Chemical Process Control
Prof. Sujit S. Jogwar
Department of Chemical Engineering
Indian Institute of Technology - Bombay

Lecture – 27
Stability of Dynamical Processes

Hello students, this week, we will be focussing on stability analysis of feedback systems. Last week we looked at what are the different types of simple feedback controllers like P, PI or PID controllers and how is their performance in terms of disturbance rejection or set-point tracking. In each of those cases what we saw was depending on the parameters of the control system like the controller gain, integral time constant or a derivative time constant, the response changes.

We also looked at a proportional controller, for example, we saw that there is always an offset when you use proportional control and that offset keeps on reducing as we increase the controller gain and in the mathematical limit of that controller gain becoming infinity that offset goes away. However, in reality, it is not always the case; you cannot keep on increasing the controller gain to infinity.

So, there is always a practical limit in terms of gain. Even though, mathematically the maximum controller gain which can be achieved is infinity many times that is not possible because of stability criteria. Sometimes, the system can become unstable even for finite values of controller gain, and that is why it is essential to know up to what limit these controller parameters can be varied.

So, in this, we will use stability analysis to compute these limits whether they may be the minimum controller gain required or the maximum controller gain which is possible in a feedback control system. So, this week will see what do we mean by stability and what is its relationship or why do we need to study it in the context of feedback control. Then we will see two different sets of techniques to assess the stability of feedback systems, one will be using the Laplace domain analysis and the other one will be using the frequency domain analysis.

So, let us get started with what do we mean by stability. We will take a few examples to assess or to get ourselves familiar with what do we mean by stability, what is instability and so on. Let us take an example of a pendulum, so we had looked at that example right in the first week as well, so let us say this is a pendulum and in this pendulum, it can have two steady states.

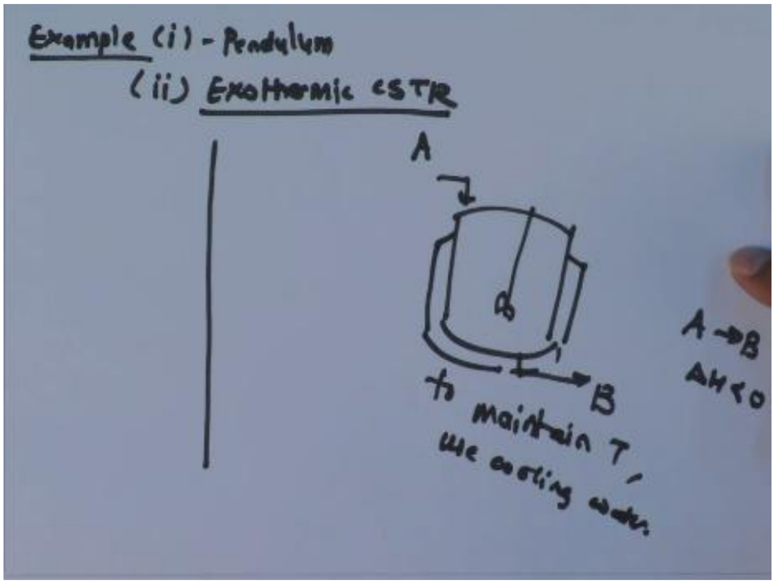
So, by steady state I mean, when it is not subjected to any other disturbance, the free position which it can achieve. So this is one of this steady state for a pendulum, and then the other steady state for this pendulum is exactly opposite of that which is a vertical position for the pendulum. Now, let us consider that we are assessing the stability of this particular point and we gave a small disturbance like this.

We will notice that eventually, the pendulum comes back to the original steady states. This type of behavior we call it as a stable behavior. This particular steady state can be called as the stable steady state, but the same cannot be said about the other steady state of this pendulum. Let us say this is my starting point, and I give a small nudge to it, then it will never remain or will never go back to the original steady state.

But it moves to the other which is the stable steady state because of that we will call this particular steady state as an unstable steady state. Now note that this is still a steady state if there is no disturbance, the pendulum will remain like this forever, but only when there is a small disturbance, it will deviate away from the steady-state point. So we have seen that this pendulum has two steady states; one is stable, and the other is unstable.

