Chemical Process Control
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Lecture - 15
Response to Step Input

So now we will look at how a second order system reacts to a simple disturbance like a step input and that will also help us classify these second order dynamical systems and we will see why that classification comes about and how it is related to the parameters of a second order dynamical system.

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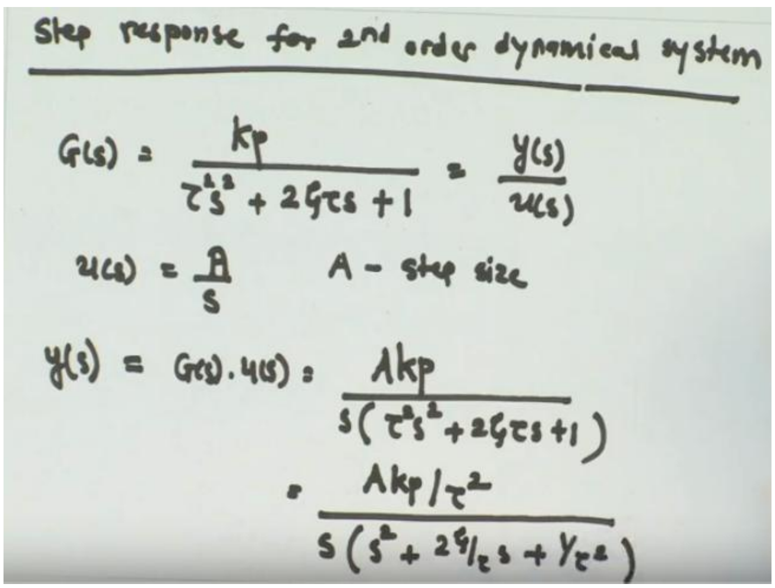

So we will now look at step response for second order dynamical system. So we will work with a general transfer function of a second order system which is $k_p/(\tau^2 s^2 + 2 \xi \tau s + 1)$ which is equal to y(s) / u(s) and we are looking at step response. So the transfer function of the input is going to be A/s where A is the step size. So you can write that the Laplace of output y is going to be G(s)* u(s) which will be $(A*k_p)/(s*(\tau^2 s^2 + 2 \xi \tau s + 1))$.

And in order to fraction this denominator we will also divide everywhere numerator and denominator by τ square. So we have $(A*k_p/ \tau^2) / (s*(s^2+ 2 \xi/\tau s + 1/ \tau^2))$. Now in order to get the response of y(t) from this Laplace transform we will be using the method of partial fractions.

So for that we will have to decompose this term into its factors. So to do that what we would be doing is we will be writing this as

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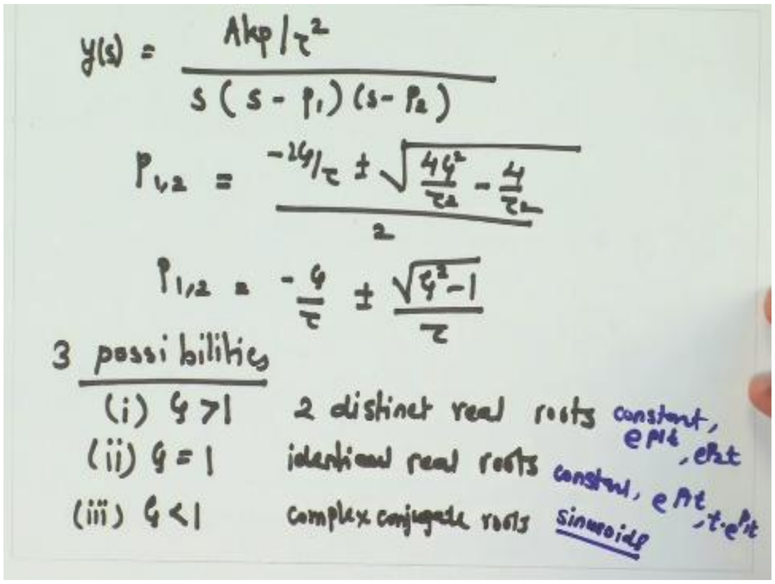

$Y(s) = (A*k_p/\tau^2) / [s (s – P_1) (s – P_2)]$ and this $P_{1,2}$ will be given by the factors of this particular polynomial which would be $(– \xi/\tau)$. So we will be writing (–b plus minus root $(\beta^2– 4ac)$)/2. So the 2, $P_{1,2}$ can be given by $(– \xi/\tau)$ plus minus root (ξ square – 1)/τ. So now you can see that we can have 3 possibilities here.

Possibility 1 is if ξ is going to be greater than 1, in that case what we will have is this term will be less than the first term and in that case we will have two distinct roots which will be real. So the partial fraction would be 1/s sorry the partial fraction would be $A/s + B/(s – P_1) + C/(s – P_2)$. So there will be 3 different, one constant term, one e raised to P_1 and other e raised to P_2 terms.

