Chemical Process Control

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Lecture – 28
Laplace Domain Analysis - Part I

Okay, welcome come back. Before the break, we saw that the relationship between stability analysis and feedback control is very interesting. Feedback control we typically use to have a disturbance rejection and it is our idea that when we implement the control system, it will be able to reject the disturbance or it will allow us the ability to move the system from one point to the other.

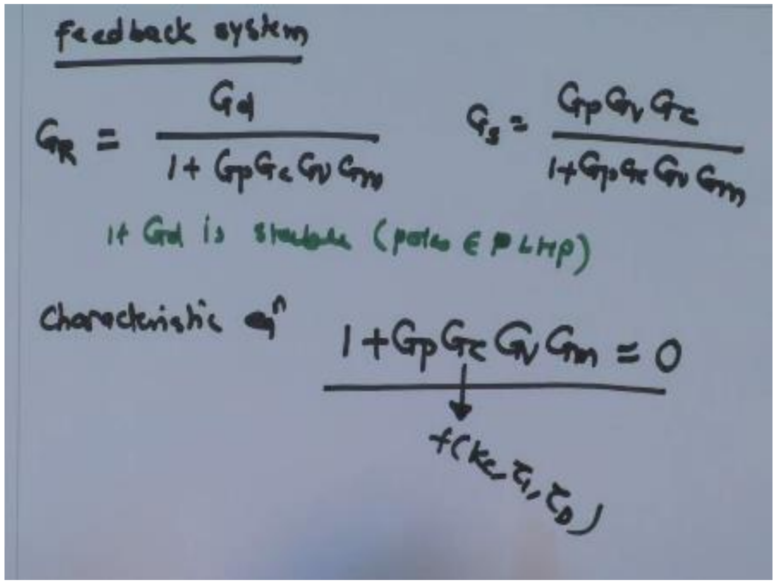
But we saw through the example of 3 CSTR that the feedback control also has a tendency that if you use a wrong value of controller parameter, it can actually destabilize an originally stable system. It is very important that you know what are the bounds up to which you can vary the controller parameters and that will be possible by doing stability analysis of the feedback system. This stability analysis can be done using Laplace domain analysis or frequency domain analysis.

We will start with the Laplace domain analysis because already we have studied the transfer function in the Laplace domain for a feedback system. Let us see how we can use that information to assess the stability of a feedback system. In week 3, we had seen, what do we mean by poles or zeros of a system and that time we had made a comment that if all the poles of the system lie on the left half the plane of the complex plane, then the system remains stable. In the sense that the system has decaying oscillation or it decaying exponential so that the system goes back to the original point or the deviations are limited, the system does not grow out of proportions. As against if the poles are on the right half plane or any of the poles is on the right half plane, the system either has growing oscillations or a growing exponential and eventually, the system will go towards instability.

In terms of stability, if we are doing Laplace domain stability analysis, the key feature which we are going to use is if all poles of the system, it may be an open system or a closed loop system, are in left half plane, the system is stable or an improved definition can be that if any pole of the system is in right half plane, the system is unstable. Because this particular statement does not include if the poles are on the imaginary axis in which case the system has sustained oscillation. And when the system has sustained oscillation, the system kind of remains marginally stable because the system oscillates around a steady state. It is not going too far away from the steady state because you can always say that the bound is the same as the amplitude of the oscillation and because of that these systems are stable. If you include the imaginary axis, the system becomes marginally stable except, if the pole is at the origin, which is an integrator and an unstable process. So, the more correct definition would be if any pole of the system is in the right half plane, the system is unstable.

All we are interested in terms of assessing the stability analysis, doing stability analysis in the Laplace domain is to find the pole and see if all the poles or any of the poles lie on the right half plane. If you recall how do we get poles of the system; so poles are the solution of an equation, D(s) = 0, this also known as the characteristic equation.

