Chemical Process Control
Prof. Sujit S. Jogwar
Department of Chemical Engineering
Indian Institute of Technology – Bombay

Lecture - 31
Frequency Domain Analysis

Welcome back. So we have now seen what do we mean by frequency domain analysis. What we do is we capture what is known as the frequency response of a system. That is we compute what is the amplitude ratio and phase. Compute the variation of amplitude ratio and phase with respect to frequency and we have seen that it can be represented as a Bode diagram or a Nyquist diagram. So let us now see how do we use this information to compute or to assess the stability of a feedback system.

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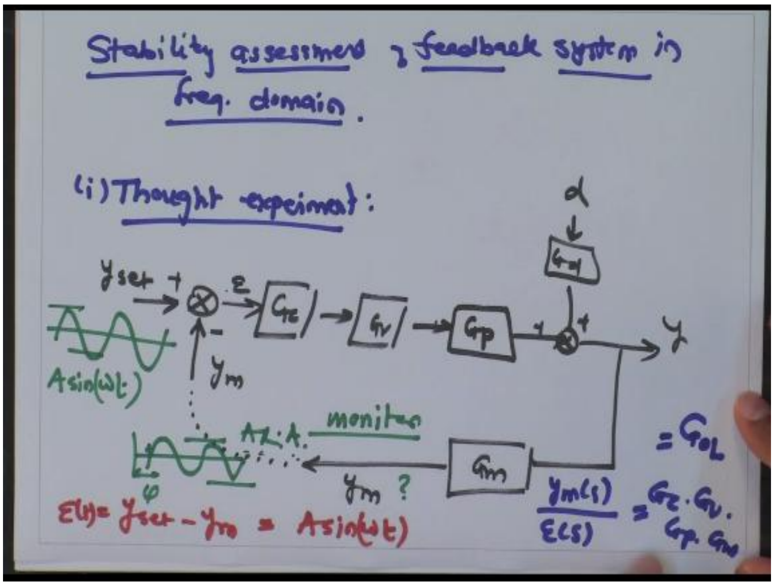

So we will see stability assessment of the feedback system in the frequency domain. So in order to do that we will consider one thought experiment so that it will help us kind of understand what are the stability conditions when we do the stability analysis in frequency domain, and we are going to see two stability criteria and both these are sort of evident from this simple thought experiment.

So let us consider an imaginary experiment of a feedback system. So let us start with a feedback system. This is our process G_p, a disturbance transfer function. This gives you output y, then you have a measurement, then you have this measured output. If we go back from here, you have the valve transfer function, before that we have the controller transfer function and this is error and error ε is the difference between the measured value and a set value.

So I have deliberately not shown this particular link because in this thought experiment we are going to inject sinusoidal input here. So we are going to say the set point trajectory is a sinusoid input. So let us say this is A*sin(ω*t). We are going to input a sinusoid here that brings in why we are in the frequency domain.

Now we monitor how does y_m look like. Although we are assessing the stability of a feedback system, the analysis or the thought experiment actually starts with the open loop, that we have opened this loop, we are not going to feedback the measured value, we are just assesseing the feedback value. So if you see what we have is in this case, the error is y_{set} - y_m.

Here y_m is not supplied, so it is considered to be 0 which is equal to A*sin(ω*t). So your error is A*sin(ω*t), and if you look at what is the transfer function between $y_m(s)$ and $\varepsilon(s)$, it is if you have to go from ε to y_m. It is G_c * G_v * G_p * $G_{m,}$ and this is also represented called as G_{oL}. So it is called an open-loop transfer function between the measured output and the error which is the product of these four transfer functions.

And we are going to give a sinusoidal input here as $y_{set,}$ but it is going to be equal to ε because y_m is 0 and as there are these four processes depending on these four transfer functions, we will see the output y_m will again be a sinusoid. So even though I am saying we want to monitor y_m, it is going to be a sinusoid with a certain phase. So it will have a certain phase, and it will have a certain amplitude ratio.

So we have already seen how do we compute these two. So we know that if this is the input sinusoid and we are interested in finding the output, then this AR and φ would be obtained by using the frequency response of this particular transfer function which is the product of these four transfer functions. So depending on the frequency, this phase, as well as amplitude ratio, would change.

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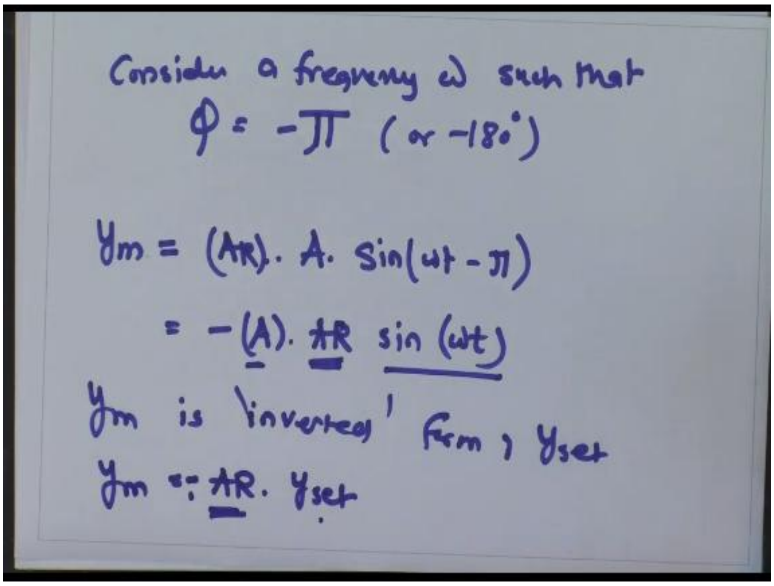

So we will consider a frequency ω such that the corresponding phase is –л or -180^o. So in that case if I say y_m will be equal to whatever is the amplitude ratio * A*sin(ω*t)+ φ(ω*t- л), so it is going to be equal to –A * AR sin(ω*t). So what we are seeing is so this y_m is inverted form of y_{set} because it is minus and we have some magnification or reduction in terms of amplitude.

