Chemical Process Control
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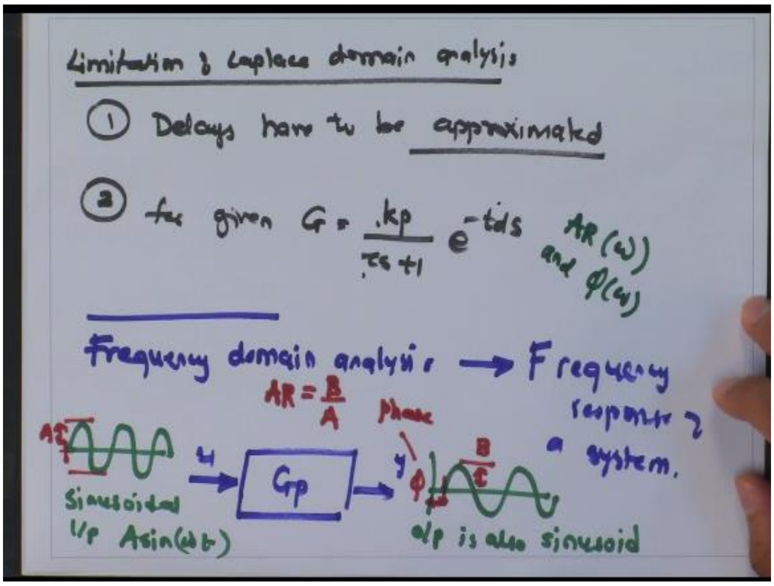
Lecture - 30
Frequency Response

Hello students. In the last lecture, we had seen that how do we assess the stability of closed-loop systems using Laplace domain and we saw that whenever the closed-loop system has any dead time, so maybe that dead time is from the process itself or it may be a dead time because of a delayed measurement. Both those cases, we cannot analyze the stability accurately using the Laplace domain.

We have to approximate the dead time by using Pade's approximation. I had said that even though you get some stability limit for that system if we implement that controller gain or slightly lower controller gain than that, you might end up still destabilizing the system and that is an approximate analysis.

In this lecture, we will see how we can correct for this and how do you perform a more accurate stability analysis by using frequency domain analysis. So that was one of the limitations of Laplace domain analysis.



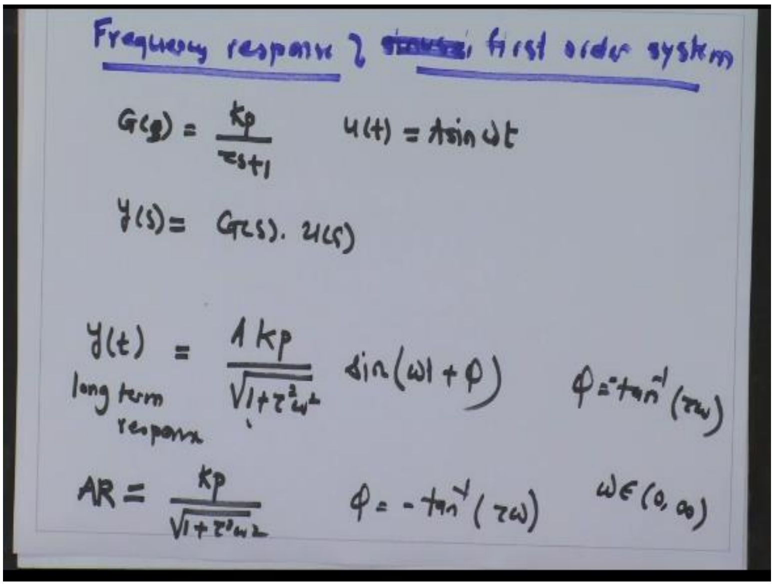
So the first one is delays have to be approximated, that means they cannot be handled accurately. The second limitation of Laplace domain analysis is that for a given transfer function, let say even though we do approximate analysis and we get the stability criteria, they are all dependent on this process parameters k_p, tau, and t_d and there is no guarantee that if there is some error or these parameters deviate which in reality is quite often the case. Then, in that case, whether that feedback system will still be stable or not, those kinds of robust analysis cannot be done using Laplace domain analysis. If there is some variation in the parameter maybe it is a plant model mismatch, it does not tell me how much is the tolerance in terms of this controller gain or time constant or a delay time which can still I can confidently say that the resulting system would still remain stable.