When we look at $\xi = 1$ however, this term will go away and both the roots are identical. So in that case we cannot directly write as $A/s + B/ P_1 + C/ P_2$. We will have to use the square term as well and in that case the response would have, a constant e^{P1*t} and e^{P2*t}. In this case what we will have is a constant, e^{P1*t} and e^{P2*t}. So these will be sort of different types of dynamics, slightly different dynamics.

The last case is if ξ is less than 1, in that case this will be square root of a negative term. So what we will have is a complex conjugate root. So this is going to give rise to sinusoids.

So depending on the value of ξ, we may have simple constant and exponents. We may have constant, exponent and some time multiplied by exponent or we can even have sinusoids. So this damping coefficient essentially gives different sorts of dynamics depending on its value compared to 1 and this kind of gives the classification of a second order dynamical system. So depending on value of ξ to be greater than 1, equal to 1 or less than 1, we have 3 different types of second order dynamical systems and we will be discussing these further after a short break. Thank you.

Hello students welcome back. We are looking at response of a second order system to a step input and based on that we will be getting the classification of second order systems.

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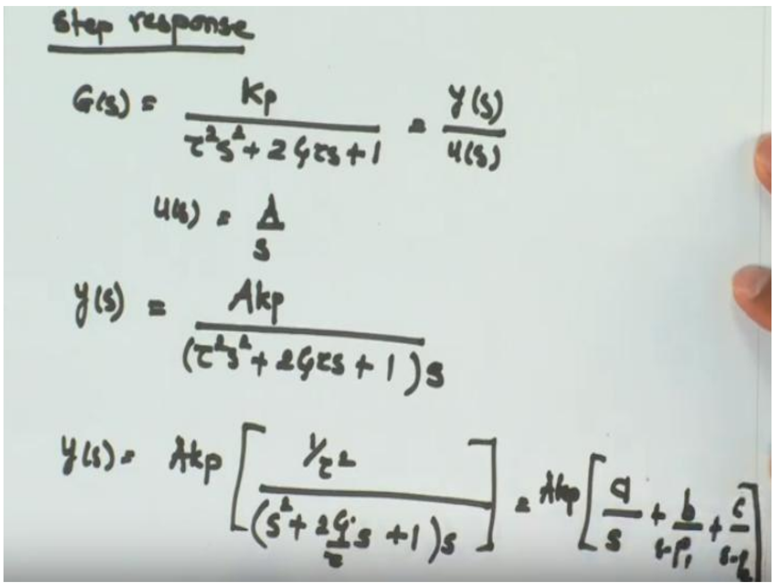

So the transfer function for the second order system in a generic form is $k_p/(\tau^2 s^2 + 2 \xi \tau s + 1)$ which is equal to y(s)/u(s) and as we are talking about a step response u(s) will be equal to A/s. So accordingly we will get the Laplace form of the output response as $A*k_p/(\tau^2 s^2 + 2 \xi \tau s + 1)$. And we will try to write it in a form $A*k_p *1/[\tau^2 s^2 + 2 \xi/\tau s + 1]$ wherein we can, write it or expand it in terms of partial fractions as some constant $a/s + b/(s - P_1) + c/(s - P_2)$ wherein P_1 and P_2 are the roots of this equation $(s^2 + 2 \xi/\tau s + 1/\tau^2)$.

And we had seen last time that depending on the nature of these roots we will get different types of dynamics. So let us consider the phase case when we have $\xi > 1$.

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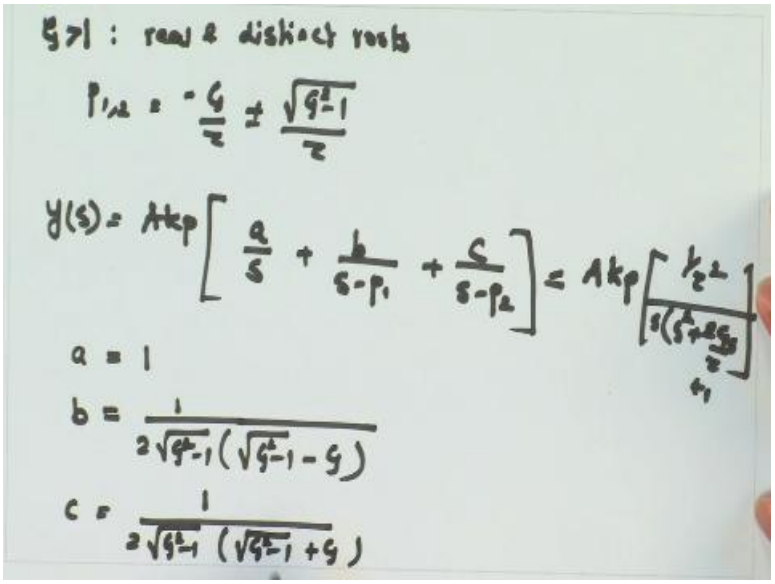

