufficient condition. If you want to have a stable configuration, then this condition has to be satisfied or whatever condition Niederlinski index gives but that does not guarantee that the resulting system will be stable. It may be stable, it may be unstable; however, if the condition is not satisfied then you can definitely say that the corresponding system will be unstable. So how do we interpret this Niederlinski index?



If the Niederlinski index is negative, then the closed-loop system under integral control will be unstable. So what Niederlinski index says is for any pair of input and output, you calculate this Niederlinski index and if that index comes out to be negative, then if you really implement that sort of pairing. If you happen to use any integral controller like PI controller or a PID controller, the corresponding system will be unstable which means you would not select, the bottom line is you would not select that kind of a pairing.

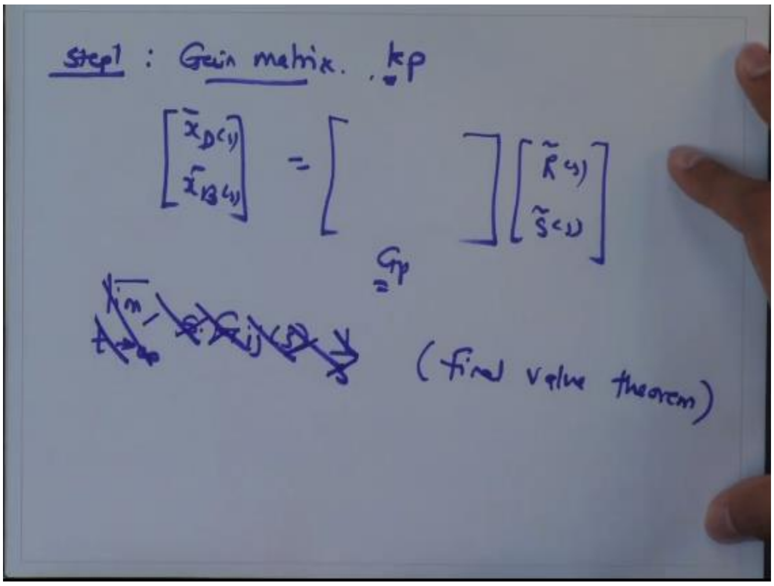
If Niederlinski index is >0, what do you do? Pairing may be stable, so the Niederlinski index does not tell you for sure whether the corresponding pairing will be stable. What it just says with confirmation is that if you have Niederlinski index which is <0, then the corresponding pairing will be unstable.

Let us now take a small example and try to see how we can find out whether the corresponding system would be stable or unstable or how do we go about really calculating this Niederlinski index.

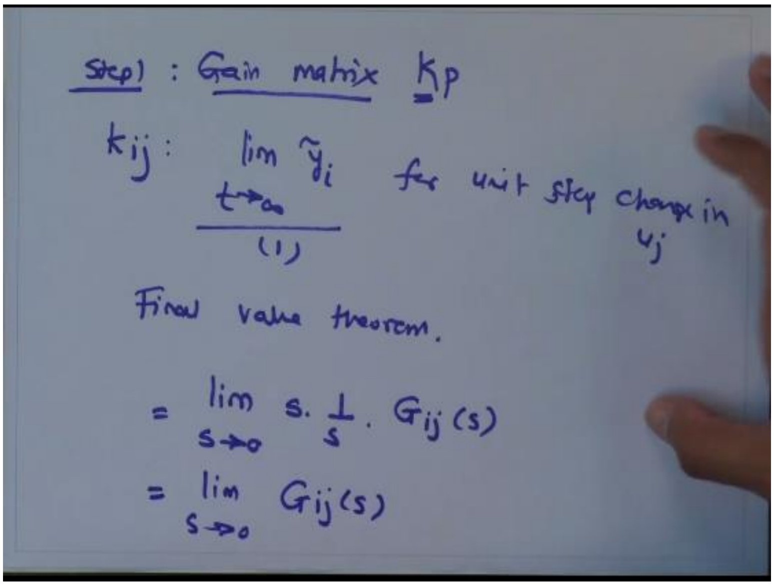


We will take an example of a binary distillation column and the corresponding transfer function matrix is given by this and this Gp is how does your top purity vary and how does your bottom purity vary as a function of 2 manipulated variables, reflux rate and steam rate in the reboiler. For this system what you can see is, you have 2 outputs, you have 2 inputs and there are 2 ways in which you can couple.