And all that we characterized based on a disturbance that if the system remains close to the steady state, it is a stable steady state and if it deviates from that steady state, we call that as an unstable steady state. Let us now consider another example of;

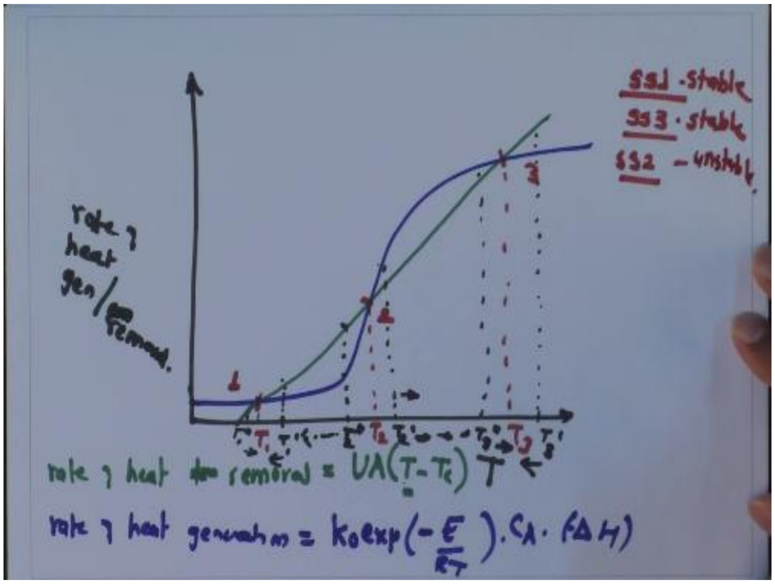
(Refer to Slide Time: 04:54)



So, we have considered one example of the pendulum, and we will consider the second example of exothermic CSTR. You might have discussed this when you had studied any reaction engineering course because this is a very peculiar system which is typically studied extensively in a reaction engineering.

We are looking at a CSTR, which is going to convert the raw material A into product B using a reaction of the form A going to B. This is an exothermic reaction, so Δ is typically a negative signifies that it is an exothermic reaction, it is going to generate heat, and in order to maintain the temperature, we need to remove this heat by using cooling water, so that cooling water is circulated in a jacket around the reactor. Now, a CSTR operates at a constant temperature, so we will see at what are the different permissible values of temperature at which this CSTR can operate. That can be determined by plotting how the rate of generation of heat changes as the function of temperature and the same way, the rate of the removal of heat.

(Refer to Slide Time: 06:36)



How does that change as a function of temperature, generation or removal? The rate of heat generation is given by the rate of reaction which will be $k_0\ e^{-E/RT}$ that is the Arrhenius form times C_A. It is the rate of reaction multiplied by ΔH, so that is the rate of heat generation, and you can see that the effect of temperature is reciprocal of inverse exponential and as it turns out, if I plot this rate of heat generation as a function of temperature, the curve which we typically get is of this form.

Now, let us look at the rate of heat removal which is given by UA(T – Tc), where T is the temperature of the reactor and Tc is the temperature of the coolant. So you can see that it is very much linear in terms of the reactor temperature. For this CSTR to operate at a constant temperature, the rate of removal of heat should be equal to the rate of generation of heat.

So, there are 3 points at which this CSTR can operate, so let us say at this point, these 2 rates are equal, and this is the third, so there are three steady state temperatures at which this CSTR can operate. We will look at now whether these are stable or unstable.

So, let us start with steady state 1, so we are looking at this steady state. Let us consider that there is some disturbance in the system because of which the reactor temperature increases. Let me just show this temperature as well, now let us consider that because of the disturbance, this temperature increases some new value, let us say this is T_1', so now, when the temperature increases we can see that the rate of removal of heat is now more than the rate of generation of heat, so there is a net deficit of energy or net deficit of heat in this reactor which will cause the reactor temperature to decrease.

So, eventually, the temperature will again come back to T1. Same logic we can apply, if the temperature of the disturbance had caused the temperature to reduce, then the rate of heat removal is less than the rate of heat generation, so there is a surplus of heat in the system, and again it will move the temperature towards T_1. So any disturbance whether the positive temperature disturbance or a negative temperature disturbance.

Both these disturbances eventually gets automatically counted. The steady-state temperature remains the same as $T_{1,}$ or the system retains to the steady state temperature T_1, same thing can be analyzed for T_3, steady state 3 as well, so we can say that steady state 1 is stable, let us look at steady state 3 and say that if there was; if the disturbance caused the temperature to increase, to T_3'.