We write the transfer function of the system, so if G(s) is the transfer function, we write it as N(s) over D(s) and we take the denominator polynomial = 0 and the roots of that polynomial will tell us the poles and depending on the pole, we will know what is whether the system is stable or unstable. Same logic we will be using in order to assess the stability of a feedback system. So, when we were talking about a feedback system, we have already derived these transfer functions in servo or regulating mode.



If we are talking about the feedback system, then in the regulatory mode, the transfer function we had derived was,

$$G_R(s) = \frac{G_d}{1 + G_p G_c G_v G_m}$$

In the servo mode, it is-

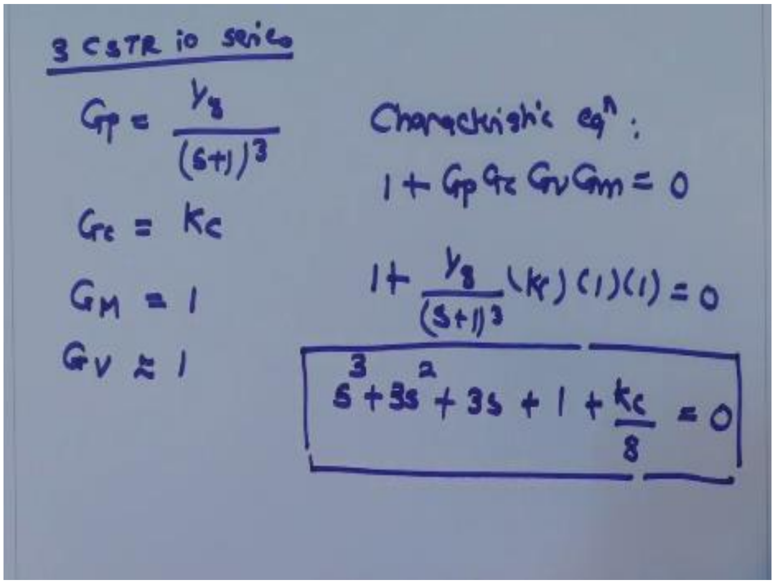
$$G_s(s) = \frac{G_p G_v G_c}{1 + G_p G_c G_v G_m}$$

Here we will at least start with the fact that G_d is stable or G_p is inherently stable. If that is the case, we just need to make an assumption that this disturbance transfer function is stable, so it is not going to add any unstable pole into the characteristic equation.

If that is the case, so G_d is stable that is poles are in the left half plane. Then this transfer function is already in N(s) over D(s) form and the characteristic equation becomes,

$$1 + G_p G_c G_v G_m = 0$$

which is the same for regulatory or servo. This is the equation which we have to monitor which also has G_c in it which will be your function of controller parameters. Using this equation, we would be able to find out whether the system is stable or unstable.



Let us now try to find out the characteristic equation for the three CSTR system, which we had just seen. So for the 3 CSTR system, we had the process transfer function as,

$$G_p = \frac{\frac{1}{8}}{(s+1)^3}$$

We have used a proportional controller, so $G_c = k_c$ and we can assume that the measurement dynamics are instantaneous, they are reliable and we can also consider the valve transfer function to be 1.