So in that case you will get real and distinct roots where this $P_{1,2}$ are,

$P_{1,2} = (-\xi/\tau) \pm \frac{\sqrt{\xi 2 - 1}}{\tau}$

So in that case what we get as y(s) as $A*k_p[a/s + b/(s - P_1) + c/(s - P_2)]$. This will be equal to $A*kp*1/\tau^2$ over the same denominator which we had earlier $s*(s^2 + 2\,\xi/\tau\, s + 1/\tau^2)$. So we can use method of partial fractions to obtain the values of a, b, and c.

So for this system what we get is,

a = 1,

$b = \frac{1}{2\sqrt{\xi^2-1}\left(\sqrt{\xi^2-1}-\xi\right)}$ ξ. And c similarly will be,

$c = \frac{1}{2\sqrt{\xi^2-1}\left(\sqrt{\xi^2-1}+\xi\right)}$

So accordingly when we substitute these we are going to get a response of the form.

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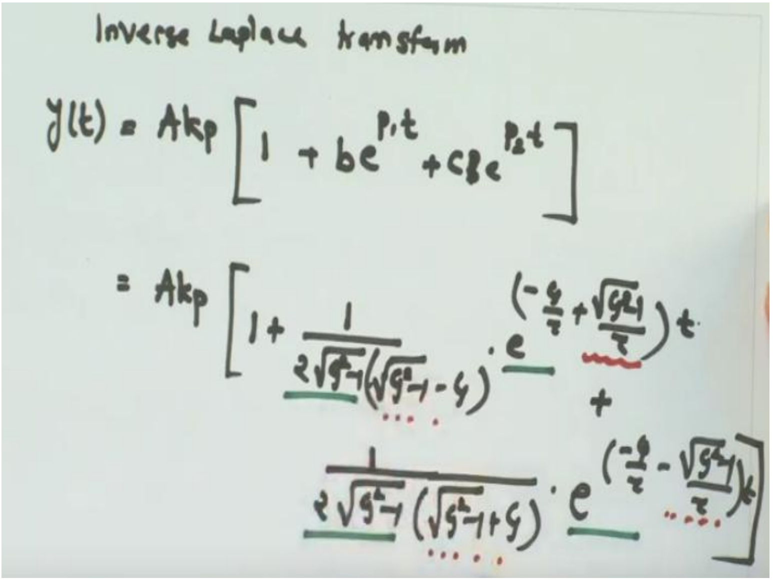
So when we take inverse Laplace we will get y(t) = A*kp and then inverse of 1/s will be a constant 1 and then we will get $b*e^{P1t} + c*e^{P2t}$. Now here we can do lot of algebraic manipulations or rearrangement and then eventually we can simplify this system.

So I will skip few steps and I can show you that the response would look like,



$$y(t) = Ak_p\left[1 + \frac{1}{2\sqrt{\xi^2 - 1}\left(\sqrt{\xi^2 - 1} - \xi\right)} e^{\left(-\frac{\xi}{\tau}\right)+\frac{\sqrt{\xi 2 - 1}}{\tau}} + \frac{1}{2\sqrt{\xi^2 - 1}\left(\sqrt{\xi^2 - 1} + \xi\right)} e^{\left(-\frac{\xi}{\tau}\right)-\frac{\sqrt{\xi 2 - 1}}{\tau}}\right]$$

So I will not show each and every step of this simplification but I will give you some guidelines about how you can proceed.

The first thing is you can take this term common. The next step would be expanding these as $e^{-\xi t/\tau}$ times $e^{(\xi 2 - 1) t/\tau}$. Similarly, we can expand this as $e^{-\xi t/\tau} / e^{\frac{\sqrt{\xi^2 - 1 * t}}{\tau}}$. We can take common terms outside. So we will be left with this term with this in the denominator and we will be leaving with this term, with this in the denominator. And then later on we can just do the addition. It will be (a – b)(a + b). So denominator again will be $\xi^2 - 1 - \xi^2$. So again the denominator of the resulting addition will be -1. So this plus would become – 1.