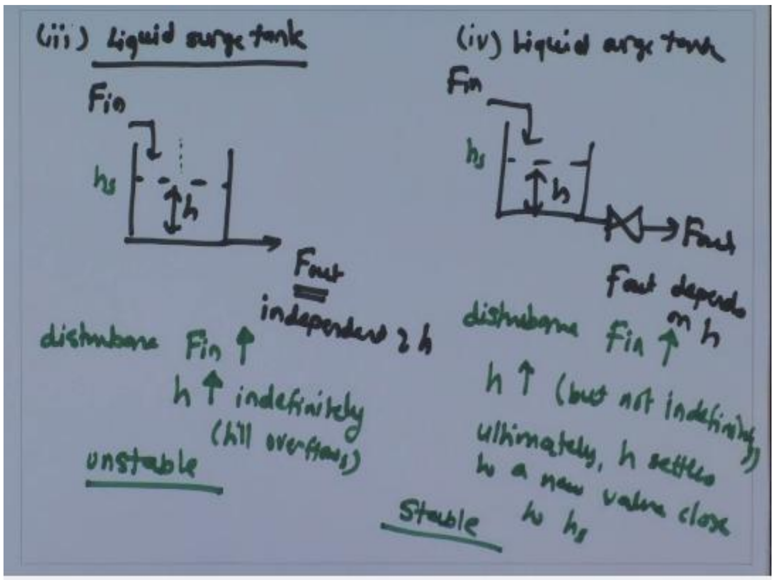
So at that point, the rate of removal of heat is greater than the rate of heat generation. Again there is a net deficit of heat into the system, the temperature will drop to T_3, and the same thing is true, if the disturbance had caused the temperature to go down which case again it would increase the temperature, so both these steady states are stable because the disturbance is not able to move the system away from the steady state.

So, let us try to analyze whether steady state 2 is also stable or not because that is the steady state which is typical of interest. So, let us say at T_2, the disturbance causes the temperature to go to T_2', so when we moved to T_2', we notice that the rate of heat generation is more than the rate of heat removal, so there is a surplus of energy or heat into the system, so that it will cause the temperature to increase.

So, you will see that increase in the temperature caused by the disturbance results in further increase into the temperature and eventually, that temperature will stabilize at T_3. If if the disturbance had reduced the temperature of the system what you can again count; find out that the rate of removal of heat is more than the rate of heat generation, so there is a net deficit of heat, and the temperature will keep on further falling down.

And eventually, it will go back to T_1, so the steady state 2 is unstable because when a disturbance affects the system, the system can no longer remain at the steady state and it moves to some different steady state.

(Refer to Slide Time: 13:00)



Now, let us close this discussion with a familiar example of the liquid surge tank. In liquid surge tank let us consider a case when the outlet flow rate can be independently manipulated, or it is independent of the height inside the tank. This is possible if there is some kind of a pump which is taking out this much flow so that flow will not depend on the height inside the tank.

Now, in this case let us consider a disturbance that F_{in} increases. Now as F_{out} is independent of height, F_{out} will remain the same as the earlier value, so there is net accumulation of material inside this tank. So height will keep on increasing indefinitely till the tank overflows. Because of this disturbance, the system no longer remains at its original steady state which was let us say of height h_s. The height keeps on increasing and eventually, the system leads to instability and it goes to overflow, so this is an unstable system. The same thing can be proved if F_{in} had decreased and then, in that case, the height will continuously decrease resulting in the dry out of the tank.

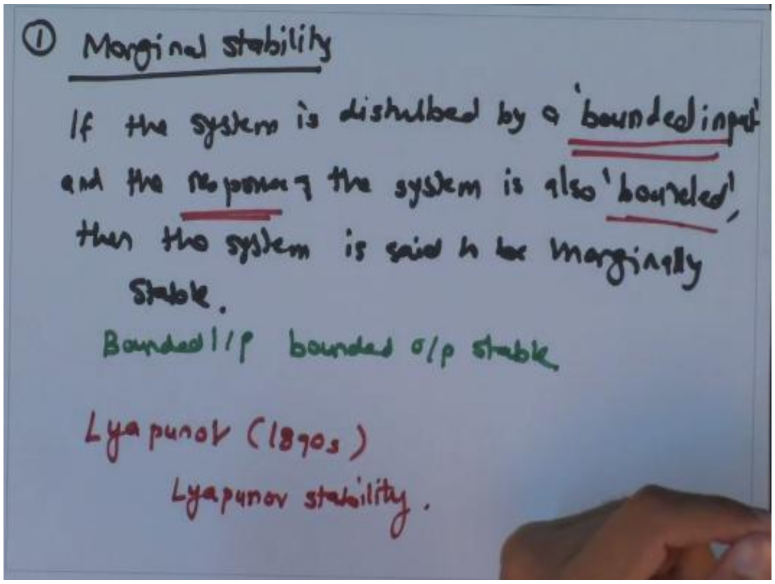
Now, let us consider a variant of this process. Same liquid surge tank, but now this F_{out} depends on h. It may be a linear dependence if the flow is laminar, or it may be square root dependence if the flow is turbulent. Let us say F_{out} depends on *h* and we look at the same disturbance scenario here that F_{in} increases, now as F_{in} increases initially, F_{out} is same as the steady state value, the height will increase or height will tend to increase, as height increases and F_{out} is proportional to the height, the F_{out} will also start increasing.