Both these limitations can be rectified if we perform the analysis in the frequency domain which is an inverse time domain. When I say frequency domain analysis, what we are essentially interested in capturing what is known as a frequency response of a system. This means we are going to calculate what is known as frequency response. When there is a frequency response of a system what I mean by that is you have a process which has certain inputs and certain output. And we say that at the input, we introduce an oscillating input, so a sinusoidal input may be of form 'A sin(ωt)' and when you do that it is hypothesized or it is like in fact true that when you put in sinusoidal input into the system, the output after sometime, after initial transient would again be a sinusoid. It may have a different amplitude and it may have a different phase, so the output is also a sinusoid.

The frequency response is based on this principle that if we input sinusoid into a process then the output will also be a sinusoid. But what is unknown is that if my input is of amplitude 'A', then the output may not be exactly of amplitude 'A'. So let us say if the amplitude is B, then we define what is known as an amplitude ratio as 'B/A'. So it is the ratio of output amplitude to the input amplitude.

Then we also calculate what is the difference in the phase between input and output. Let say input was 0 at this point; however, the output has this much lag, so this is called as the phase and we will see later that both these amplitude ratio and phase are functions of frequency. So when we talk about frequency response, what we are actually interested in is calculating 'AR' as a function of omega and this phase as a function of omega. So when we get this, that is known as the frequency response of a system.

Now we had actually seen a response of a sinusoidal input to one system and it was a first-order dynamical system. When we were studying first-order systems in a way back in week 2, we had seen how does it respond to a sinusoidal input and if you recollect what was our answer.



So if I want to calculate the frequency response or frequency response of the first-order system, then what we had done was we had seen the process transfer function is,

$$G(s) = \frac{k_p}{\tau s + 1}$$

We had used input as ‘A sin(ωt)’. Then by using Laplace domain analysis, we had calculated what is y(s) which will be,

$$y(s) = G(s)\, u(s)$$

Taking the Laplace of this and eventually, we expanded it using partial fractions. Then we obtained as the long-term response was,

$$y(t) = \frac{k_p}{\sqrt{1 + \tau^2 \omega^2}} \sin(\omega t + \phi)$$

Where

$$\phi = -\tan^{-1}(\tau\omega)$$

So that is what we had obtained long back. In this case, if we define the amplitude ratio as the amplitude of output to the amplitude of input, we get the amplitude ratio as,

$$AR = \frac{k_p}{\sqrt{1 + \tau^2\omega^2}}$$

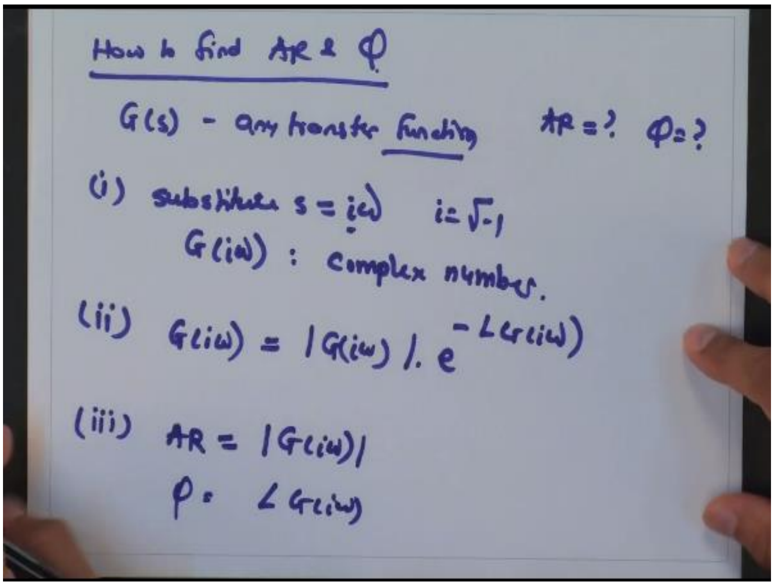
and the phase as,

$$\phi = -\tan^{-1}(\tau\omega)$$

You can see that both these amplitude ratio and phi are functions of omega. When we do the frequency response analysis, we compute this AR and phi for all possible ranges of omega that is for all omega from 0 to infinity. The motivation behind that is if you have studied Fourier series in mathematics, then you can recall that any periodic function can be represented as the sum of sinusoids with the time frequency.