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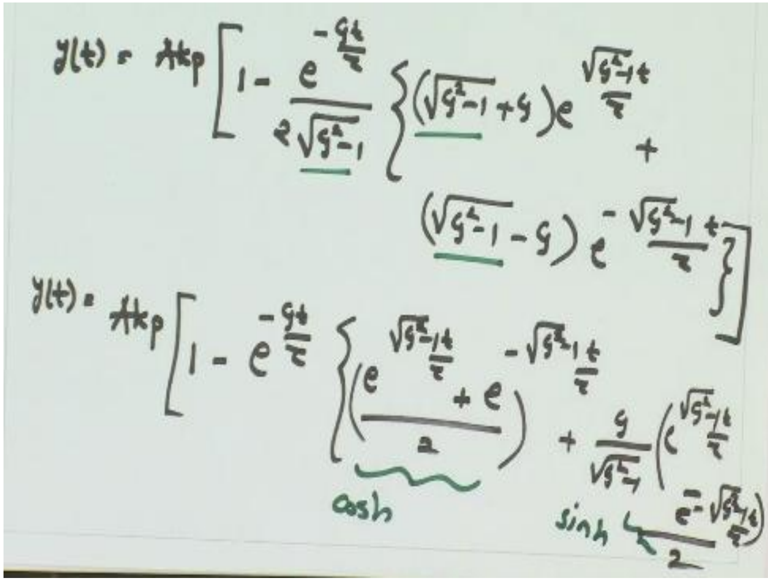

So once you do all these simplifications what you would end up with will be,

y(t) = A*kp[1 – ($e^{-\xi\, t/\tau}$)/(2 $\sqrt{\xi^2-1}$)*{}]. So you will be left with this and again what we can notice now is which we can now simplify again as y(t) = A*kp[1 – ($e^{-\xi\, t/\tau}$){}] and we will take the remaining thing inside. So in this case you can observe that we have this $\sqrt{\xi^2-1}$is common in both the sides. And then it will get cancelled with this $\sqrt{\xi^2-1}$. So what we would eventually left with will be,

$$\frac{e^{\frac{\sqrt{\xi^2-1}t}{\tau}}+e^{\frac{-\sqrt{\xi^2-1}t}{\tau}}}{2}$$

and similarly for the other case what we would have is $\xi/\sqrt{\xi^2-1}$ and then it will be the other term which will be,

$$\frac{e^{\frac{\sqrt{\xi^2-1}t}{\tau}}-e^{\frac{-\sqrt{\xi^2-1}t}{\tau}}}{2}$$

Which we can note that will be a cosine hyperbolic form and this other thing is sin hyperbolic. So the final response of this system will be given as

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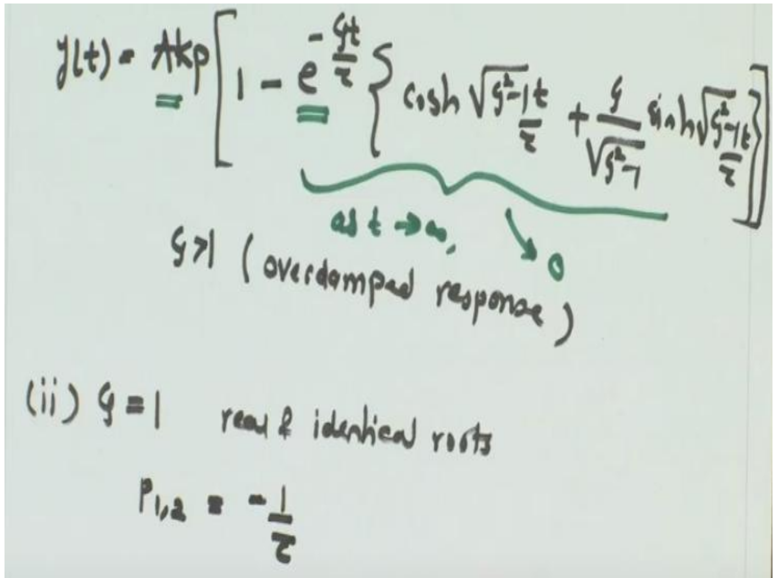

$$y(t) = Ak_p e^{\frac{-\xi t}{\tau}}\left\{cosh\sqrt{\xi^2 - 1}\frac{t}{\tau} + \frac{\xi}{\sqrt{\xi^2 - 1}} sinh\sqrt{\xi^2 - 1}\frac{t}{\tau}\right\}$$

So you can note that the response of a second order system when ξ is greater than 1 would be given by this term which has the overall multiplication factor of A^*k_p and you can very well appreciate that A^*k_p would eventually lead you to the final value of the response.

So when t goes to infinity this entire term will go to 1 or this, so this entire term will vanish so as t tends to infinity this entire term would go to 0 because there is a decaying exponential. This is the case when ξ is indeed greater than 0. And the final value of the response would then be given by A^*k_p. I would show the response of this system in some time when we have looked at all the other criteria's in terms of the values of ξ. So for now this is the response when we have $\xi > 1$ which is also known as an over damped response and I would also show you why it is called as over damped response when we look at the other two cases as well.

The second case is when ξ is indeed = 1. In that case we get real and identical roots which will be $P_{1,2} = -1/\tau$. So in that case as we have a double root we will have to use a slightly different version of partial fractions.