But if you look at overall signal y is equal to y_{set}, so this is actually AR-AR*y $_{set}$. That is why I said it is inverted because there is a minus sign, there is a gain of AR, or otherwise, it looks very much like y_{set}. So now coming back to our thought experiment. Now let us see that we have this particular frequency ω at which phase is -л so that this is -AR * y_{set} and at that whenever you have that we have found that frequency.

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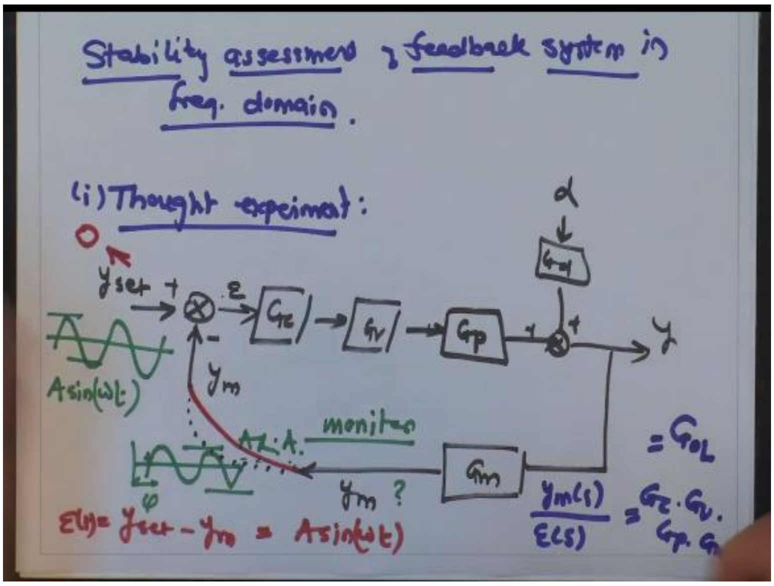

You now close the loop so at that particular point, we are going to close the loop and we are going to make it 0.

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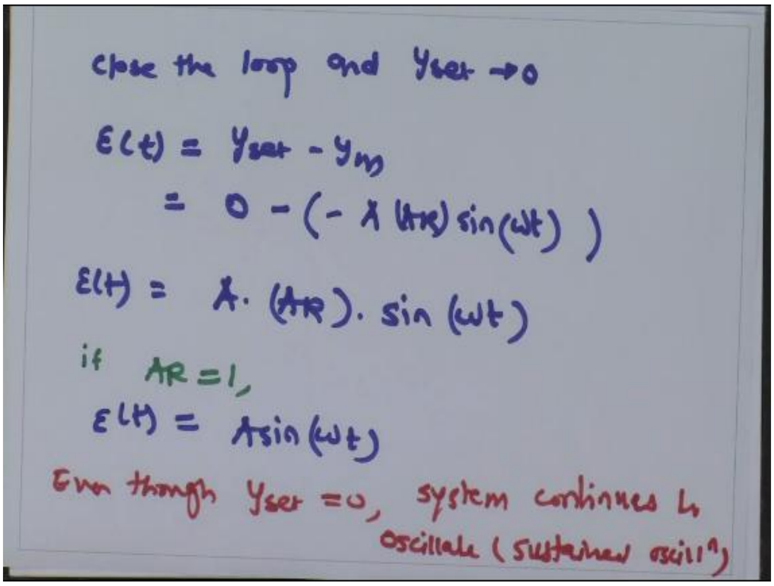

So what we are going to do is we are going to close the loop and y_{set} goes to 0. Now let us look at what would be the error. Error is still y_{set} - y_m so now y_{set} is 0 and y_m we have just calculated as –A*AR*sin(ω*t). So this is equal to A*AR*sin(ω*t). So you can see that earlier our error was A* sin(ω*t). Now it looks very similar to that even though the set point is not there, we have still had an error as A* sin(ω*t) which is multiplied by AR.

So let us consider that if AR = 1 then ε(t) = A* sin(ω*t). So you see that ε is A* sin(ω*t)which is the same as what we had put in earlier. So that means before closing the loop, the system was oscillating at A* sin(ωt) with frequency ω, and the moment I close the loop and even though there is no set point change, so when I say y_{set} = 0 all these are deviation variables, that means the system is not subjected to any external conditioning and still this ε = A* sin(ω*t), it will go through all these transfer functions and what comes out is still A* sin(ω*t) and then the system will keep on oscillating at A* sin(ω*t). So the implication of this is that even though y_{set} = 0, the system continues to oscillate even though the original source of the oscillation is gone.

So that means if I have a phase of –л and if the corresponding amplitude ratio is 1, then I am going to get sustained oscillations.