In that case, the characteristic equation becomes 1 + Gp Gc Gv Gm = 0. It means,

$$1 + \frac{\frac{1}{8}}{(s+1)^3}\, k_c\,(1)(1) = 0$$

which you can simplify and get as,

$$s^3 + 3s^2 + 3s + 1 + \frac{k_c}{8} = 0$$

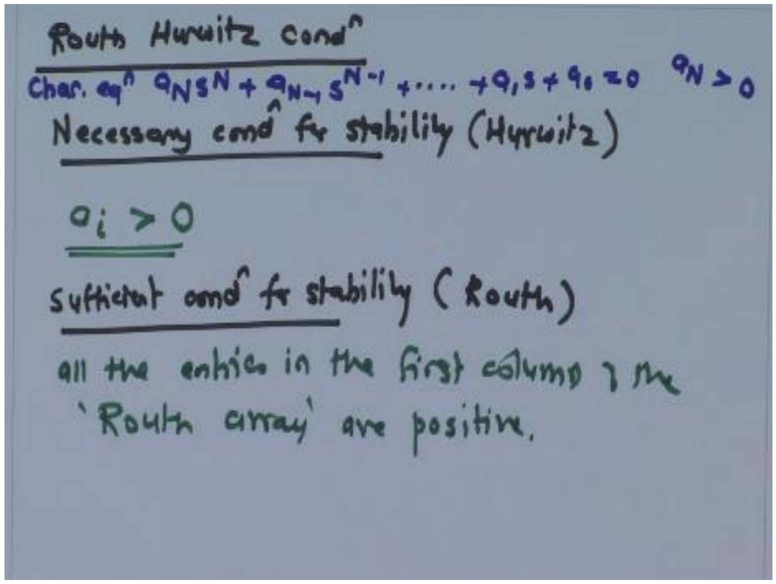
So this is our characteristic equation. You can see that the characteristic equation includes this controller parameter. So, the poles are a function of the controller gain. That is how the controller gain has an effect on the stability or instability of the system.

So, all the poles or some of the poles of the system would depend on the controller gain. Now here we see a point that we have reached a cubic equation which will have three poles and we will have to solve this analytically for different values of kc in order to assess whether that particular kc value is stable or unstable. So the Laplace domain analysis can become tricky and numerically time consuming.

In order to simplify this, especially the techniques were developed almost about a century ago when the digital computers were not there to assist in terms of finding a numerical solution or in general, all the analytical solutions of this kind of higher order polynomial systems. The researches at that time came up with simplified methods in order to assess the stability or what we are interested in this particular system is that all we are interested in to know that the poles of the system do not lie in the right half plane.

As long as we can ascertain that irrespective of the value of the pole of the systems, we can tell whether the system is stable or unstable. So, in summary, we are just interested in finding out what is the typical range or whether the poles lie on the right half plane or the left half plane. In order to do that, there is a simple method which was; is a simple condition; set of conditions which were developed about a century ago by two scientists known as Routh; by the name Routh and Hurwitz.

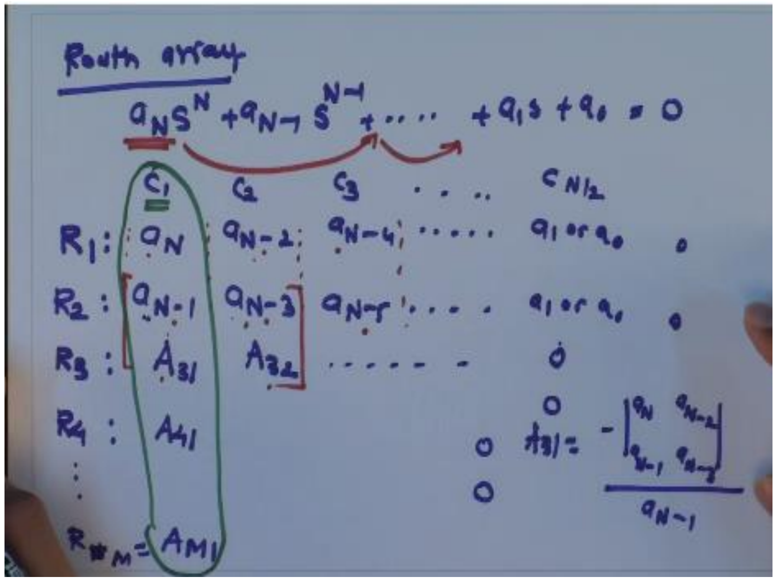
We will see now what is that method and how it can be applicable here. That avoids us that eliminate the need to solve these polynomial systems in order to assess stability.