So what essentially in the frequency response we are doing is that by computing this AR and phi at all possible values of omega, we are ensuring or we are finding out how this system would respond to a sinusoid of any frequency that means it can respond to any periodic function. Then later on when we develop the stability analysis theory we will see that then it will be able to capture this particular frequency which is going to destabilize the system and what are the corresponding conditions for stability and so on.

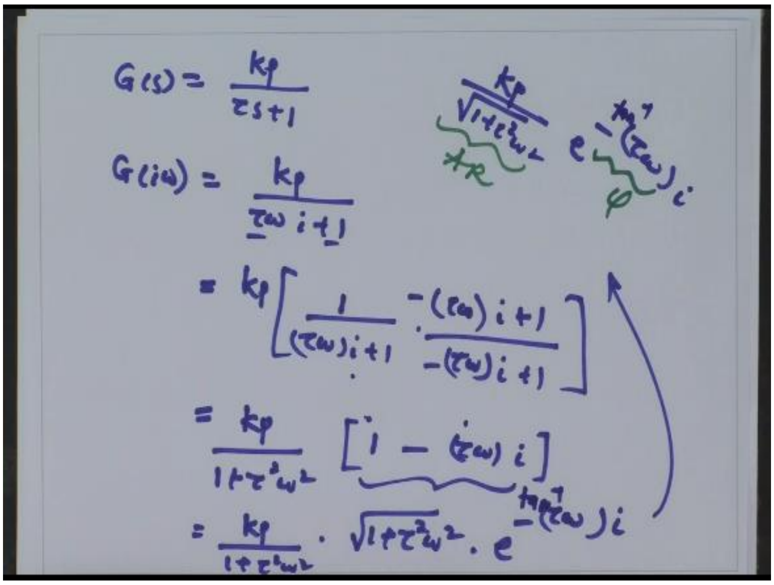
That is the motivation behind using frequency response and going from all the values and all we are interested in is how this AR and phi are functions of omega. Previously for the first-order system, we had seen how do you compute AR and phi. Let us try to find a general method of how do you find AR and phi.



We had seen a very fundamental method of finding this is that you take the transfer function of the process, you take the transfer function of input as ‘A sin (ωt)’ and the Laplace of that. Find the Laplace transform of the output and invert it and it becomes a very laborious process especially if the transfer function or the process has a much bigger or higher order transfer function.

I will now give you a very simple method of finding this AR and phi for any given transfer function. Let us say if we have a transfer function G(s), so this is any transfer function and we want to find out what is an AR and what is the phi and by that I mean what are these functional forms as a function of omega. So what we do is we first substitute s=iω where ‘i’ is the complex number of the root of -1 and ω is any frequency.

So when we do that, we are going to get is $G(i\omega)$ which is a complex number. Any complex number if you recall your complex number mathematics, can be represented in the amplitude angle form as the amplitude of that number times e raised to -angle of that number. So you represent a complex number in the amplitude angle form and then amplitude ratio is the amplitude of this complex number. The phase is the angle of this complex number. Let us now see what do you get if we apply to the first-order system.



If

$$G(s) = \frac{k_p}{\tau s + 1}$$

We substitute $s = i\omega$. We get,

$$G(wi) = \frac{k_p}{\tau\, \omega i + 1}$$

Now we have to represent it into amplitude angle form. We will take out k_p as common multiplying by the complex conjugate of the denominator, we will get-

$$G(wi) = k_p \left[\frac{1}{\tau\, \omega i + 1}\ \frac{-\tau\, \omega i + 1}{-\tau\, \omega i + 1}\right]$$

This is (A+B) (A-B), so we will get,

$$G(wi) = \frac{k_p}{1 + \tau^2 \omega^2}\ (-\tau\, \omega i + 1)$$

Now in order to find the amplitude, I can write it as the root of 1+tau square omega square into e raised to –tau omega i. The final form what you will get is,