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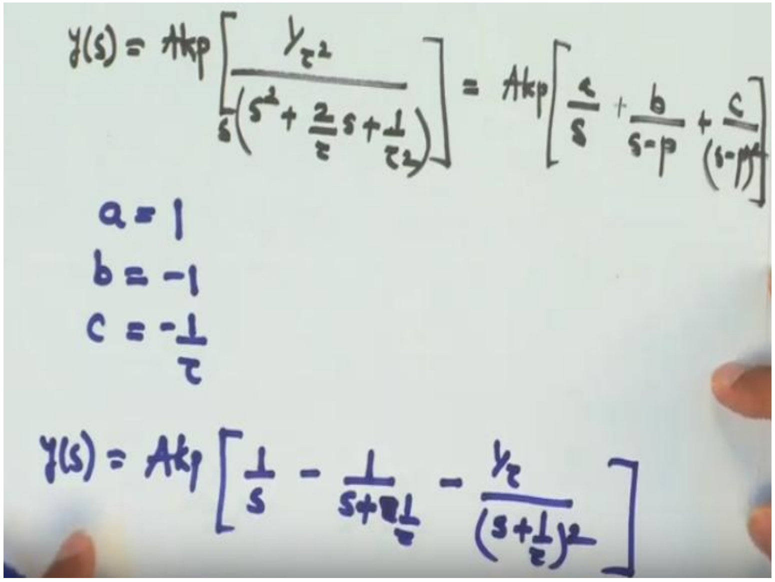

So what we would be writing as,

$$y(s) = A{*}k_p \frac{1/\tau 2}{s2 + 2\,\xi/\tau\, s + 1/\,\tau 2}.$$

This is we are going to write as $A{*}k_p{*}[a/s + b/(s – P)$ + the double root which is s – P square]. So in this case when we use partial fractions, what you are going to end up with is a = 1, b will come out to be equal to -1 and c will come out to be – 1/τ. So when we substitute this we will get y(s) = $A{*}k_p\ [1/s – 1/(s + 1/\tau) – 1/\tau\ /(s + 1/\tau)^2]$.

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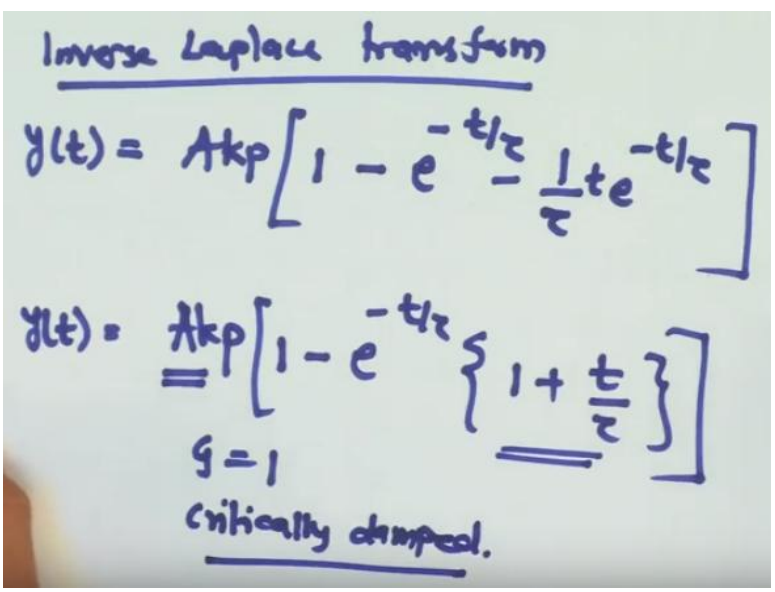

And now we can take Laplace inverse for this to get y(t) which will be $A{*}k_p$ Laplace inverse of 1/s would be a constant 1 and then Laplace inverse of 1/(s + 1/τ) will be an exponential term so

$e^{-t/\tau}$ and the last term is $- 1/\tau /(s + 1/\tau)^2$. So it is like the inverse of that will be t/ τ times $e^{-t/\tau}$ which we can condense and write as $A^*k_p [1 - e^{-t/\tau} (1 + t/\tau)]$.

So you can see that the form of the response is similar to the response for the case when ξ was greater than 1. So this is for ξ = 1. So we have still the first ultimate value as A^*k_p. So even this response the final value of this will reach A^*k_p. We have 1 – e raised to a decaying exponential term $e^{-t/\tau}$ followed by a term which is a little bit simpler in this case as there are no cosine hyperbolic or sine function. So we have simple linear function as 1/t + τ. So this is the case when ξ is = 1 which we also call as critically damped. And we are one step from coming to the origin of these terms as over damped or critically damped and we will consider the last case when ξ is indeed less than 1.