Eventually, there will come a time when the height inside the tank is sufficient enough that the new F_{out} which is proportional to this that new height will be equal to F_{in}, so the increase in the height, so there will be increase in height but not indefinitely and ultimately, height settles to a new value close to h_s. So what we have seen that when the disturbance is affecting this particular system, the system is not really unstable that it is indefinitely moving to the boundary of the system.

The height does indeed move from the original steady state, but it remains close enough to the original steady state, so this kind of system will still be stable. We will see why this kind of stability is different from the stability which we saw for the pendulum example or a CSTR example where the system had returned to the steady state. So, to do that let us now look at the formal definitions of stabilities.

So, the stability is defined in 2 broad terms; one is the weaker definition of stability, and other is the stronger definition of stability, so let us start with the weaker definition of stability which is also known as marginal stability.

(Refer to Slide Time: 17:46)



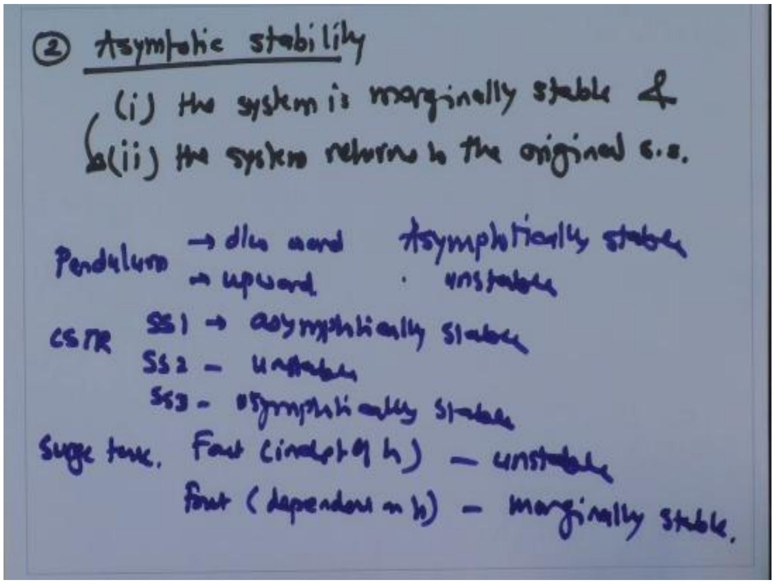
Marginal stability:- if the system is disturbed by a 'bounded' input like for example, a step input (that we have made a finite change into the input) and the response of the system is also 'bounded' then, the system is said to be marginally stable. The key points here in this definition are that you are giving a bounded input and if the output of the system or the response or output of the system is also bounded then the system is called to be a marginally stable system.

It is also known as bounded input bounded output stable. All it says that if I give any bounded input and the system remains close to the steady state, then that type of system is known as the marginally stable system. The liquid surge tank which had the outlet flow proportional to the height of the tank is a marginally stable system. If you look at the steady state of the pendulum which was like this as well as the CSTR steady state 1 and 3, in all these 3 cases, the output remains or came back to the original steady state.

That means it was still bounded or it was close to the original steady state whether closeness being = 0, so the marginal stability also present in those type of systems. So, this particular definition of stability was given by a scientist Lyapunov back in the 1890s when he was studying the stability of planetary orbits. You can see that the connection between the theory which was developed to study the orbits of different celestial bodies. We will be using this theory to assess the stability of Chemical Engineering systems.

So, he was the guy who is responsible for defining this type of stability notion. This is also known as Lyapunov stability, and this is the weaker definition of stability, there is also a stronger definition of stability known as asymptotic stability.

(Refer to Slide Time: 21:30)



So, what do we mean by asymptotic stability? This is the stronger definition of stability, so it says that if the input is bounded, the response is also bounded, and in addition to that the system returns to the original steady state. So this is a stronger definition of stability because of the second criteria, it says that not only the output remains bounded, but it goes back to the original steady state.