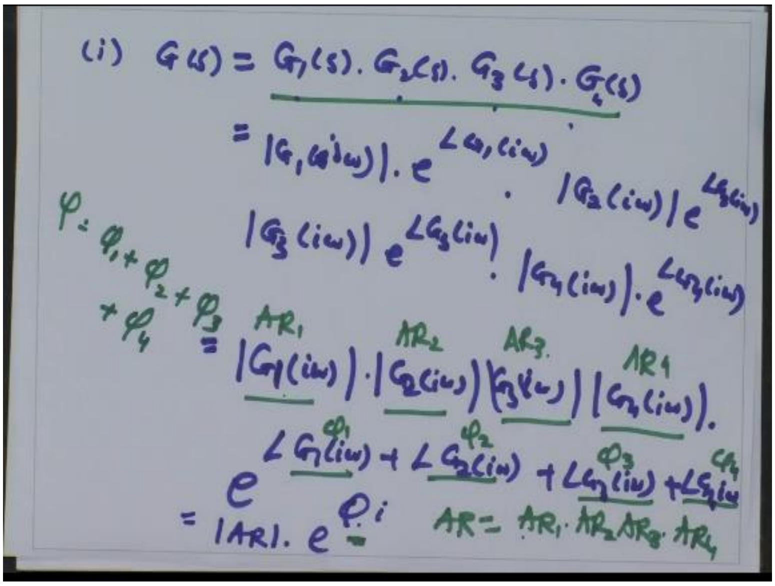
$$G(wi) = \frac{k_p}{\sqrt{1 + \tau^2 \omega^2}}\ e^{-\tan^{-1}(\tau\omega) i}$$

As per the previous definition what we are getting AR as,

$$AR = \frac{k_p}{\sqrt{1+\tau^2\omega^2}}$$

You recreate the same values of AR and phi as we had obtained by using the inverse Laplace method. This method is a very simple method to find out the frequency response or amplitude ratio and phi for any transfer function.

Before going forward let us try to find out the frequency response for some other processes. So we will try to establish a few results.



So one is if I have a process function transfer function as a product of multiple transfer functions which is very common in feedback systems when we have multiple systems transfer functions in series. So what we are going to get is how do we get the amplitude ratio and phi for such a case. Let us represent each of these as the amplitude ratio and phi, so this will be-

$$G(s) = G_1(s)\; G_2(s)\; G_3(s)\; G_4(s)$$

$$G(i\omega) = |G_1(i\omega)|\, e^{\angle(G_1(i\omega))}.|G_2(i\omega)|\, e^{\angle(G_2(i\omega))}.\ |G_3(i\omega)|\, e^{\angle(G_3(i\omega))}.|G_4(i\omega)|\, e^{\angle(G_4(i\omega))}$$

Which on simplification we will get as $G_1(i\omega)$ G_2 $(i\omega)$ G_3 $(i\omega)$ and G_4 $(i\omega)$ times e raised to power all the summations of these angles. So what we are seeing now is this is AR_1, this is AR_2, this is AR_3 and this is AR_4 in above slide and this entire product we are going to write as so this eventually we are trying to say is amplitude ratio times phase.

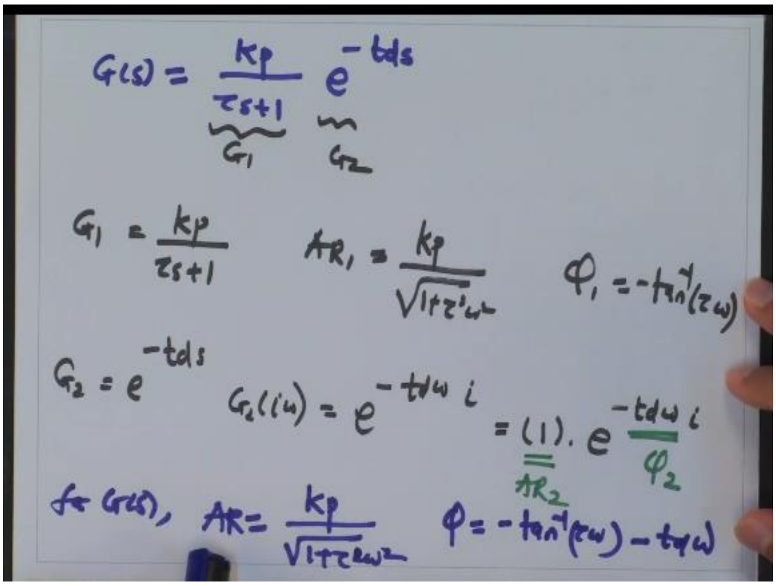
So based on this what we can write is,

$$AR = AR_1 . AR_2 . AR_3 . AR_4$$

If we have transfer functions in series, then the amplitude ratio of the resulting combination is the product of all these amplitude ratios. And when we talk about the phase, this phase phi, these are all phase 1, this is phase 2, this is a phase of 3, this is a phase of 4. A phase is actually equal to the summation of the phases.

