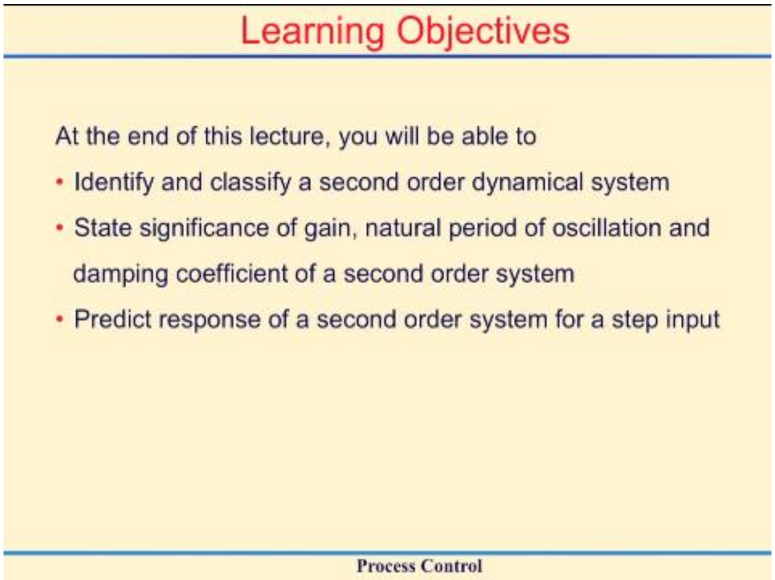
Chemical Process Control
Prof. Sujit S. Jogwar
Department of Chemical Engineering
Indian Institute of Technology-Bombay

Lecture - 13
Introduction to Second Order Dynamical Systems

Welcome to week 3 of this course. In the previous week, we looked at the simplest of the dynamical systems which were first-order systems. In this week, we will look at second order systems as well as higher order systems which we would be approximating as first order plus dead time and also we will look at some additional different types of dynamics or which come under the umbrella of numerator dynamics. So we will start with the second order systems.

So at the end of this part of the lecture, the objectives are that you should be able to identify as well as classify a second order dynamical system.

(Refer to Slide Time: 01:05)



Learning Objectives

At the end of this lecture, you will be able to

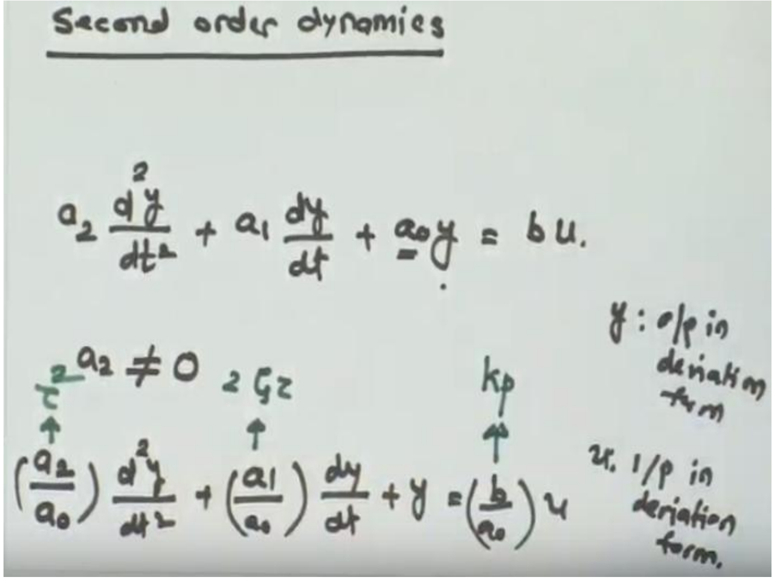
- Identify and classify a second order dynamical system
- State significance of gain, natural period of oscillation and damping coefficient of a second order system
- Predict response of a second order system for a step input

Process Control

For the second order systems, there are three parameters. One is the process gain, natural period of oscillation and the damping coefficient. So you should be able to tell the significance of each of those, for the system at hand and lastly, you should be able to predict how a second order system responds to a step change. So let us start with what characterizes a second order dynamical system.