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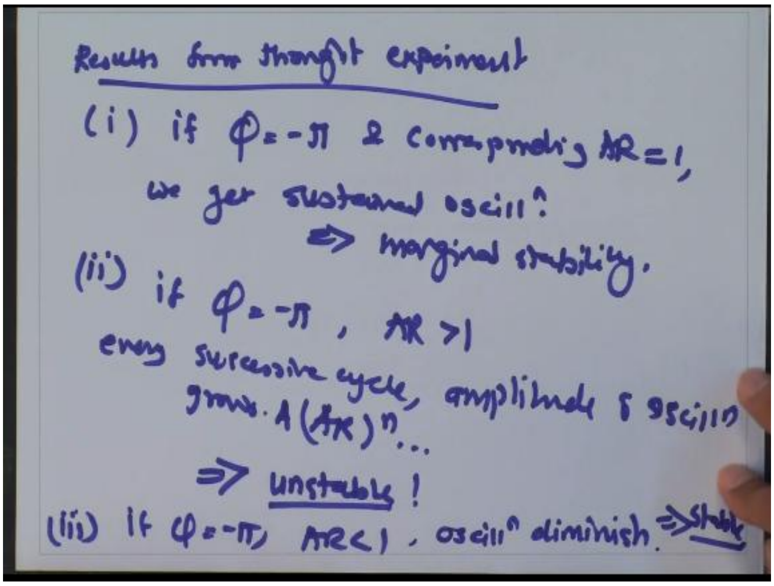

So what are the results from the thought experiment. If phase is -л and corresponding AR=1, we get sustained oscillations and if you recall from the previous definitions of stability, this is as good as marginal stability and it tyлcally represents the limit of stability. Now let us consider if for the same φ =л, AR is >1. So we go back to this same figure. Before closing the loop, we know that this y_m for this recirculating error is going to be A*AR* sin(ω*t).

So what does that mean? We close the loop, we made the y_{set} to be 0 and if AR is >1 if I put in A* sin(ω*t) what I get is something which is more than A* sin(ω*t). Then, that goes into this next cycle, what we will get is even higher than that. So every successive cycle amplitude grows because you will get $(AR)^n$. So the amplitude will keep on becoming $A*(AR)^n$.

So what does that mean? The system is unstable because the system will have growing oscillations and similarly you can also show that for the corresponding case, AR <1, in that case, every successive oscillation would have a smaller magnitude. So oscillations diminish and what you have is a stable system.

So simply by looking at what happens when your phase is –л, through this thought experiment, we can claim that if at this particular frequency if amplitude ratio is 1, the system has a stability limit, is at the stability limit. If that corresponding amplitude ratio is >1, the closed-loop system is unstable, and if that corresponding amplitude ratio is <1, then you have a stable system.

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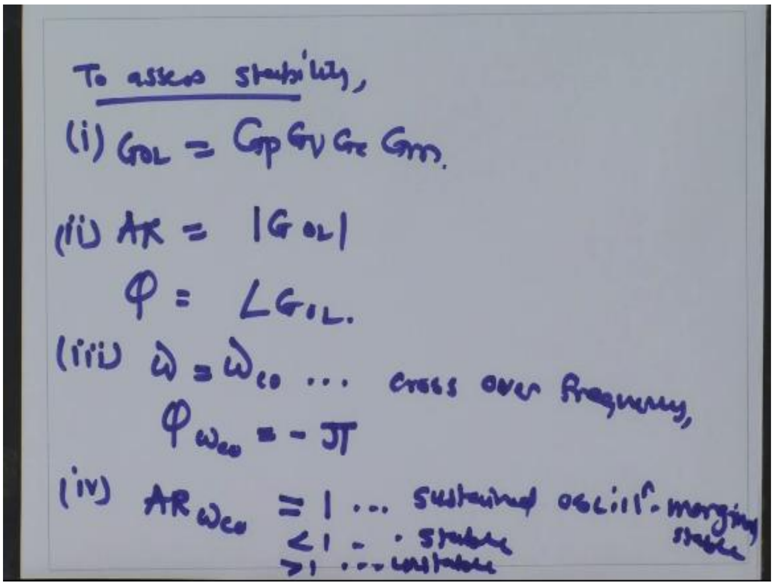

So what all we need to do is in order to assess the stability, you start with the open-loop transfer function which is Gp Gv Gc Gm. Then you compute its amplitude ratio and φ which is the amplitude of G_{oL} and angle of G_{oL}. Then, you compute ω equal to what we call as a crossover frequency such that $\varphi_{\omega\ \text{crossover}}$ = -л and then you compute AR at ω crossover. If it is equal to 1, you get sustained oscillation or marginal stability.

If it is<1, it is stable. If it is >1, unstable. So this principle we will be using in order to formulate the two stability criteria in frequency domain, the first of which is known as a Bode stability criteria, is presented here.

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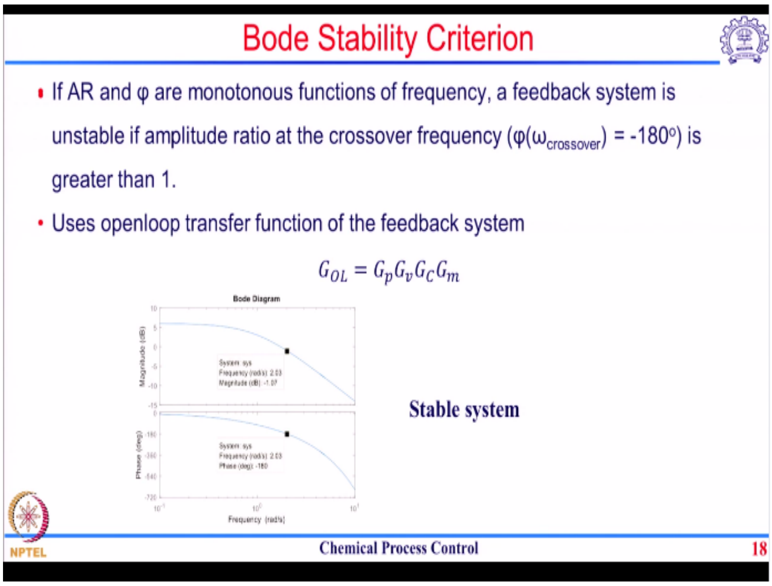

So you can see that in the Bode's stability criteria, it starts with an assumption that if AR amplitude ratio and φ are the monotonous functions of frequency which are tyлcally most of the cases is most of the times is the case. The feedback system is unstable if the amplitude ratio at the crossover frequency which is when phase is equal to -л or -180^o >1. So that is the Bode stability criteria and in order to compute that we need this G_{oL} which is G_p G_v G_c and G_m.