It is a very important result that if we have a series combination of multiple transfer functions, then the final amplitude ratio is the product of amplitude ratios and the final phase is the summation of all the phases.

Let us use this for very commonly used transfer function which is first-order plus dead time.



If your transfer function is,

$$G(s) = \frac{k_p}{\tau s + 1} e^{-t_d s}$$

If you recall these are the types of transfer functions which prompted us to use frequency response because Laplace domain analysis was not possible or Laplace domain stability analysis was not possible for such kind of a system and we had to approximate this. Let us see now what happens when we have the frequency domain analysis for this system.

We will call,

$$G_1 = \frac{k_p}{\tau s + 1} \; and \; G_2 = e^{-t_d s}$$

G_1 is our very own friend first-order system, so we have

$$AR_1 = \frac{k_p}{\sqrt{1 + \tau^2 \omega^2}} \; and \; \phi_1 = -\tan^{-1}(\tau\omega)$$

In order to find its AR and phi of G_2, we have to substitute s=iω. So,

$$G_2\,(i\omega) = e^{-t_d(i\omega)}$$

Which fortunately is in the amplitude angle form. So what we have is the amplitude ratio of a delay is 1.

That means if you have a delay, then if this sinusoid is input, you will get the same sinusoid as the output which makes sense. However, the phase is different obviously because we are delaying a sinusoid, so accordingly the phase is going to be different than the input.

For a first-order plus dead time system, here for G(s),

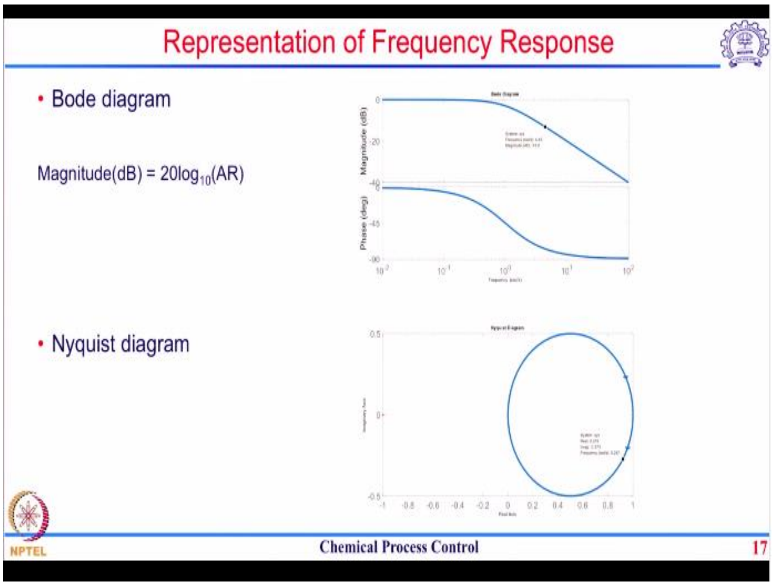
$$AR = AR_1 . AR_2 = \frac{k_p}{\sqrt{1 + \tau^2 \omega^2}} .1 = \frac{k_p}{\sqrt{1 + \tau^2 \omega^2}}$$

The amplitude ratio which is the product of amplitude ratios which we get as same as the first-order system. The phase we get is,

$$\phi = -\tan^{-1}(\tau\omega - t_d\omega)$$

You can see that the dead time is going to affect only the phase of the system and not going to affect the amplitude ratio.

With that groundwork, we will now move on to how do we represent the frequency response.



The frequency response can be represented in two forms. We will see the representation of frequency response and the first is a Bode diagram. The Bode diagram, of so in all these cases we have to represent AR as the function of omega, phi as a function of omega. In Bode diagram, we will have two sets of plots, one will be for AR and one will be for phi and so on one of the axes we plot the log of amplitude ratio versus log of omega. So we will be having a log-log plot between amplitude ratio and omega and I will come to the point why we need a log-log plot.