(Refer to Slide Time: 01:33)

The second order dynamical system is a system where the output of that system is given by a second order ordinary differential equation. So the general form of that will be,



Second order dynamics

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = bu.$$

$a_2 \neq 0$

$$\overset{\tau^2}{\overset{\uparrow}{\left(\frac{a_2}{a_0}\right)}} \frac{d^2y}{dt^2} + \overset{2\xi\tau}{\overset{\uparrow}{\left(\frac{a_1}{a_0}\right)}} \frac{dy}{dt} + y = \overset{k_p}{\overset{\uparrow}{\left(\frac{b}{a_0}\right)}} u$$

y : o/p in deviation form

u. i/p in deviation form.

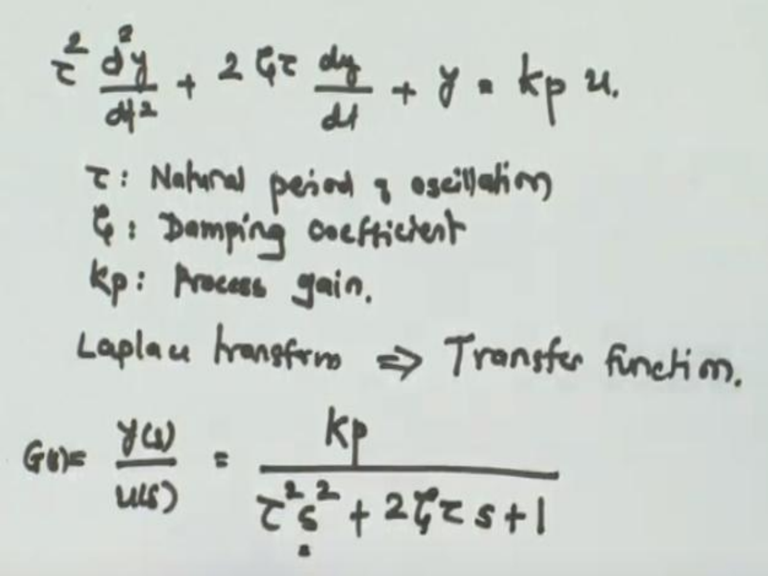
$a2 * d2y/dt2 + a1 * dy/dt + a0 * y = b * u.$

Where y is my output in deviation form and u is my input in deviation form, and obviously, the condition is that a_2 should not be equal to 0. Because if a_2 is equal to 0, what we have left is a first order differential equation, so this will be a first order system, and then all these other constants are there. We will typically write this in a form which was similar to what we did for a first order system, that is, we divide all over by a_0 so that we want the coefficient multiplying by to be 1. And then the way we represent this, we represent this as a square of a parameter τ, τ^2

But mind you this τ even though we are using the same symbol as time constant for a first order process, most of the time this τ is different from the τ which we had used for first order system in terms of significance only in one particular case it has the same significance as that of first order time constant. Its name is also different. It is called the natural period of oscillation.

Why it is called that way, we will look at it when we look at the response of the system. But for all other purposes, the symbol is the same, but the meaning is different. If $a1/a0$ we will represent it as $2*\xi*\tau$. So the τ is the same as this and we have a new variable ξ which is known as the damping coefficient. And this is the same as the earlier nomenclature of gain k_p. So a standard form of second order dynamical system is that if we just write it down again, it will be

(Refer Slide Time: 04:32)



$$\tau^2 \frac{d^2y}{dt^2} + 2\zeta\tau \frac{dy}{dt} + y = k_p u.$$

τ : Natural period of oscillation
ζ : Damping coefficient
k_p : Process gain.

Laplace transform ⇒ Transfer function.