So it follows directly from the thought experiment that you compute the frequency at which ω at which φ is –л and compute the corresponding amplitude ratio. If it is >1, the system is unstable. If it is <1, it will be stable. So here it is represented as a Bode diagram for a stable system, you can see that. At this particular frequency, you would see that the phase is -л and then you move up into the first figure, you compute what the corresponding AR is.

This is a Matlab plot, so in Matlab, AR is given as in a decibel unit, so it is 20 log of amplitude ratio. So if the amplitude ratio is equal to 1, this decibel will be 0. So as long as this decibel value is positive which means AR is >1, the system is unstable. Here this decibel value is negative, so the system is stable. So by just naturally following from this thought experiment what we have found is that using the Bode stability criteria all we need is compute the corresponding crossover frequency and check the amplitude ratio.

If it is >1, the system is unstable. Then, so the major assumption in the Bode's stability criteria is that this AR and φ are monotonous functions of frequency and by that we mean as we keep on increasing ω from 0 to infinity, AR and φ both change in the same direction and for a first-order system, second-order system this is indeed the case for most of the time that AR and φ both will decrease like this figure.

But in some cases, this condition is not satisfied. So what do we do? So in that case, we have to look at more rigorous stability criteria which are given by Nyquist, and that is why there are two diagrams.

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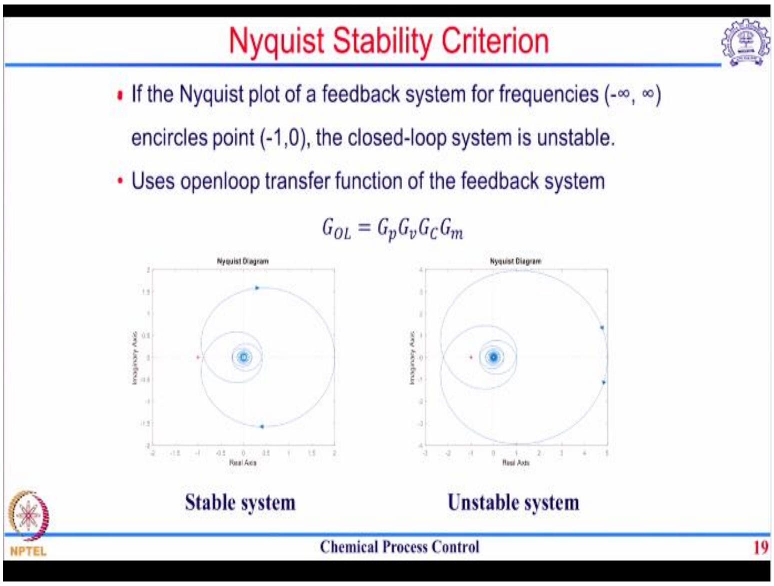

One is given for Bode and one for Nyquist. So the Nyquist stability criteria also follows from the same thought experiment; however, it is more general compared to the Bode stability criteria. What you see in the Bode's criteria or from the thought experiment is that in order to assist the stability all we need is this signal should be sort of an inverted form of this, so this inversion will happen when you have a phase of –л.

But the same thing will also happen at the phase of -3л or -2(n+1)л. So this Nyquist plot then looks at all these successive periodic values of frequency when these phases are multiples of (2n+1)*л and then takes into consideration. So if AR and Φ are, particularly if AR is not a monotonous function of ω, then just satisfying this condition at -л is not sufficient, you have to look at the condition at every successive (2n+1)* л and ensure that all those AR's are also <1.

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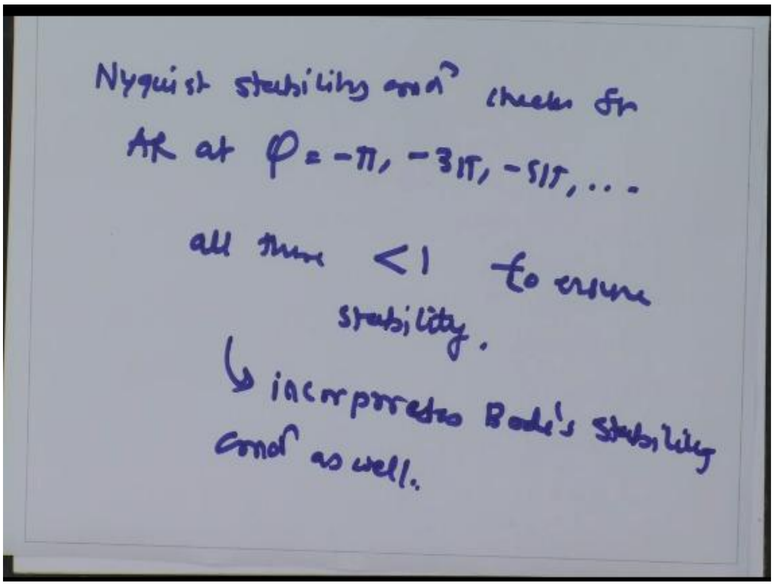

So the Nyquist condition checks for amplitude ratio at φ = -л, -3л, -5л and so on and it says that all these should be <1 to ensure stability. So in a way, it incorporates Bode's stability condition as well, and the way this stability condition is specified is using a Nyquist diagram. It says that if you generate a Nyquist plot of a feedback system and here you are computing the Nyquist response for negative frequencies as well. This is so that the Nyquist plot has a good shape. So what you do is, you draw the Nyquist plot for all these frequencies, and if it encircles point (-1,0) then the closed-loop system is unstable. So now what is so special about this point (-1,0)?