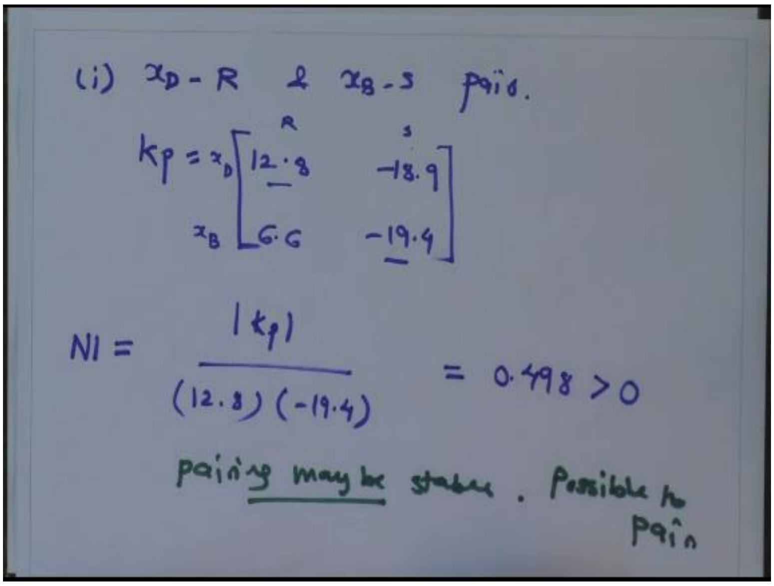
So 2 possible pairs, one is x_D-R and x_B-S and the other pair is x_D-S and x_B-R. In one pairing what you can have is you can control the top purity by using reflux rate and the bottom purity by changing the steam flow. The other possible pairing is controlling the distillate purity with the steam flow and the bottom purity by the reflux flow. Let us now see how do we go about calculating Niederlinski index for this. First you have to find out the steady state gain matrix.



So whatever transfer function we have right now in order to get the steady state gain what we have to find out is limit t tending to infinity, the value of the final output when we gave step change the input which typically for any gain, it would come out to be s times $G_{ij}(s)$ into unit step chain so it will be 1/s. This is the final value theorem. We have to use final value theorem so continue. Let us start with how do we calculate the gain matrix first.



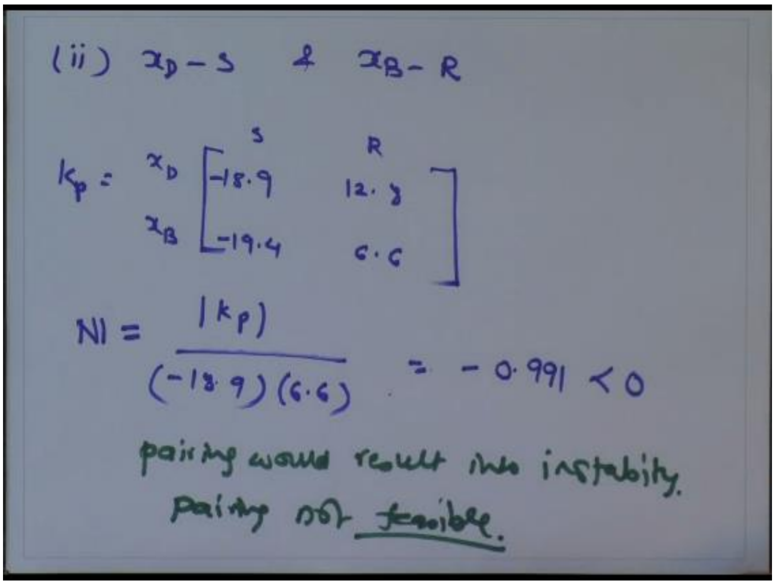
The gain matrix represents the gains between the 2 outputs and the corresponding inputs. For any input-output pair, so if I want to calculate k_{ij}, it will be given by limit t tending to infinity how y_i changes for a unit step change in the u_j, so that will be divided by 1. You can find that by using final value theorem and that will be eventually given by limit s tending to 0, s into unit step change so 1/s and the corresponding final output will be $G_{ij}(s)$. So what it essentially boils down to is limit s tending to 0, $G_{ij}(s)$, that way you can calculate the gain matrix.



For this particular example let us see if we consider the first option of x_D-R and x_B-S pairs and then the corresponding gain matrix will be you can say this will be for R, this will be for S and this will be for x_D and x_B. The gain between R and x_D would be 12.8, gain between S and x_D would be -18.9. The gain between x_B and R will be 6.6 and the corresponding gain between x_B and S will be -19.4.