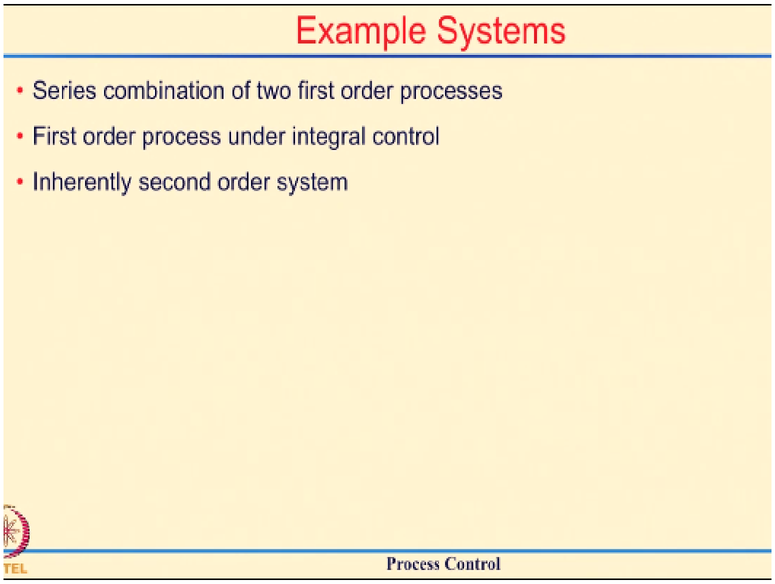
$$G(s) = \frac{y(s)}{u(s)} = \frac{k_p}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$\tau^2 d2y/dt2 + 2 * \xi * \tau * dy/dt + y = kp * u$. So this is a standard form for a second order dynamical system where τ is known as the natural period of oscillation. Zeta is known as damping coefficient and k_p is known as process gain. So like the first order systems we will also take a Laplace transform that will give us the transfer function for the second order process. If you take the Laplace of this particular equation and write it into a transfer function notation.

So what we get is a transfer function,

$G(s) = y(s)/u(s) = k_p/(\tau^2 * s^2 + 2 * \xi * \tau * s + 1)$. So it is second order, it is quadratic in terms of the Laplace variable s.

(Refer to Slide Time: 06:09)



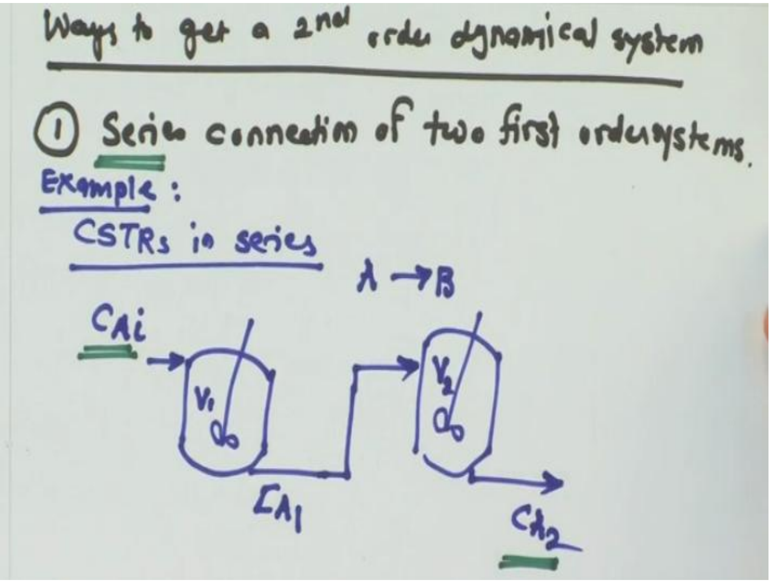
Example Systems

- Series combination of two first order processes
- First order process under integral control
- Inherently second order system

Process Control

So let us now look at what are the different cases when we get a second order dynamical system if we try to recall what we had done for the first order process. What we had was that the first order process is the process which is characterized by its ability to store material or inherit material or energy. Similarly, we will try to see what are the different physical phenomena which are going to give rise to a second order dynamical system. As it turns out, there are three ways to get a second order dynamical system. We will cover them one by one.

(Refer to Slide Time: 06:54)



Ways to get a 2nd order dynamical system

① Series connection of two first order systems.

Example :

CSTRs in series

$A \to B$

So the first way, so ways to get a second order dynamical system. The first one is a series connection of two first-order processes or systems. So now keep in mind this is a series connection. We had looked at a few examples in the previous week where there was a parallel connection of first-order capacities. Anywhere there is a parallel connection, then the overall order of the system remains the same.

Now here we are talking about the series connection, the output of one speeds the as the input of the other. So when we have such kind of an arrangement which is also very common in our chemical industry, what we get is a second order system. So it is like the addition of two first-order processes. In series, it is going to give us a second order system. We will consider an example which you might have seen in your reaction engineering course which is CSTRs in series.