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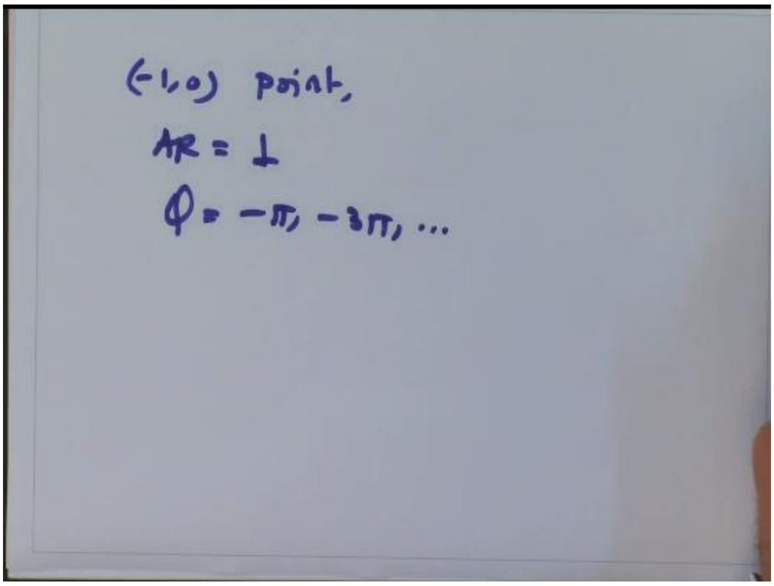

So (-1,0) point says its amplitude ratio is 1 and the phase is –л, -3л and so on. So this represents all these points at which you are going to have this inversion of the signal and so therefore if your Nyquist plot does not encircle this particular point which means we will have a stable response. So here is a figure, so this particular figure shows this is the -(-1,0) point which is shown in red.

So for this particular Nyquist plot has, the AR is not a monotonous function of ω, but you can see that (-1,0) lies outside all this. So it is a stable system. Whereas for this particular system this (-1,0) point has been encircled using this circle. So it is going to be an unstable system. So this is the stability assessment in the frequency domain.

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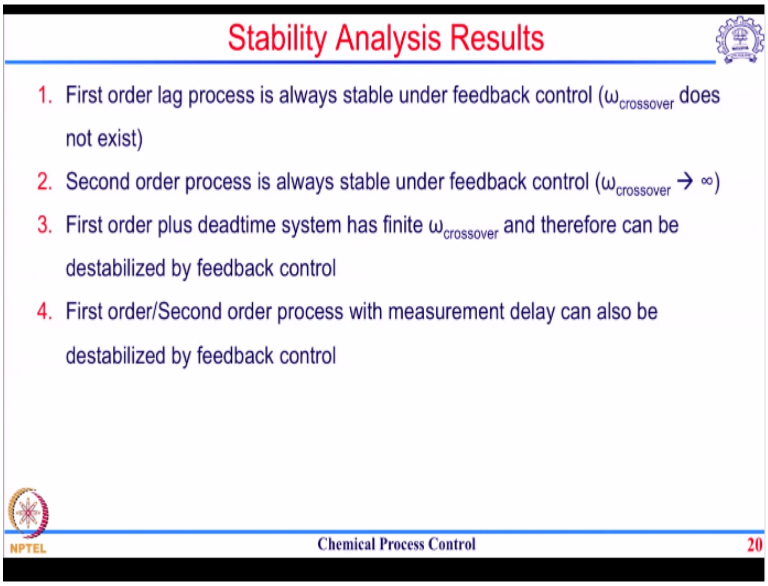

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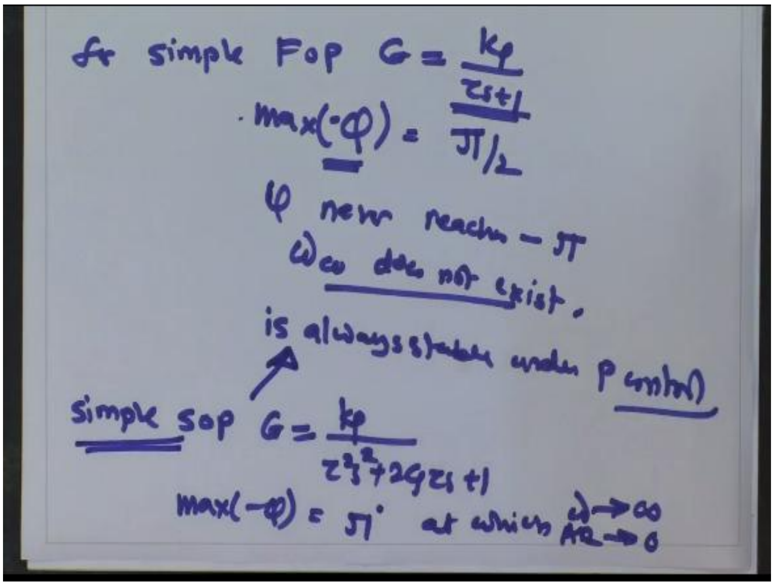

So now if you want to get some results from this what we would see that, for a first-order process, where G is kp/(τ s+1). So the maximum φor the maximum phase lag we are going to get is л/2 because the phase is -90^o, so because of that the phase will never reach, so the phase never reaches –л. So ω crossover does not exist. As ω crossover does not exist, this simple first-order process is always stable under P control.