These are 2 conditions which will help us in terms of deciding whether this closed-loop system is going to be stable or unstable. The first condition is a necessary condition for stability, so this has to be satisfied and this was given by Hurwitz and these conditions are defined for a characteristic equation of the form; $a_N s^N$, so N^{th} order polynomial in s, where $a_N > 0$ or a_N is positive.

For a characteristic equation of this particular form, for the system to be stable, the necessary condition is that all these a_i's are positive. So all this a_1, a_2 all the way up to a_N, all this should be positive then only the system can be stable. If any of those are negative, the system will be unstable and if any of those are 0, then this condition; this particular analysis; the Routh-Hurwitz criteria cannot be used in that particular system. The system may be stable, the system may be unstable.

Then let us say if all these coefficients are positive which most of the times would be the case then, we need a sufficient condition for stability which was given by Routh and that condition is; all the entries in the first column of the so called Routh array are positive. So, in order to apply this sufficient condition for stability, we have to compute something known as a Routh array.



We will see how to construct a Routh array for any characteristic equation of this form. We have $a_N s^N$, so the Routh array, the first row of the Routh array will be designated by R_1, it will include all the alternate coefficients of this polynomial equation in s, starting with the highest power.
So we will start with a_N, so the first row will include a_N and then all the alternate entries from that. So after a_N, it will be the entry with s^{N-2}, then s^{N-4}, all the way up to whether it will be a_N, a_1 or a_0. In the second row, will be the remaining coefficients, so we will start with a_{N-1}, a_{N-3}, a_{N-5}, all the way up to a_1 or a_0 depending on N is even or odd. So all the coefficients inside this characteristic equation are represented in the two rows by taking the alternate coefficients.

And depending on the order of this particular characteristic equation, we will have multiple columns, this is column 1, 2, 3 all the way up to let us say N - N/2 columns. And if you go back to the condition in of Routh, it says all the entries in the first column of the Routh array are positive. So even though we are constructing this entire matrix of multiple rows and columns, in the end, we are going to be interested only in this column 1.

So, we have populated the two rows of this Routh array by using the coefficients of the characteristic equation. Let us now see how to we populate the remaining entries of this Routh array. So this first entry of this R_3, let us call it as A_{31} and this A_{31} can be calculated as,

$$A_{31} = -\frac{\begin{vmatrix} a_N & a_{N-2} \\ a_{N-1} & a_{N-3} \end{vmatrix}}{a_{N-1}}$$

So, it is a_{N-1} time a_{N-2} – a_N times a_{N-3}. So it is actually negative of this determinant divided by this entry a_{N-1}. So that is how we will define A_{31}. Then, we look at A_{32}, it will be a similar procedure but now, the matrix we will be focusing on will be this, so it will be

$$A_{31} = -\frac{\begin{vmatrix} a_{N-2} & a_{N-4} \\ a_{N-3} & a_{N-5} \end{vmatrix}}{a_{N-3}}$$

now a_{N-3} into a_{N-4} – a_{N-2} a_{N-5} divided by a_{N-3}. So that is how all this third row will be populated and wherever let us say we reach this particular point in order to form this determinant, we will have to consider these other entries to be 0, so automatically this entry will also become 0.