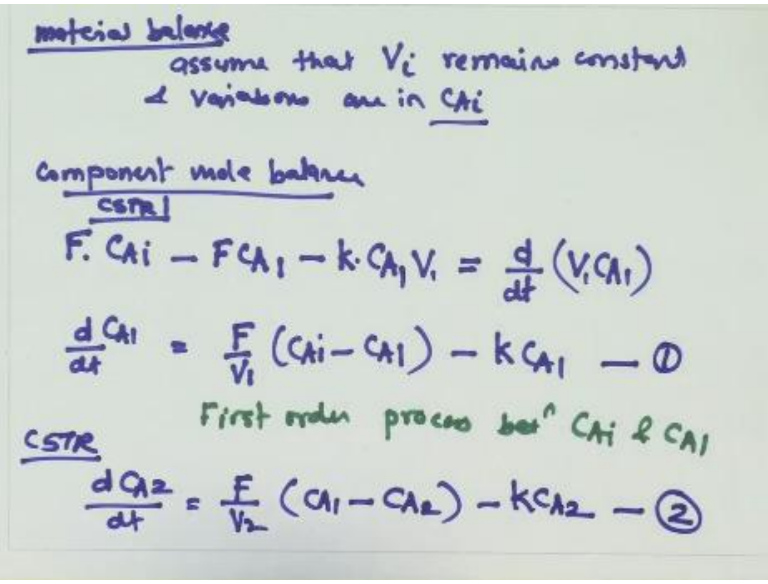
So CSTR is a first order dynamical system. We will see that in a minute. So let us see what this particular system is. So we have one CSTR. Let us say the input coming in is C_{Ai}. The volume is V_1. Everywhere the reaction which we are going to consider is a simple reaction A going to B. The output of this is going to be C_{A1} and what we have is this output goes to a second CSTR which has a volume of V_2 and what you get out is the final product which is C_{A2}.

So now what we are interested in is what is the effect of variations in inlet composition on the final variations on the final product purity. You can see that when we change C_{Ai}, it is going to have some impact on C_{A2}. So we will try to quantify that and as we are in the topic of second-order dynamical systems what I should be able to impress upon you is that this variation indeed follows second-order dynamics. We will try to formulate a process model for this particular system.

So we will start with material balance. Now for simplicity here I will assume that volume of the CSTR remains constant. We will not consider variations coming from inlet flow. These can be considered and you can still show that it is though the system still follows second order dynamics but just for the constraint of time and for just a proof of concept we will assume that volume remains constant and variations are in C_{Ai}. These are the variation we are going to consider.

So what we will be writing is a component mole balance. So the total material coming in would be F which is the total flow rate times C_{Ai}. This is for CSTR 1. The material going out of the system would be F*C_{A1}. The rate of consumption would be k*C_{A1}*V. And lastly, there is no rate of generation. So this has to be equal to the rate of change of content of that particular species which is V_1*CA1.

What we would get is,



material balance

assume that V_i remains constant

& variations are in $\underline{C_{Ai}}$

Component mole balance

CSTR 1

$$F \cdot C_{Ai} - FC_{A1} - k \cdot C_{A1} V_1 = \frac{d}{dt}(V_1 C_{A1})$$

$$\frac{dC_{A1}}{dt} = \frac{F}{V_1}(C_{Ai} - C_{A1}) - kC_{A1} \quad — ①$$

First order process betn C_{Ai} & C_{A1}

CSTR

$$\frac{dC_{A2}}{dt} = \frac{F}{V_2}(C_{A1} - C_{A2}) - kC_{A2} \quad — ②$$

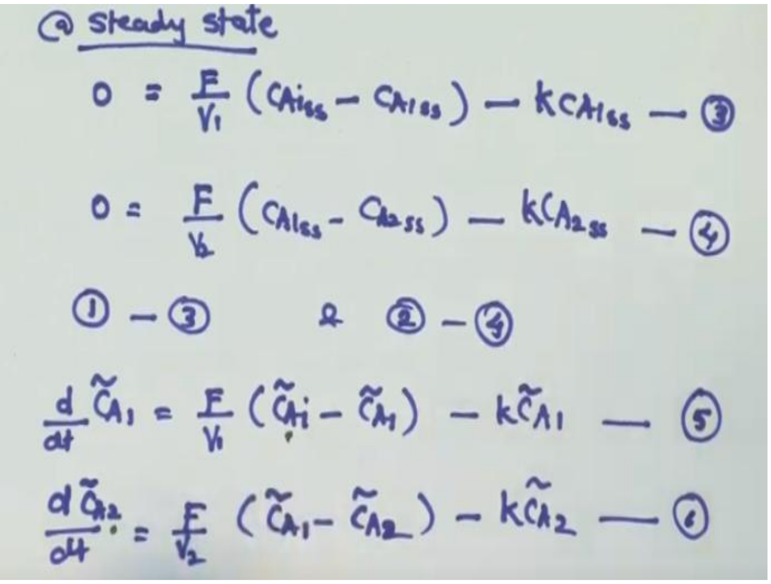
d(C_{A1})/dt = (F/V_1)* (C_{Ai} – C_{A1}) – k*C_{A1}. So this also has to be 1. You can see that this is a first order process between C_{Ai} and CA1. So our first CSTR is a first order process between its input concentration and output concentration. So similarly for CSTR 2 which is exactly the same except that it has a different volume. Now we can write that d(C_{A2})/dt = (F/V_2)* (C_{A1} – C_{A2}) – k*C_{A2}.