That means, the system automatically rejects the effect of disturbance and goes back to the original steady state. For the pendulum, the downward position as well as CSTR steady state 1 and 3, all these 3 are asymptotically stable steady states. So when you have asymptotic stability, it automatically means this system is marginally stable, so if we look at all the examples which we have seen, downward pendulum position; steady state is asymptotically stable, the upward position is unstable.

Then we look at CSTR steady state 1 is asymptotically stable, SS2 was unstable, SS3 was again asymptotically stable. Then we looked at the surge tank where F_{out} independent of height, it was unstable, and when F_{out} is dependent on height, it is marginally stable. We will now be using these definitions whenever we are trying to assess the stability.

So, let us now see why we need to care about stability analysis and what is its relationship with the feedback control? Right in the first week when we were looking at what are the functions of a feedback system, one of the important function of a feedback system was to stabilize an unstable process, so feedback control can stabilize any unstable process, like the unstable process which we have studied.

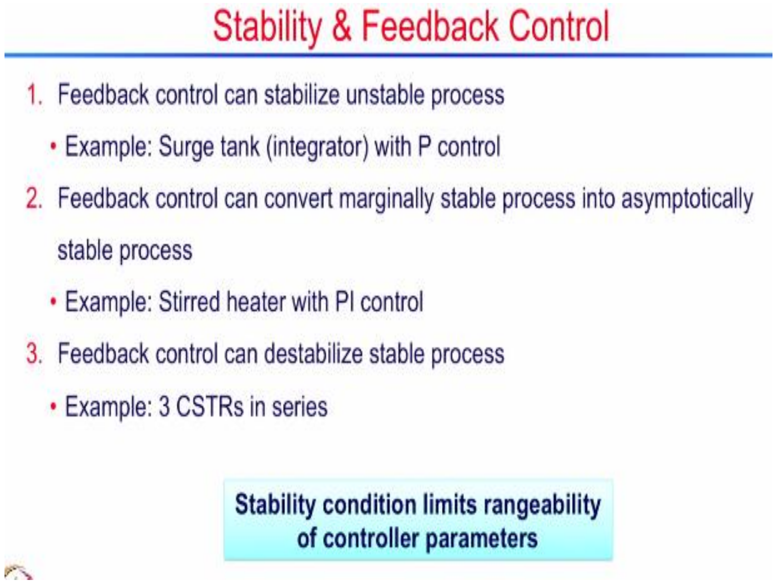
So, we had seen that this particular liquid surge tank system is unstable and if we simply have a feedback control where we are controlling this outlet flow by using let us say P control, PI or PID control, then any disturbance in F_{in} can be regulated, and the system can be made stable. If you use the simple P controller then, the system will become marginally stable, if we use the PI type of control or PID control where the offset can become 0, we can stabilize it at the original steady state, so the system can be made asymptotically stable.

So, one of the important characteristic to study stability is if we want to stabilize any unstable process like an integrator then a stability analysis will tell us what the minimum amount of feedback which is required. It will tell us the minimum value of controller parameters which are essential to stabilize that process then, stability; the feedback system can also convert a marginally stable system into an asymptotically stable system.

Again, as we look at the surge tank example, this particular system is marginally stable because we do not have any control on that. But let us say if I change this valve opening using a PI controller where it opens or closes the valves depending on deviation of this height from the steady state by using a PI control or PID control, I can maintain this height irrespective of the disturbance, so I can convert this marginally stable process into an asymptotically stable process.

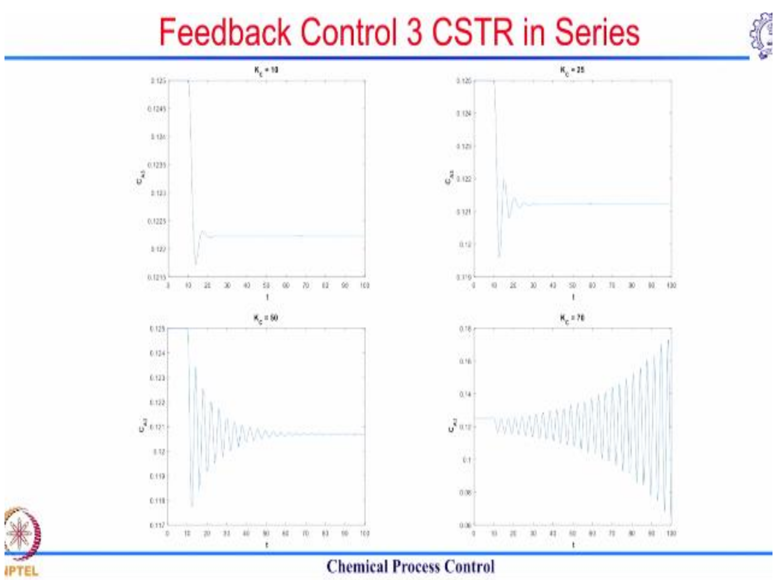
And lastly, one of the most important area, why we should be studying stability is that other than these advantages of converting an unstable process into stable one or marginally stable into asymptotically stable, feedback control can also destabilize a system.