The second one is normal versus log plot, so it is a phase versus log of omega. We are going to plot two things. The Bode diagram is a log of AR versus log of omega plot and phi versus log of omega plot. Depending on which book you are referring to or which simulation software you are using and there might be different variations of these log-log plots as in it may be a log to the base 10 or it may be your decibel conversion.

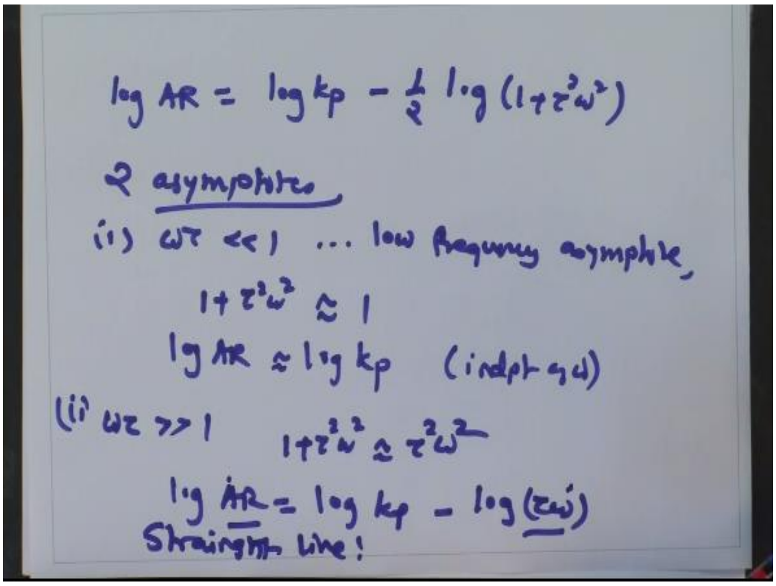
All those things are possible, the idea is it will just shift the curve up and down. Many times you would also have a log of omega tau rather than omega as the x-axis. So these are just different variations, it will just shift the curve either up or down or left or right. But essentially what you are interested in the Bode diagram is how is the log-log variation between omega and AR and normal versus log variation of phase.

Let us see if we have a first-order system.

$$G(s) = \frac{k_p}{\tau s + 1}$$

$$AR = \frac{k_p}{\sqrt{1 + \tau^2 \omega^2}} \; and \; \phi = -\tan^{-1}(\tau\omega)$$

So let us try to see how does the Bode diagram look like here.



If you take the log of AR, we will have,

$$\log AR = \log k_p - \frac{1}{2}\log(1 + \tau^2\omega^2)$$

Now if we take two asymptotes are possible here when omega tau is very much <1, so that is a low frequency asymptote. Then, we will see that $(1 + \tau^2\omega^2)$ be roughly equal to one. So this second term would vanish and we will have a log of AR is roughly equal to the log of k_p. It will be a constant and independent of omega. So it will be a horizontal line in the Bode diagram.

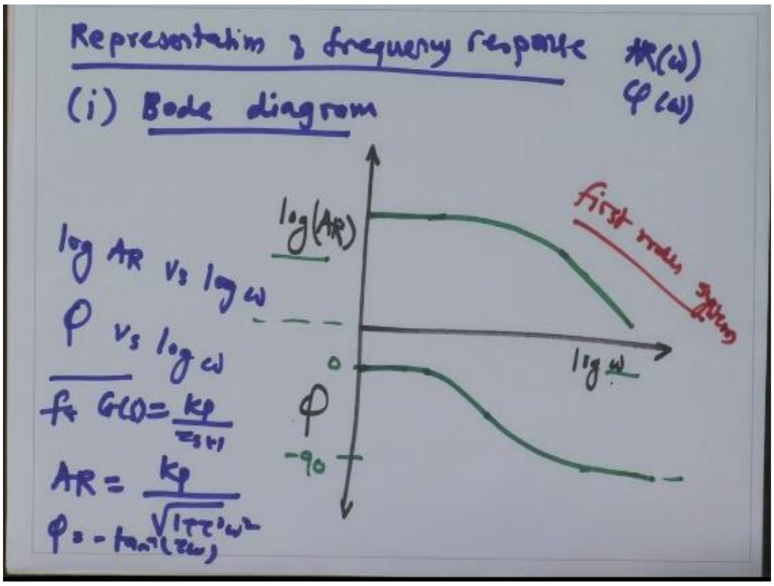
When we consider the other as asymptotes where omega tau is very much >1, in that case, we can approximate,