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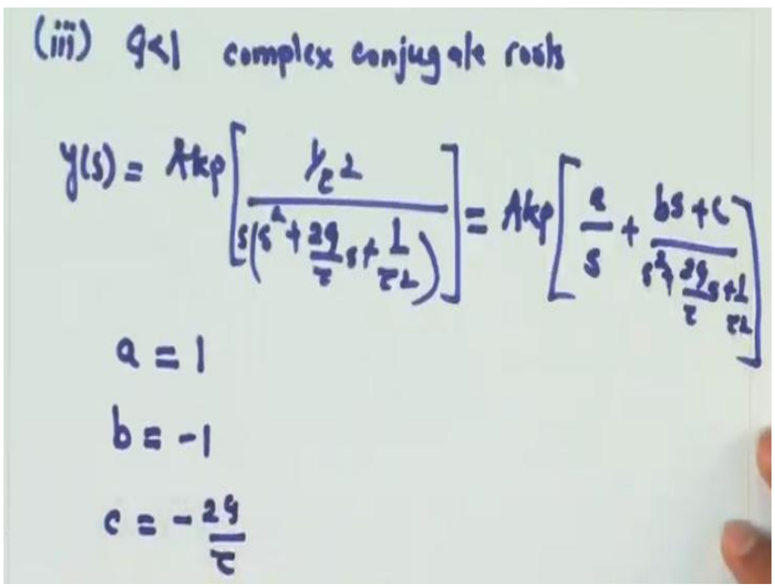

So in that case what we have is complex conjugate roots. So we have,

$$y(s) = A^*k_p \frac{1/\tau 2}{s^2+2\frac{\xi}{\tau}s+\frac{1}{\tau^2}}$$

This we are going to write in this case as,

$$y(s) = A * k_p \frac{1/\tau 2}{s2 \; + \; 2\,\xi/\tau\, s \; + \; 1/\,\tau 2} = A * k_p \left[\frac{a}{s} + \frac{bs + c}{s^2 + 2\frac{\xi}{\tau}s + \frac{1}{\tau^2}}\right]$$

So for this case the method of partial fractions would give us a = 1, b will come out to be -1 and c will come out to be -2 ξ/τ.

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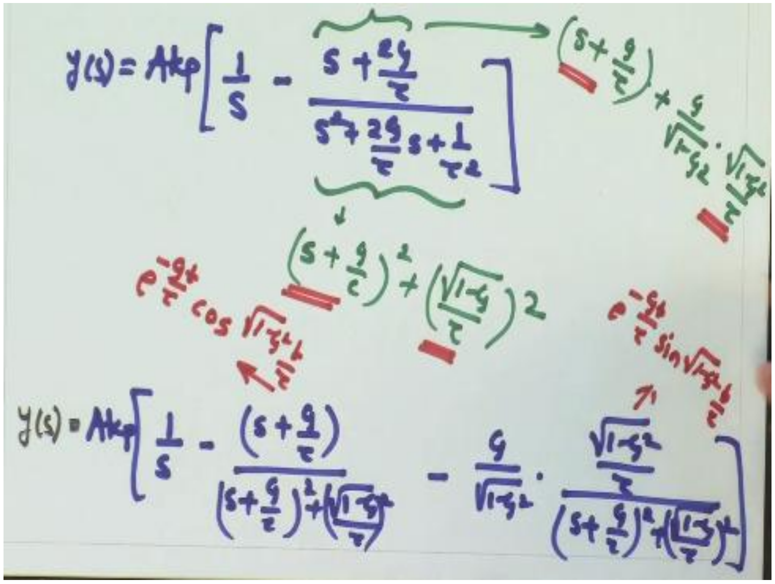

So substituting these values what we are going to get is

$$y(s) = A*k_p\ [1/s – (s + 2\ \xi/\tau)/(s^2 + 2\frac{\xi}{\tau}s + \frac{1}{\tau^2})].$$

Now again I will not show you all the steps of this derivation but we are going to write this as $(s+a)^2 + b^2$. So we will write it as denominator as $(s + \xi/\tau)^2 + (\sqrt{1-\xi}/\tau)^2$.

And numerator we will split as $s + \xi/\tau$ which is similar to this. And the other term we will try to write in terms of this $\sqrt{1-\xi^2}/\tau$. So that we will write as plus this remaining ξ/τ we will write as $\xi/\sqrt{1-\xi^2}$ times $\sqrt{1-\xi^2}/\tau$. So the idea is we are going to get one factor which is same as this and the other factor which will give me this.