(Refer to Slide Time: 27:18)

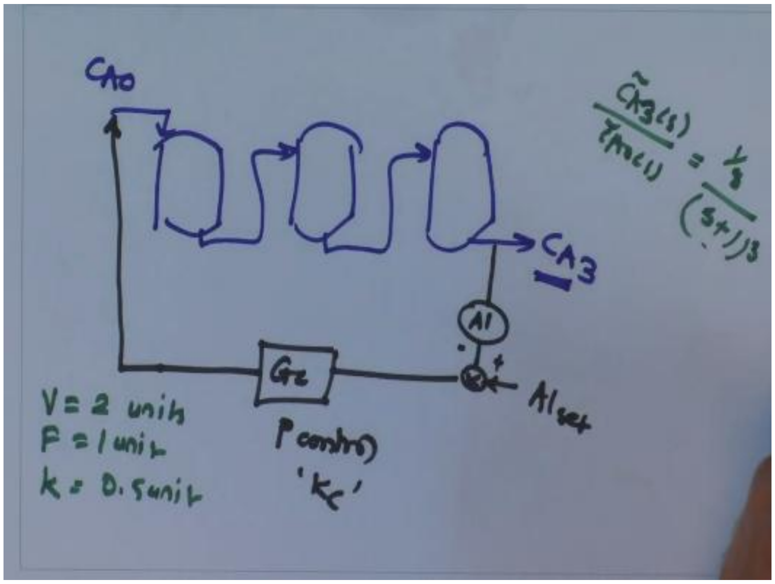


So, here I have shown a simple plot of 3 CSTR's in series.

(Refer to Slide Time: 27:22)



(Refer to Slide Time: 27:30)



Let us consider that, we have 3 CSTRs in series, where this is the input, and this is output, and to control this outlet concentration, we have a feedback system. So we will measure this concentration, then there is some set-point value, we have a controller, and it is going to change this composition and let us say we are using a P control which has a controller gain as the input.

Now, without any controller, this is the stable process because if I have a finite change in the inlet concentration, the outlet concentration will also change by a finite amount. The corresponding transfer function let us say if I take V = 2 units, F = 1 unit and the rate constant has 0.5 unit in corresponding units then, the transfer function $C_{A3}(s)/ C_{A0}(s)$ has come out to be $(1/8)/(s + 1)^3$, so it is a stable process, all the poles are at -1.

So, we know that this open loop system is stable. Now let us say I introduce a P control so that I want to control this outlet concentration in the presence of any disturbance. Here I will show you the plots when we want to change this C_{A3} to a new value and let us see how the controller performs. When I use a small value of controller gain let us say like $k_c = 10$, you can see that the system moves towards the new steady state without much of oscillation and this response is stable.

If I increase this controller gain let us say 25, the oscillations increase, and the offset also decreases. If we increase it to 50, you can see that the oscillations have increased tremendously and all we are doing is increasing the controller gain because we want to reduce the offset. Lastly, let us say if I use a controller gain of 70 an interesting thing happens, you can see that the system no longer remain stable, so all these oscillations rather than being decaying they are actually increasing the oscillations and the system is going to move towards instability.

So we here, what I want to convey to you is that there is always some limit in terms of the controller parameters which you can use which is clearly highlighted from this example. If I keep on increasing the gain, after some point the system just becomes unstable which is the very worse situation which as a process engineer, you can land yourself in, that the original process was stable.

Because you added a controller on that, you have made it unstable. The system would have performed better in the absence of a controller. So in summary what it tells me that stability limit is very important because it will kind of put a limit on how much is the rangeability of the controller parameters. We can take a small break here, and after we come back, we will see how is this relationship between the stability and feedback control parameters.

And we will see how this can be assessed and how can we compute this bounds on the controller parameters, thank you.