Now let us calculate Niederlinski index. It will be the determinant of kp divided by the product of the 2 diagonals which is 12.8 and -19.4. So if you do that you will find out the corresponding Niederlinski index is 0.498 which is >0. So the conclusion which you can draw from here is the pairing maybe stable that is possible to pair.

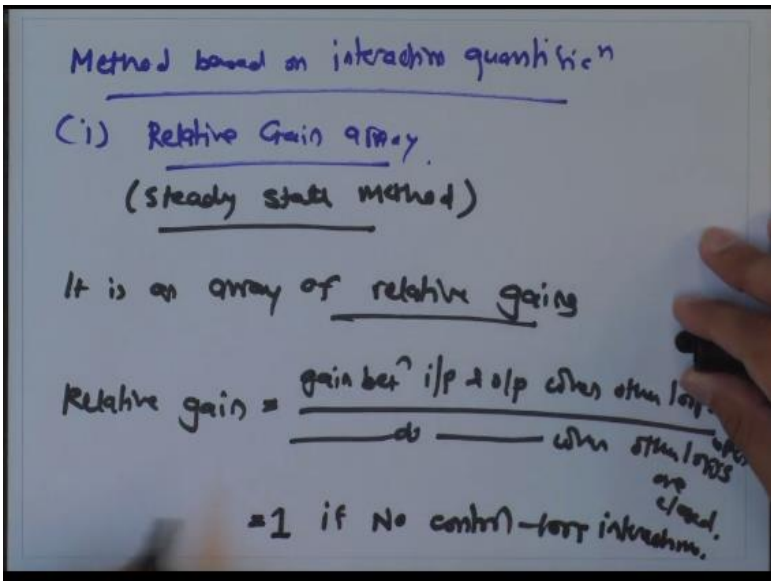
Let us now consider this other pair.



We will have x_D-S and x_B-R. Correspondingly you will have x_D, x_B, S, and R. Again, between S and x_D, is -18.9, this gain is 12.8, is -19.4 and this is 6.6. Again, will calculate Niederlinski index, the determinant of kp over (-18.9*6.6) and in this case, the determinant comes out to be, Niederlinski index comes out to be 0.991 which is <0 whereas Niederlinski index is <0, so pairing would result into instability or pairing not feasible.

From a stability point of view what you can see is this particular pair is not feasible. So essentially this exclusion leads you to the conclusion that for this column you can use the previous pairing which was x_D-R and x_B-S. So that is how Niederlinski index can be used to sort of reject or exclude the pairings which may result into instability.

Now it is not always possible that you would be able to exclude all the other alternatives giving rise to the final pairing. Because of that, another method is used that is based on interaction and this is known as a relative gain array. It tells, captures the interactions because of all possible pairings and you will try to select the pairing which would result into minimum interactions. We will look at the second method.



This method is based on interaction quantification and the method which we are going to use is a relative gain array. Now let me point out to you that both these Niederlinski index and relative gain array, both these are steady-state methods. All you require is the steady state gain of the process and based on that it will be able to calculate or it will be able to give you guidance about what pairings are possible, which pairing would give you minimum interactions and so on.

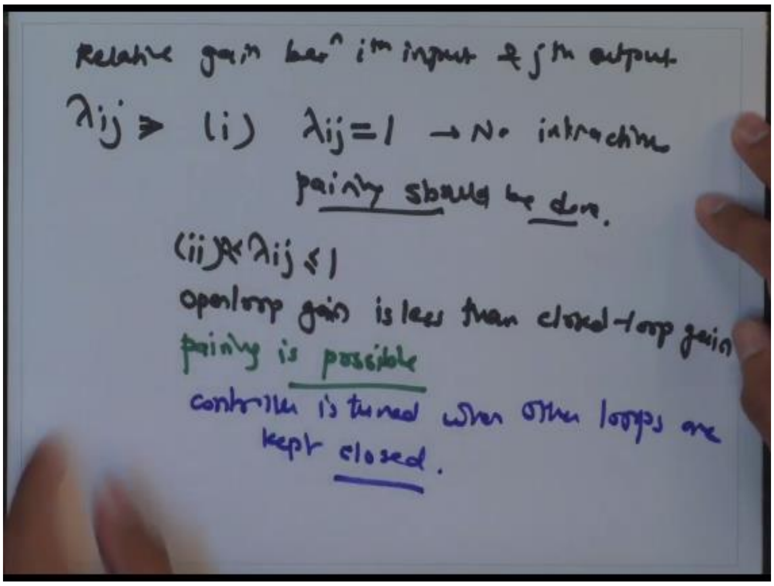
So that by no means is the ultimate method. A lot of times what has been observed is that whatever analysis you do is based on the steady-state methods, may not always be effective from a dynamic point of view and because of that lot of new methods have come up which would work on the actual taking the dynamics into consideration while pairing these variables; however, these steady-state methods do give you a very good starting point. A lot of times it requires a minimum amount of information from the process. With these advantages generally, a relative gain array is very popular in terms of doing the pairing between variables. Let us now see what is this relative gain array method.