Because when you have a P control, it is just going to add to the gain of the system. So it has no effect on the phase. Now when we look at the simple second-order process, G is kp/($\tau^2 s^2$ +2 ξ τ s+1). What you will see is that the maximum phase lag is л at which ω tends to infinity and AR tends to 0. So the ω crossover is actually an infinite frequency at which a second-order system will give you phase of –л and the corresponding AR is always 0.

So even this second-order system is also always stable under P control. So if you recall the example which I had shown you, 3 tanks in series because even if I would had considered a single tank or two tanks in series without any delay, those systems will always be stable. So only when you have any system which is higher than the second-order without a delay that will have some limitation in terms of the proportional controller gain.

Any first or second-order system will not have any finite ω crossover. But the case changes when you have any delay.

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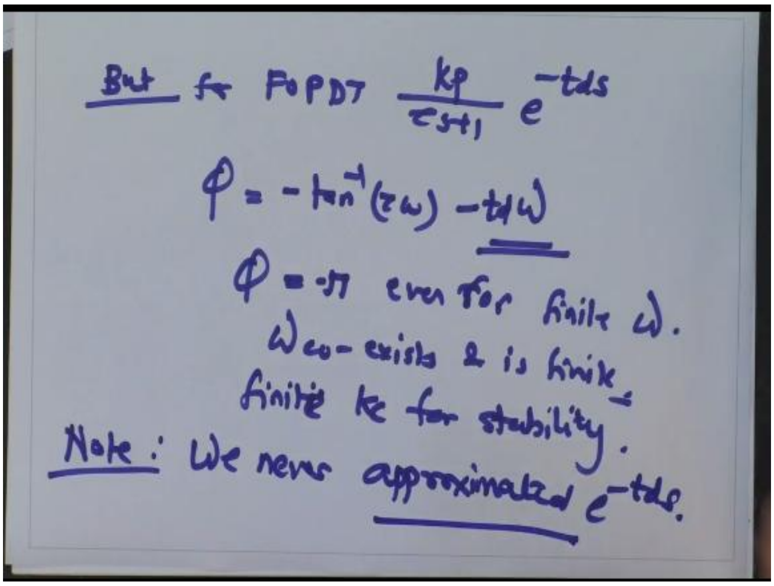

So if you have, a "first-order + dead time" when you have [k_p/(τ s+1)]* (e–^{td*s}), the phase is -tan^{-1}(τω) –t_d*ω and because of this term φ = -л even for finite ω. So ω crossover exists and is finite, and therefore you also have finite k_c for stability. So the moment you add any time delay, you are going to have some restrictions on the stability limit.

Note that we never approximated e^{-td*s}, so all the analysis which we have carried preserved the transfer function e^{-td*s}. Therefore, the stability analysis which we are going to get from this is going to be accurate, or it is going to be more accurate compared to a Laplace domain analysis. So let us now complete this discussion by looking at the same tutorial problem of the Blender.

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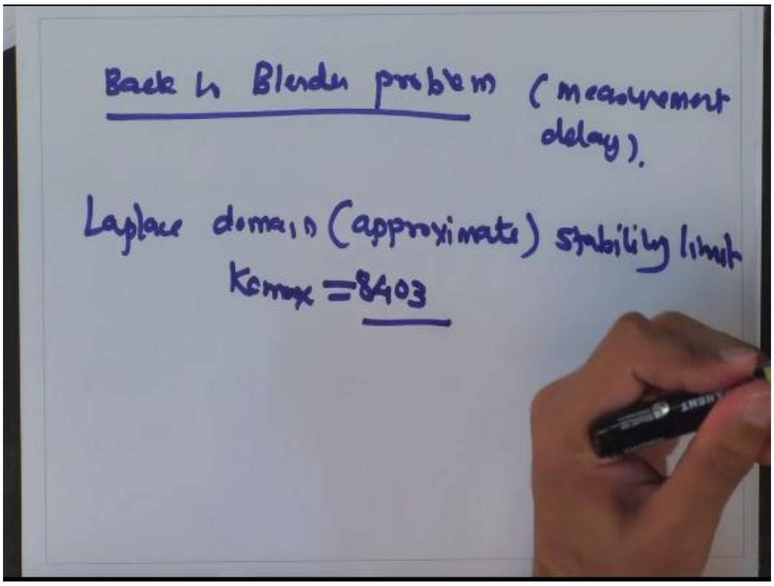

So the Blender problem, so back to Blender problem which had measurement delay, so the Laplace domain analysis, as approximate stability limit was $k_{c,max}$=8403. So now let us look at what happens.

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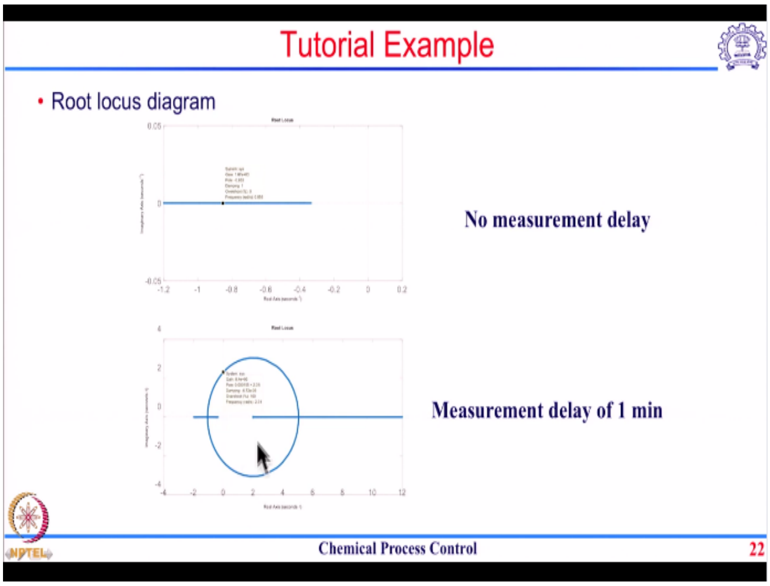

So here is the root locus diagram for that particular system when there is no measurement delay, it is a simple flat line. It's all the poles move towards negative infinity, it is all stable because it is a first-order process. When you have a measurement delay, and we approximated by Pade’s approximation, you can see that the root locus diagram does cut the imaginary axis at these two points when the gain is 8403. So that was the stability limit computed by using Laplace domain analysis.