So we have two first order processes which are connected together through this term C_{A1}, because you can see that whenever there is a change in inlet composition which is C_{Ai}, it is not going to directly affect C_{A2}. Because if you look at the equation 2 you will see that there is no term for C_{Ai}. So C_{Ai} is not going to directly affect the concentration of second CSTR.

Whenever you change C_{Ai}, it is going to first change C_{A1}. So it is going to affect the first tank directly. It will cause changes in C_{A1} and those changes in C_{A1} are now going to drive the changes in C_{A2}. So that is how it is a series connection between two first order capacities. So let us now try to formulate the final transfer function between C_{Ai} and C_{A2}. For that, we will first have to write these equations in deviation forms.

(Refer Slide Time: 14:10)

So we will be writing the equations at steady state. So at steady state what we have is for first CSTR it is,



@ steady state

$$0 = \frac{F}{V_1}\left(C_{Ai_{ss}} - C_{A1_{ss}}\right) - kC_{A1_{ss}} \quad — ③$$

$$0 = \frac{F}{V_2}\left(C_{A1_{ss}} - C_{A2_{ss}}\right) - kC_{A2_{ss}} \quad — ④$$

① – ③ & ② – ④

$$\frac{d}{dt}\tilde{C}_{A1} = \frac{F}{V_1}\left(\tilde{C}_{Ai} - \tilde{C}_{A1}\right) - k\tilde{C}_{A1} \quad — ⑤$$

$$\frac{d\tilde{C}_{A2}}{dt} = \frac{F}{V_2}\left(\tilde{C}_{A1} - \tilde{C}_{A2}\right) - k\tilde{C}_{A2} \quad — ⑥$$

$$0 = (F/V_1) * (C_{Aiss} - C_{A1ss}) - k * C_{A1ss}.$$

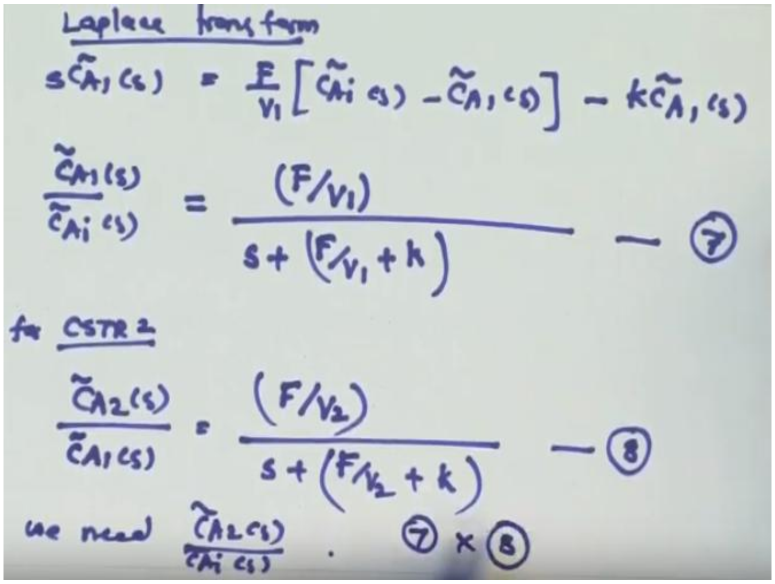
And similarly, for second CSTR, we will have,

$$0 = (F/V_2) * (C_{A1ss} - C_{A2ss}) - k * C_{A2ss}.$$

So if we do 1 – 3 and 2 – 4 we will get the deviation form of these equations which looks like the, we will call it as equation 5.