The relative gain array is an array as the name suggests of relative gains. First, we need to define what is the relative gain. The relative gain is the ratio of two gains, so it is the gain between input and output when other loops are open. That means there is no control into the system, all the manipulated variables are free to vary and that is and when if you calculate the gain between input and output under those conditions, then what you will get is the numerator of this and the same gain between input and output when other loops are closed.

What it essentially means is, it tries to capture the deviation between input-output gain when the other loops are closed or other loops are open. Naturally, if you see that if there are no, you can imagine that if there are no interactions between the controllers, then the gain which you obtain between the input and output when other loops are closed or open it is irrelevant and therefore the relative gain will be 1 when there are no interactions.

It will be equal to 1 if no control loop interactions and as you move away from 1, both in terms of fractions that if your open loop gain is less than the closed loop gain. The more you deviate from 1 in both the directions what you would realize is that it will increase the interactions between the control loops and therefore what this relative gain array method tells you is once you find the array you will try to pick up elements which are closer to 1. That way you can ensure that there is minimal interaction between the control loops.

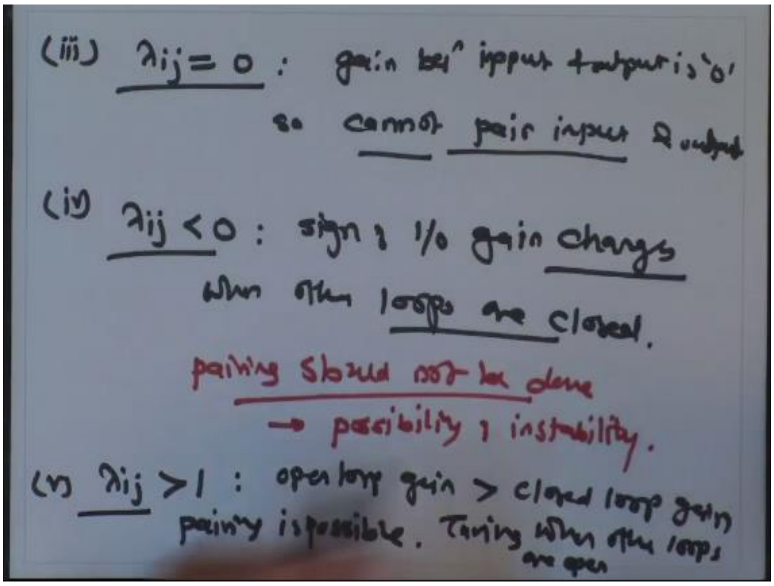
Let us now try to see what are different possibilities and under what conditions you can pair the variables, under what conditions you cannot pair the variables. So we will try to see what are the different cases which can arise from a relative gain.



The relative gain between i[th] input and j[th] output is generally given by λ_{ij} and let us say the first condition is λ_{ij} is equal to 1. So what I previously described to you was that the gain between input and output input j and output i is not dependent on whether the other loops are closed or open. That means there are no interactions and therefore pairing should be done.

Let us consider another case that relative gain is between 0 and 1. In that case, what you are seeing is that the open-loop gain is less than closed-loop gain that is why the relative gain has become less than 1, so in this case pairing is possible. So the conclusion here is pairing is possible and in this case, what you would do is generally when you want to tune a controller, you try to tune the controller when the gain of the process is maximum so that the controller gain would be minimum. Therefore here the maximum gain is in the closed loop, so the controller is tuned when other loops are kept closed.

So we will see the significance of this later on when we talk about how do you pair these multi-loop SISO controllers but here the pairing would be possible and you would try to select a value of λ_{ij} which is closer to 1 rather than which is closer to 0 because that will tell you that the interactions are minimum.