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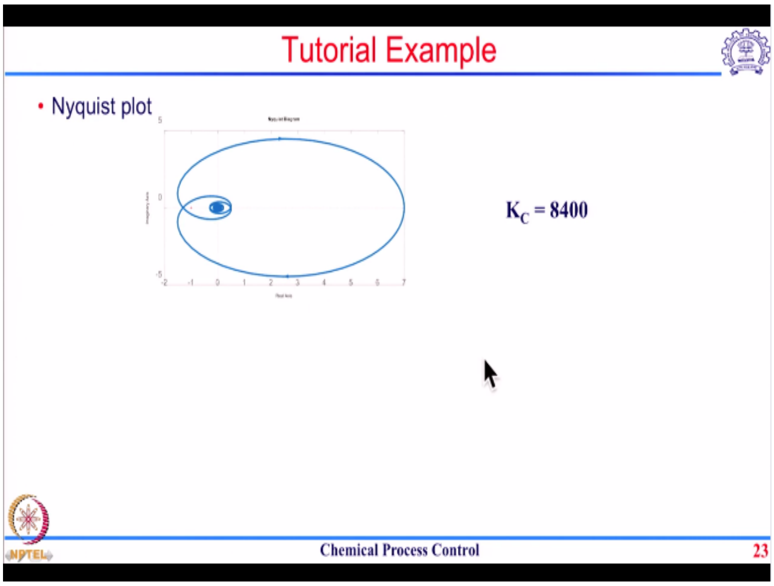

When we move to the Nyquist plot even though we consider any gain which is less than the stability limit which in this case is 8400, you will see that it is encircling the point (-1,0). So this frequency domain analysis is correctly pointing out that this particular controller gain is unstable.

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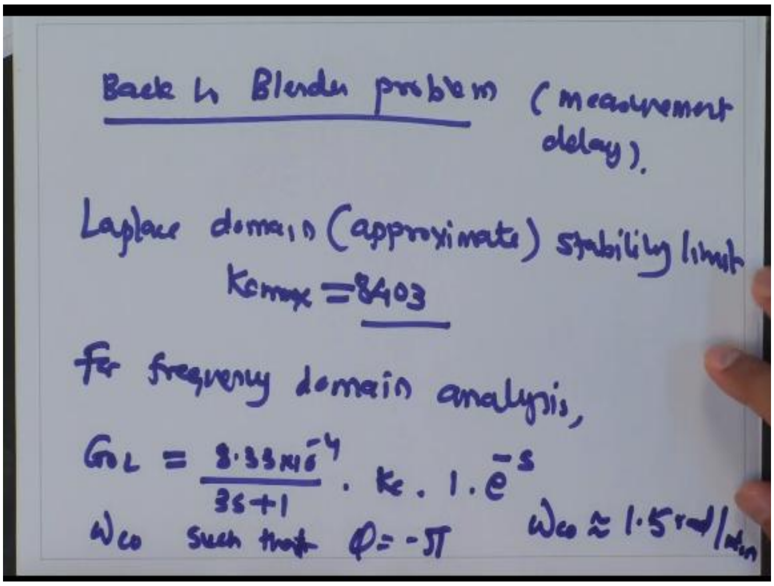

And if we do the frequency domain analysis for this system, what we will analyze is that for frequency domain analysis our G_{oL}, in that case, is $(8.33*10^{-4})/(3s+1)$. So this is Gp, Gc is Kc, Gv is 1 and G_M is $e^{-s,}$ so it is more or less like a first-order plus dead time system. You can compute ω crossover such that φ = -л so that ω crossover roughly comes out to be 1.5 radians per whatever is the time unit.

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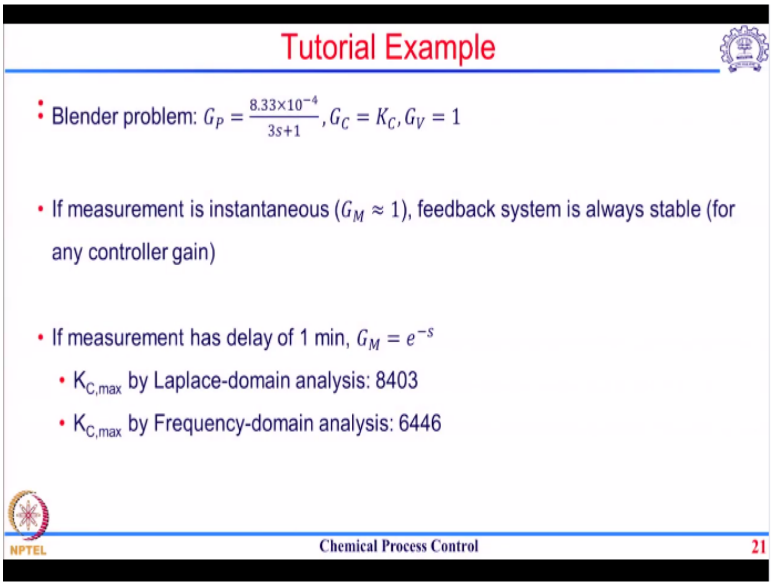