So based on this the way we will write this entire expression as

$$A * k_p \left[\frac{1}{s} - \frac{s + \xi/\tau}{(s + \xi/\tau)^2 + (\frac{\sqrt{1-\xi}}{\tau})^2} - \frac{\xi}{\sqrt{1-\xi^2}} * \frac{\sqrt{1-\xi^2}\Big/\xi}{(s + \xi/\tau)^2 + (\frac{\sqrt{1-\xi}}{\tau})^2}\right]$$

So essentially what we have written it as $(s + a)/[(s + a)^2 + \omega^2]$ and this we have written as $\omega//[(s + a)^2 + \omega^2]$.

So the inverse of this would be of the form $e^{-\xi t/\tau}\cos(\sqrt{1-\xi^2}\, t/\tau)$ and inverse of second term is going to give me $e^{-\xi t/\tau}$ sine of the same angle $\sqrt{1-\xi^2}\, t/\tau$. So now we know inverses of each of these terms. So we can finally take the inverse Laplace.

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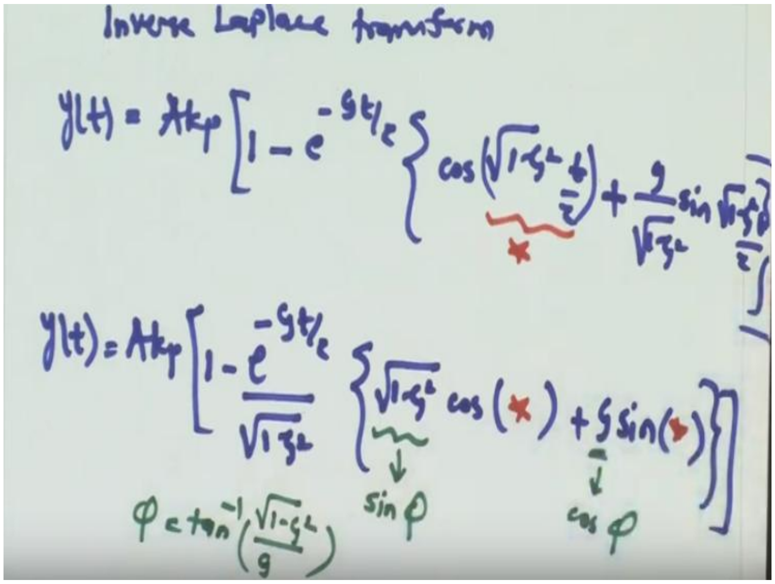

Y(t) would be = Ak p[1 – $e^{-\xi t/\tau}$ {}], $e^{-\xi t/\tau}$ we are going to take common. We will have cosine of $\sqrt{1-\xi^2}\, t/\tau$ + $\xi/\sqrt{1-\xi^2}$ sine of $\sqrt{1-\xi^2}\, t/\tau$. Which we can further simplify as Ak p[1- $e^{-\xi t/\tau}$ and we will take this $1-\xi^2$ out. So what we are going to have is $\sqrt{1-\xi^2}$ cosine of this angle + ξ times sine of this angle].

Where I am simply writing this as * and at this point what we would try to say is we will take some angle φ and we will say as ξ < 1 root of 1 – ξ square is also < 1. So we will call it a sine of some angle φ and this ξ we will call as the cosine of the same angle φ and this definition of sin and cosine is possible as long as we satisfy the identity $\sin^2\varphi + \cos^2\varphi = 1$ which indeed is satisfied here.

So that φ is = $\tan^{-1}(\sqrt{1-\xi^2}/\xi)$. So this becomes sin A*cos B + cos A* sin B, the final form of the response will be sin (A + B) and the final response we are going to get out of this is Y(t) = A^*k_p.

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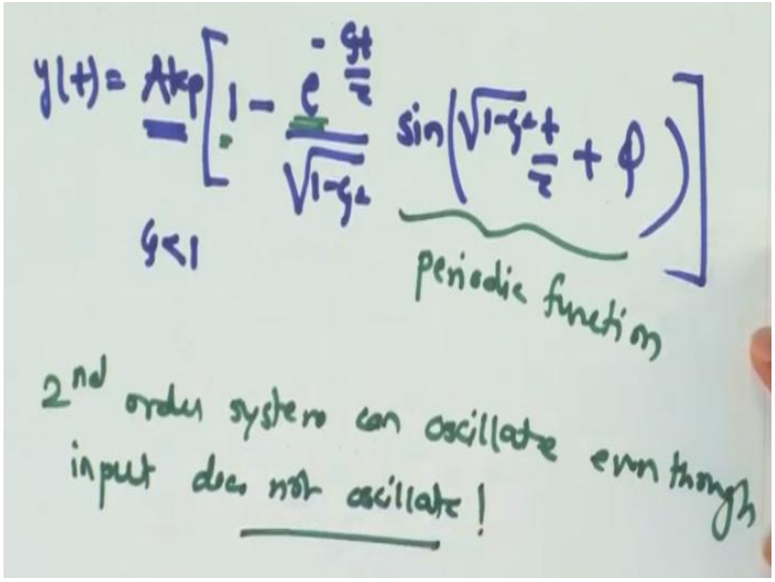

You will notice that it also has a similar form of A*k_p[1-($e^{-\xi t/\tau}$ /$\sqrt{1-\xi^2}$)sin ($\sqrt{1-\xi^2}$ t/τ)+ φ. So this is the response of the second order system when ξ is < 1 for a unit step change and now here you can see that we have the ultimate value which is A*k_p. We have this constant. We also have an exponential decaying function. But what makes this response interesting is the presence of a periodic function, a sinusoid.