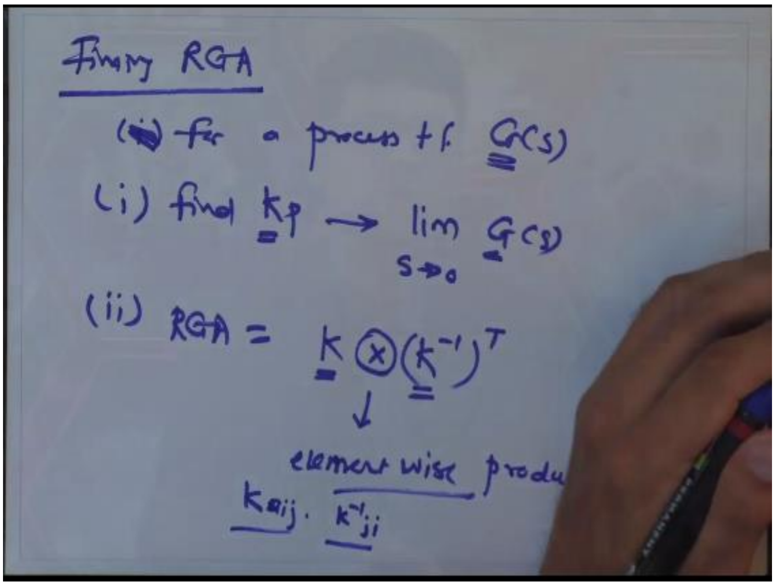
Then, let us consider the third case. We will consider λ_{ij} is equal to 0. When you say λ_{ij} is equal to 0, that means the gain between input and output is 0, so cannot pair the two inputs. You will not pair anything which has lambda=0.

The fourth case let us consider is λ_{ij} is negative. In that case, what you can note is the sign of input-output gain changes when other loops are closed. This is one of the most serious cases of interaction which you can find in the control system that a λ_{ij} is negative which means if my open loop gain is positive, my closed loop gain is negative or vice versa. So what it means is let us say if you design a control system considering that the other loops are open and you get a positive process gain, so that your controller gain is negative. In such a case and your controller gain is also positive because the product has to be positive and after some time you make the other controllers closed, so what happens is the corresponding process gain reverses its sign. So when the other loops are closed, then the gain from positive it becomes negative and as you are using the same controller which was earlier, the controller gain is still positive and the system will instantly become unstable.

This is one of the most serious cases of interactions which you can encounter that the process gains between the input and output change its sign when the other loops are closed or open. So pairing should never be done in such a case because this results in a very large amount of interactions and the possibility of instability.

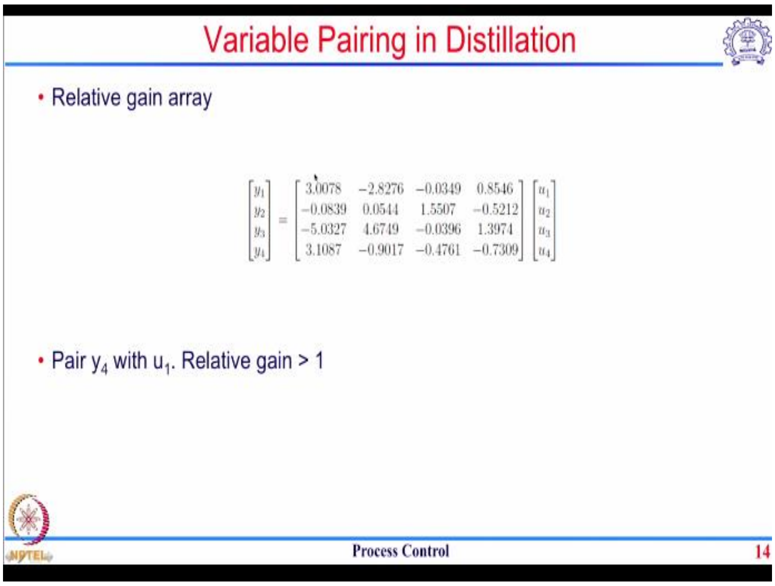
Lastly, we will consider the other side that λ_{ij} is greater than 1. In that case, your open-loop gain is greater than closed-loop gain. So pairing is possible and the controller should be tuned when other loops are open because as I said you would want to tune the controller when the gain between input and output is maximum. That is maximum when the other loops are open, so you will do the tuning when other loops are open.