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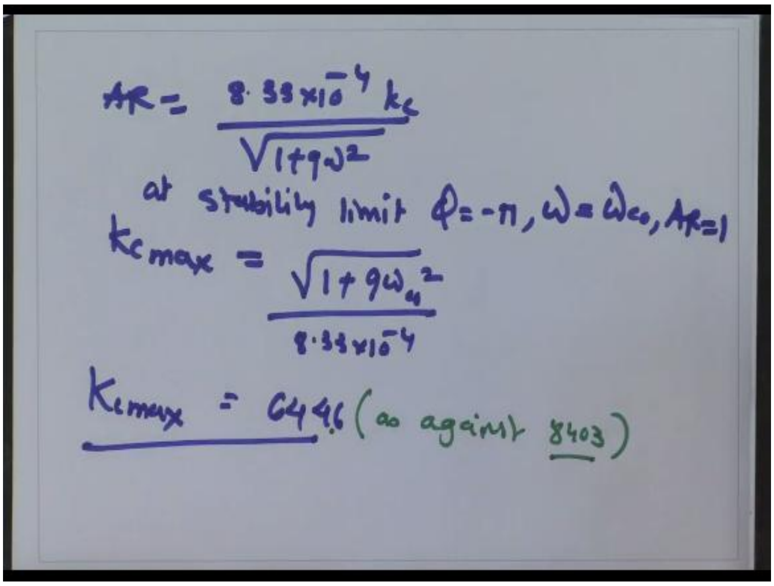

And correspondingly if you equate and your AR is=$8.33*10^{-4}$ $k_c/\sqrt{(1+9\,\omega^2)}$. So $k_{c,max}$ will be given by at stability limit φ = -л, ω =$\omega_{crossover}$, AR should be equal to1. So substituting it to be 1, you will get $1+9\,\omega^2_{crossover}/8.33*10^{-4}$ which comes out to be 6446. So the correct stability limit, in this case, is 6446 as against what we have found using the Laplace domain analysis which was 8403.

So if we have used any gain between 6446 and 8403, the Laplace domain analysis is going to say that the system is stable, but the frequency domain analysis is going to say that the system is unstable and if you simulate the system, you will indeed find that the maximum stability limit is close to 6446.

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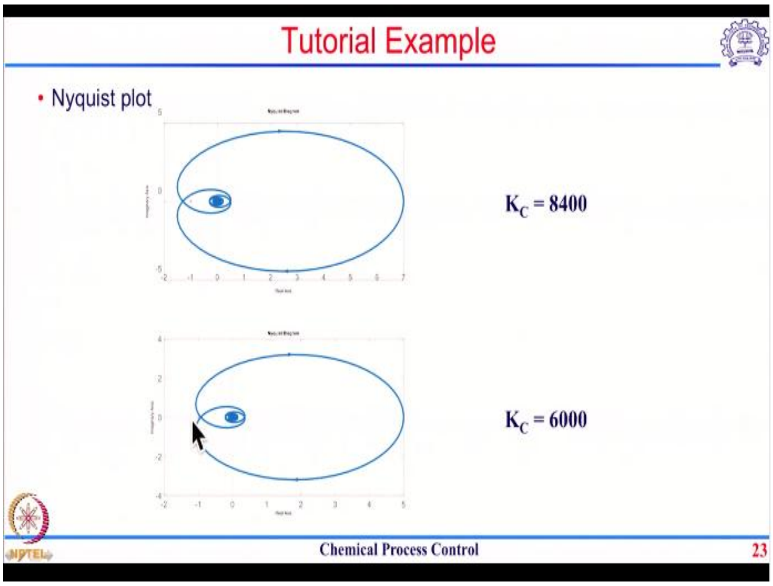

And here is the corresponding Nyquist plot or when you use any k_c which is <6000, you will see that this (-1,0) point is outside this Nyquist plot, so the system is going to be stable.

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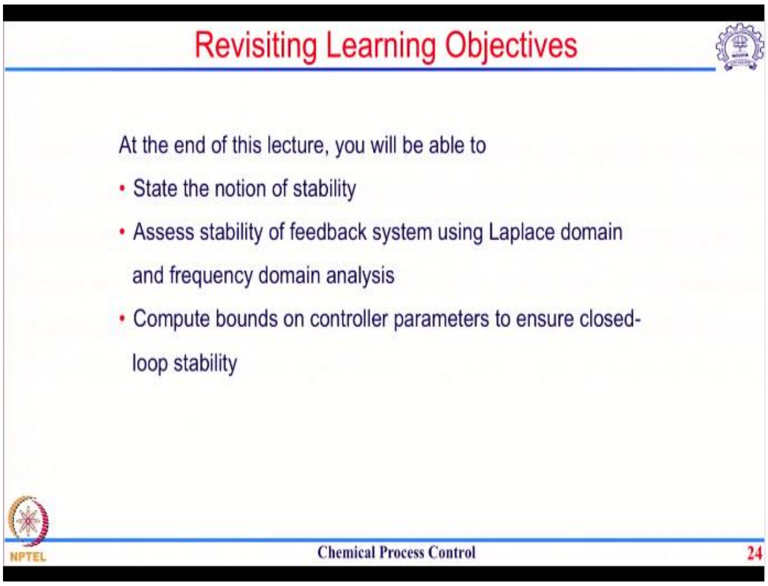

So we have now seen that how do we accurately capture the stability analysis of systems where there is significant dead time and that we do by using a frequency domain analysis which will be like capturing the amplitude ratio and phase as a function of ω and then finding out typically the crossover frequency would suffice the job, and then you compute what is the amplitude ratio at that particular cross frequency.

If it is <1, the system is stable, and if it is >1, the system is going to be unstable. So that is about how do you assess the stability of feedback systems using frequency domain analysis. Thank you.