$$(1 + \tau^2\omega^2) = \tau^2\omega^2$$

We will have,

$$\log AR = \log k_p - \log(\tau\omega)$$

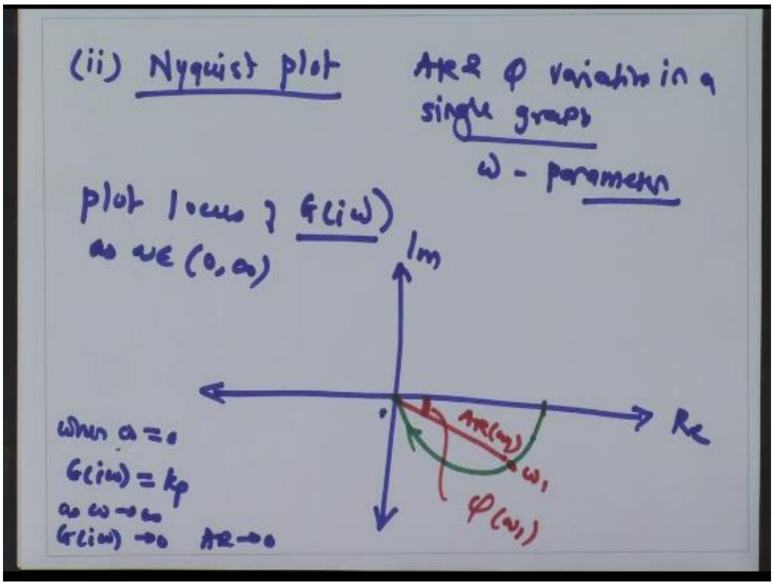
So, it is a straight line in a log-log plot. That is why of in the Bode diagram we have a log-log plot between AR and omega because then this gives you a straight line.



If we go back to a previous figure, what we have seen is at the low frequency we have almost a straight line, horizontal line and at very high frequencies we have a straight line with a slope of -1. Typically for a first-order system the frequency response, the first part of the Bode diagram will look like this and the second part is simply the tan inverse of tau omega. So when omega is=0, we have phase equal to 0.

Here we cannot show the omega=0 is that other negative infinity. So this phase is going to approach 0 as you have very low frequency. So that will be the asymptote and when omega tends to infinity its tan inverse infinity, so what we will have is phase will be equal to -90. So the other extreme is -90. So this response looks like this. So that is how a Bode diagram for a first-order system will look like and it can be constructed for any transfer function.

Now let us look at the second way of representing frequency response and that is known as a Nyquist plot.



Here instead of computing two figures, we are actually going to compute a single to represent AR and phi in a single graph. So in order to do that what we do is we take frequency as a parameter and then we plot locus of the complex number which was used to obtain this frequency response. So we take G of ($i\omega$) as a complex number and we plot this complex number in a complex plane.

This is the complex plane and you simply plot how does this G of ($i\omega$) move into this complex plane as omega goes from 0 to infinity. Let us see how would it look like for a first-order system. For a first-order system when omega is 0, in that case, what we have is AR is equal to so if we see this figure, so when omega is=0 what we are going to get is AR is=k_p and this will be tan inverse 0 so it will be 0.

So we have a complex number which is 1 or k_p. So when omega=0, G of ($i\omega$) is=kp, so it is a number which can be seen here. It will remain here and then when omega goes to infinity, your complex number goes to 0 because AR goes to 0. Eventually this trajectory is going to end at this point; however, for all the values of omega between 0 and infinity, the phase is always going to be negative and the maximum phase is going to be 90. So this will look like, the trajectory will look like this as you increase omega. At any point in this graph, let us say if I am interested in this particular point, it will have a certain frequency and the distance of this point from the origin is the amplitude ratio at that particular location or that frequency. If this is omega 1, this is AR of omega 1 and this particular angle will tell me the phase. So this is the phase at omega 1. So using a single figure we can represent the entire frequency response and this is known as a Nyquist plot.

We will take a short break here and when we come back we will see using this Bode diagram or a Nyquist plot and in general frequency response, how do we assess the stability of a feedback system. Thank you.