Now at this point, there are 2 ways to show that this particular system between C_{Ai} and C_{A2} is the second order dynamical system. One way is that we will eliminate the C_{A1} from these 2 equations. The way you can do that is by differentiating equation 6 which will give me second-order derivative with respect to C_{A2} and it will have first-order derivative with respect to C_{A1}. Then $d(C_{A1})/dt$ will be substituted from 5 into equation 6 and eventually what we can get is the second order term in terms of C_{A2}. We will have some contribution coming from C_{Ai}. Now as it turns out, it becomes very lengthy. What we will use is a simpler method of getting a transfer function which is by taking Laplace of both the equations and then eliminating $C_{A\,1}$ Laplace from that. So we will take a Laplace transform of these 2 equations.

(Refer to Slide Time: 16:45)



Laplace transform

$$s\tilde{C}_{A1}(s) = \frac{F}{V_1}\left[\tilde{C}_{Ai}(s) - \tilde{C}_{A1}(s)\right] - k\tilde{C}_{A1}(s)$$

$$\frac{\tilde{C}_{A1}(s)}{\tilde{C}_{Ai}(s)} = \frac{(F/V_1)}{s+(F/V_1+k)} \quad — ⑦$$

for CSTR 2

$$\frac{\tilde{C}_{A2}(s)}{\tilde{C}_{A1}(s)} = \frac{(F/V_2)}{s+(F/V_2+k)} \quad — ⑧$$

we need $\frac{\tilde{C}_{A2}(s)}{\tilde{C}_{Ai}(s)}$. ⑦ × ⑧

So Laplace transform as we are working with deviation variables we do not have to worry about the initial condition. So we will have,

$$s*C_{A1}(s) = (F/V_1)*[C_{Ai}\ (s) – C_{A1}\ (s)] – k*C_{A1}\ (s)$$

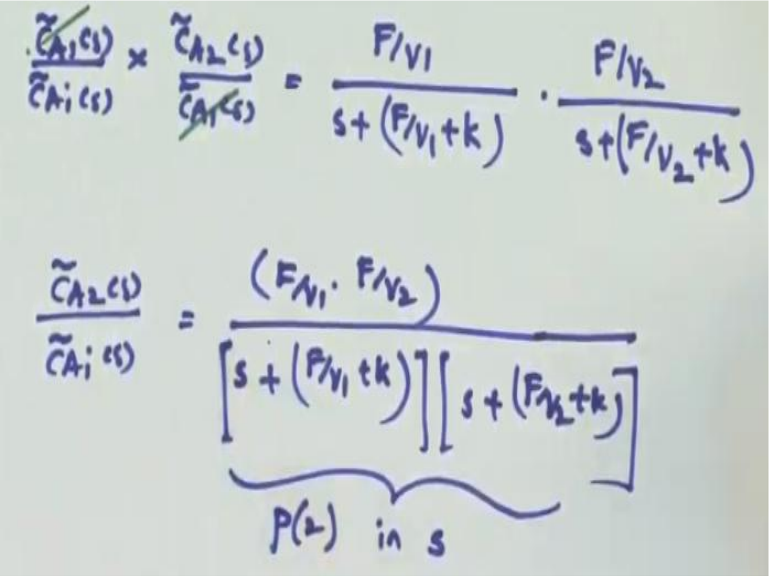
which we can write as, C_{A1} (s)/C_{Ai} (s).

So we will write a transfer function for the first CSTR which will come out to be,

$$C_{A1}\ (s)/C_{Ai}\ (s) = (F/V_1)/(s + (F/V_1 + k)).$$

Let us call it equation 7. So this is the transfer function between changes in the inlet concentration to changes in the first CSTR concentration. So similarly for the second CSTR, we will get if you do the same analogy, the equations are similar. So we can simply write this other transfer function by comparing it with equation 7. So now we have 2 transfer functions. One between C_{Ai} and C_{A1} and the other one is between C_{A1} and C_{A2}. And what we need is C_{A2}/C_{Ai}. So we can simply multiply equation 7 and 8 and then this C_{A1} (s) will get canceled out from numerator and denominator. So we do 7 * 8.