So by finding the values of these relative gains, you can find out whether a particular pairing is possible or not and then later on you would try to select a pairing which gives you relative gains which are very close to 1 and that way you would be able to choose this multi-loop SISO system which gives you minimum interactions. Let us now see how you can find the relative gain array for any system.



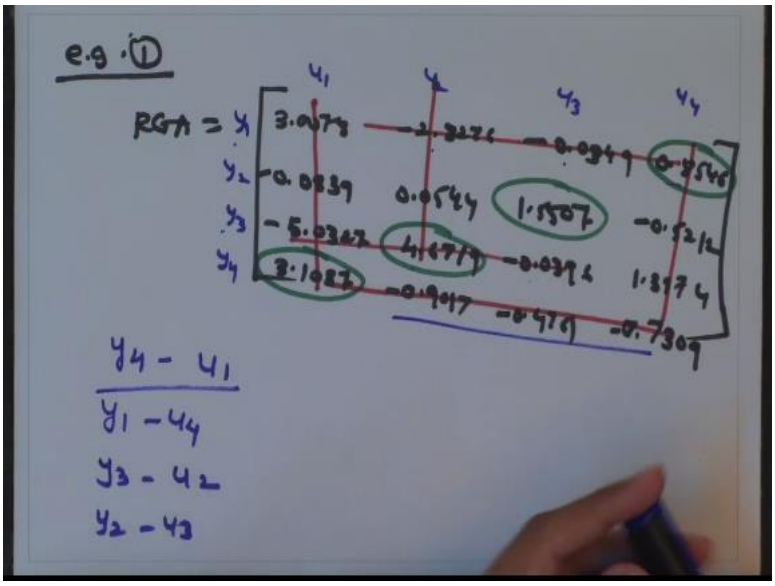
In order to find RGA, what you start with is for a process transfer function G(s) like the one which we saw earlier. The first thing is to find the gain matrix kp. This we have already seen, it is limit s tending to 0 G(s). So you find out the gain matrix and then the RGA is this gain matrix with an element by element multiplication with K inverse transpose. So this is an element-wise product.

This is not a matrix product which we are talking about. So it will be a_{ij}, so this will be K_{ij} multiplied by K^{-1}_{ji}. Every element will be like this and once you do that multiplication whatever the matrix which you get would be the relative gain array. Let us now see some example systems. I have one example system here.



As you can see that there are 4 outputs here and there are corresponding 4 inputs and this is the relative gain array which has been obtained by using a similar method. Now you can see that what we want is we do not want any negative relative gains, so if you look at the fourth output y_4, then the 3 relative gains are all negative, so the only possible pairing here is y_4 with u_1.

So, therefore, you would fix y_4 and u_1, so this thing is gone. Let us now see. Let me just show it to you on paper as well.



Example 1, the corresponding RGA is this. As I said for y_4, these 3 are negative, so you cannot pair y_4 with u_2, u_3 or u_4. So y_4 with u_1 is the only possible pair. Accordingly, now this pairing is done. Now your u_1 is done and your y_4 is also done. The remaining matrix is this 3 x 3 matrix and again we will try to find out what is the possible pairing. Now in the second step, you can see that in terms of y_1 what you have is this is negative.

The other gain is negative, so the only nonzero entry is this. So you will have to pair y_1 with u_4. When you do that pairing what you will end up with is now you have used u_4 and you have used y_1, so now that is the 2 x 2 system which is remaining. Now within that, you can see that for y_3, u_3 the corresponding relative gain is negative, so you will not want to pair that. You will want to pair y_3 with u_2 and automatically once that is done, the only remaining thing is y_2 with u_3 and that is nonzero. So automatically you have made sure that the corresponding system is not unstable.

The 4 relative gains which are of importance here are y_4-u_1, so this is 1. y_1-u_4, so this is the second relative gain. y_3-u_2 and this one. So you can see that these are the 4 relative gains with which you are pairing.

None of them are negative, so the pairing is possible and you can see that the corresponding numbers indicate the interactions. The minimum interactions are obtained for this particular controller which is y_1-u_4 which is the closest one and that is the only value which is less than 1. So that controller will be tuned when the other loops are closed whereas all these other 3 controllers would be tuned when the other loops are open.

That way you can find out the pairings between inputs and outputs and so we will take a short break here and after the break, we will continue this discussion. Thank you.