So we are going to have oscillations inside this system. So that is essentially a characteristic of a second order or even higher order system which was not at all present in a first order system. So this second order system can oscillate even though input does not oscillate. So this was not possible in a first order system. When we talked about a first order system, if we give a sinusoidal input you will get sinusoid at the output. So if you oscillate the input the output will oscillate. But when the input was constant then the output would never oscillate. Not the case with the second order system. So when ξ is < 1, even though we have made a step change, the output is going to oscillate and what we would actually see is.

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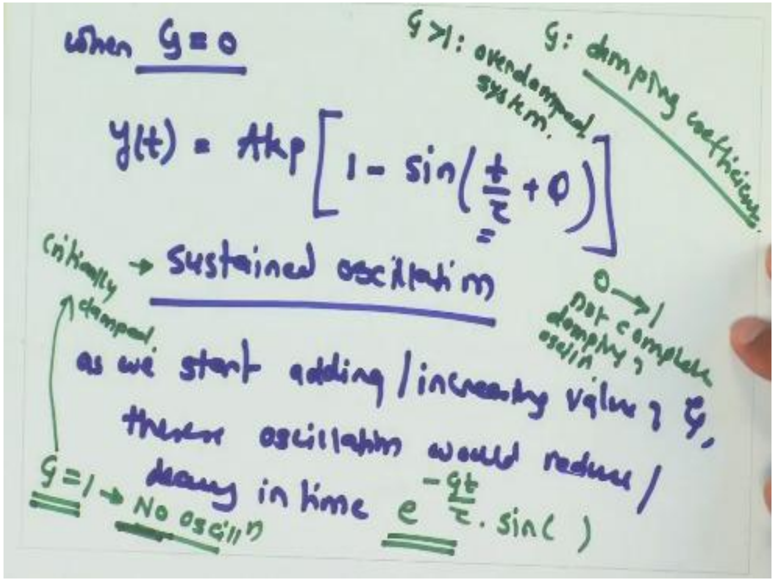

So when we say this coefficient is = 0 in that case we will get the response which is of the form $y(t) = A*k_p\ [1 - \sin(t/\tau + \varphi)]$ which in this case will be 0, which in this case will be infinity or pi/2, time will be infinity. So what we are going to see is sustained oscillation. So the output is a pure sinusoidal and then it continuously oscillate with this radian frequency of $1/\tau$ and as we start adding or increasing value of ξ, these oscillations would reduce or decay in time.

Because we are going to add contribution to this decay factor of $e^{-\xi t/\tau}$ which multiplies the sinusoid. So as ξ keeps on increasing from 0 the contribution of this or the contribution of the sinusoid will keep on reducing and this will happen till we have $\xi = 1$. So in that case there are no oscillation. We had seen this case when $\xi = 1$ the response did not contain any sinusoid; it was a pure linear function in that curly bracket.

So what we are going to see is as we are increasing the value of this parameter ξ, we are going to dampen the oscillations. So that is why this ξ is also known as a damping coefficient. So it is damping the oscillations and the genesis of this name damping coefficient and this damping is 0 when we do not have any value to this. So when the value is 0 there is no damping to the oscillation, we get sustained oscillations.

And as we start increasing the value from 0 to 1 we are progressively going to dampen these oscillations but this damping is not complete. So not complete damping of oscillation when we are between 0 to 1. That is why it is known as under damping. When we reach the value of 1 all the oscillations, that is the minimum value of ξ which is required to dampen all the oscillations. That is why this case is known as critically damped.

And lastly when we go for value greater than 1, in that case there are no oscillations to dampen. So we have already dampened all the oscillations. We are unnecessarily over damping the system and that is why when ξ is > 1 we call it as an over damped system. So that is the genesis of the name damping coefficient and also the classification of the second order system based on the value of this damping coefficient. If the value is between 0 and 1 we call it as an under damped system.

When the value is = 1 we call it a critically damped system and when the value is > 1 we call it as over damped system. So we will take a short pause here and after that we will try to look at the example of that manometer and how it performs under these different cases of damping coefficient and as well as we will look at some of the characteristics of the second order response. Thank you.