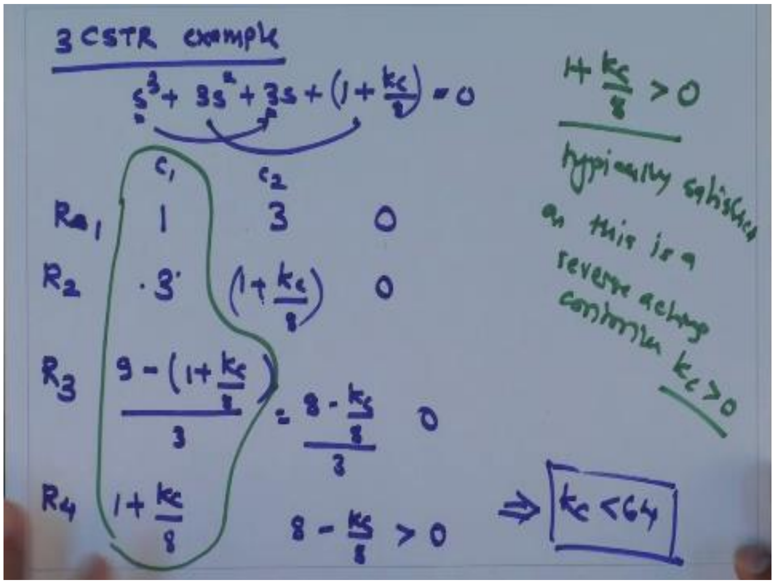
So, every time we finished 2 rows, we will be reducing the number of columns which has nonzero entries. Then we look at row 4 and all these subsequent calculations are very similar. Now when we are calculating the first entry of R_4, let us call it A_{41} that will be given by the corresponding negative determinant of this, so it will be,

$$A_{41} = -\frac{\begin{vmatrix} a_{N-1} & a_{N-3} \\ A_{31} & A_{32} \end{vmatrix}}{A_{31}}$$

A_{31} times this a_{N-3}, A_{32} times a_{N-1} divided by A_{31}. This calculation will be repeated subsequently and we will keep on reducing the number of 0's every 2 rows, one of the columns will become 0 and all the way we will return some value as R_M will be A_{M1}. After that, all the entries below will become 0. So that way progressively, we can calculate the Routh array and in order to apply this condition of stability, what we will be doing is; we will be focusing on this particular first column.

And if all the entries in this first column are positive, then the full system is stable. If any of those entries is 0, then this particular method fails or it is not able to assess whether the system is going to be stable or unstable. And if any of those entries is negative, then the system; the closed-loop system will be unstable because one of the poles will be on the right half plane.

So, let us now try to apply it to the three CSTR example.



For which the characteristic equation was,

$$s^3 + 3s^2 + 3s + 1 + \frac{k_c}{8} = 0$$

The first row of Routh array will be this coefficient which is 1 and then the alternate one which is 3 and there is nothing else after that. In the second row will be starting with this 3 and then this $1 + \frac{k_c}{8}$. This is column 1, this is column 2.

Now before proceeding, we did not look at what is the Hurwitz condition. So, for this particular system, the Hurwitz condition faced that all the coefficients should be > 0 which means $(1 + \frac{k_c}{8})$ should be > 0 and which is typically satisfied as this is a reverse acting controller. So kc> 0 is typically the value which we select, so automatically $(1 + \frac{k_c}{8})$ would always be >0.

Now, let us look at how to we populate the remaining rows, so let us say row 3 will be 3 times 3 which is 9 –$(1 + \frac{k_c}{8})$ divided by 3 which comes out to be (8 – kc/8) /3.

And then the other entry will be just 0. And when we go to row 4, it will be this particular matrix; 3 $(1+\frac{k_c}{8})$; (8 – kc/8)/ 3 and 0, so that last entry will become $(1+\frac{k_c}{8})$. So, based on the Rouths array, the stability conditions say that all the entries in this column should be > 0, so this is automatically satisfied by the Hurwitz condition. So the only additional condition which we are getting for this system is 8 – (kc/8) should be > 0, which means kc should be < 64.

So, the stability condition on this particular system is that the controller gain, if I am using a proportional controller should be < 64. If you recall or if you go back in the video and previous video and see the plots for which I had shown you the responses, one was kc = 50 where we had decaying oscillations and one of the plots was for kc = 70 where we had the growing oscillation which was unstable. If you plot it for kc = 64 actually, you will find that it is the stability limit at which you get the marginal stability.

So, we will take a short break here and when we come back, we will look at another method of finding the stability limit other than using this Routh Hurwitz criterion.