(Refer to Slide Time: 19:12)

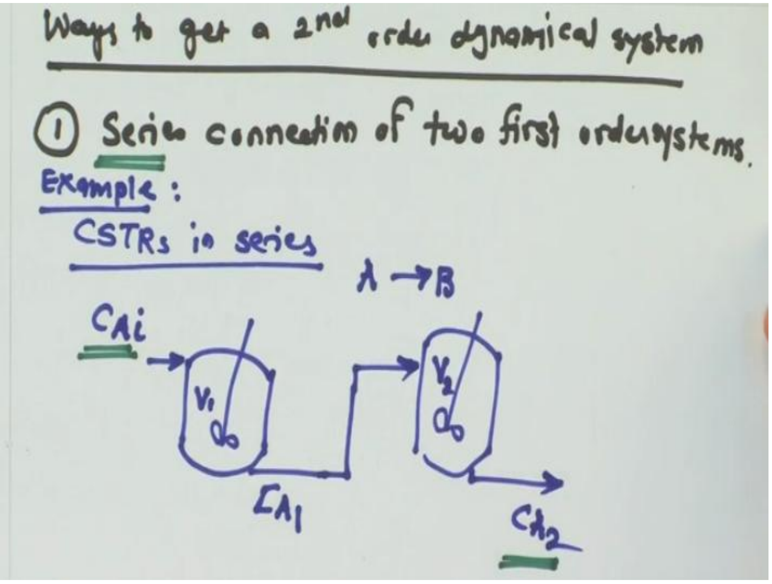


$$\frac{\cancel{\tilde{C}_{A_1}(s)}}{\tilde{C}_{Ai}(s)} \times \frac{\tilde{C}_{A_2}(s)}{\cancel{\tilde{C}_{A_1}(s)}} = \frac{F/V_1}{s+(F/V_1+k)} \cdot \frac{F/V_2}{s+(F/V_2+k)}$$

$$\frac{\tilde{C}_{A_2}(s)}{\tilde{C}_{Ai}(s)} = \frac{(F/V_1 \cdot F/V_2)}{\underbrace{[s+(F/V_1+k)][s+(F/V_2+k)]}_{P(2)\text{ in } s}}$$

The final transfer function we get is a quadratic polynomial in s. So it is a second-order dynamical system. So here I am not trying to put it into the general form of ($\tau^2 s^2 + 2\,\xi\tau s + 1$). Because for that, I need to have this coefficient to be 1 which would require me to take this particular (F/V1 + k) common from this side and (F/V_2 + k) common from this side and then go to the top and accordingly your gain will also have that multiplication factor. But the point is, there is a s square term. So it is going to be a second order dynamical system.

(Refer to Slide Time: 21:10)



Ways to get a 2nd order dynamical system

① Series connection of two first order systems.

Example :

CSTRs in series

What we have seen is that when we connect 2 first-order capacities which in this case was these 2 first-order tanks what we got is a series connection what we got is a second order dynamical system. Now there are 2 ways in which this series connections can be done. This particular type of a series connection is known as a non-interacting type connection because what we are going to see is the direction of effect of the disturbance is one way. So when we have any changes in the reactor 1 they are going to get translated to reactor 2. However, if there are some changes in reactor 2 they are not going to get propagated back to 1. So the direction of interaction is only in one direction. The second tank does not interact with the first one. That is why this type of connection is also known as a non-interacting series connection.

You may be wondering, can we have an interacting series connection between these 2 tanks. Yes, we can. Let us say there was some sort of a recycle going from this particular stream back to the reactor. In that case, any variations in reactor 2 would also affect reactor 1 and these kinds of interconnections, series connection will be called as the interacting series connection between two first-order systems. So these series connections can be interacting or non-interacting. If you refer to textbooks you will see that typically when we have interacting systems then the systems are slower compared to non-interacting systems. So that was one way of getting a second order dynamical system.

Let us look at the second way of getting a second order dynamical system which is a first order system under feedback control. So what we are seeing is to get a second order system we actually have to start with the first order system. That is why in the last week I had said that first order systems are the primary or the basic fundamental type of dynamical system. In order to get a second order system you start with the first order system. So here we are going to look at a first-order system under feedback control.

In the first week we had looked at what is a feedback control. Just to recap, feedback control is a control mechanism wherein you measure the controlled variable and then looking at the deviation of that controlled variable from its desired value you change the manipulated variable. So you take a feedback from the system and make a change to the manipulated variable. So here the system which we are going to consider is the simple surge tank which we have seen as a first order system.

So the example we are going to consider is surge tank under a specific type of feedback control which is integral control. So we will take a short break here, and after that, we will try to analyze this type of system. Thank you.