Chemical Process Control

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Lecture - 12
Response to Sinusoidal Input

Let us now look at the response to another important input which is a sinusoidal input. So let us consider again the transfer function

$$G(s) = \frac{k_p}{\tau s+1}$$

And change in input is $Asin(\omega t)$. Note that when we are writing this y and u, these are deviation variables. So we are always considering a deviation from the steady state. So let us say we give a sinusoidal input to this first order system which has an amplitude of A and frequency ω. So in that case, the Laplace transform of input will be,

$$L[Asin(\omega t)] = \frac{A\omega}{s^2+\omega^2}$$

Let us now see what the effect of this is. So we will have,

$$y(s) \ = \ G(s) \ * u(s)$$

and we will have,

$$y(s) = Ak_p \frac{\omega}{(\tau s+1)(s^2+\omega^2)}$$

So again we will have to use partial fractions. So, in this case, we will be breaking it up as,

$$y(s) = Ak_p\left[\frac{\alpha}{\tau s+1}+\frac{\beta s+\gamma}{s^2+\omega^2}\right] = \ Ak_p \frac{\omega}{(\tau s+1)(s^2+\omega^2)}$$

The reason for such kind of partial fraction is that we already know the Laplace inverse of $\frac{1}{\tau s+1}$ as e^{-at}. Now we have also seen that the Laplace of sine function will be $\frac{\omega}{s^2+\omega^2}$. So there is a constant in the numerator, and the denominator has $(s^2+\omega^2)$. So this $\frac{\omega}{s^2+\omega^2}$ will give us some sinusoidal function and then the Laplace of $cos(\omega t)$ is $\frac{s}{s^2+\omega^2}$. So this $L^{-1}\left[\frac{\beta s}{s^2+\omega^2}\right]$will give us a cosine of a function.

If we simplify this using the method of partial fractions what we get,

$$\omega = \alpha(s^2+\omega^2) + (\beta s + \gamma)(\tau s + 1)$$

So in order to get alpha, beta, and gamma we will have to use different values of *s*.

Let $s = \frac{-1}{\tau}$,

$$\alpha = \frac{\tau^2\omega}{1+\tau^2\omega^2}$$

Let $s = 0$,

$$\gamma = \frac{\omega}{1+\tau^2\omega^2}$$

Lastly, we can substitute s = 1,

$$\beta = \frac{-\tau\omega}{1+\tau^2\omega^2}$$

So we can substitute all these into the original equation and try to see how does the output look like.

We will get,

$$y(s) = Ak_p\left[\frac{\omega\tau^2/1+\tau^2\omega^2}{\tau s+1} + \frac{-\tau\omega s+\omega/1+\tau^2\omega^2}{s^2+\omega^2}\right]$$

Now let us try to individually look at all the three terms. So let us call $\frac{\omega\tau^2/_{1+\tau^2\omega^2}}{\tau s+1}$ as term 1 and $\frac{-\tau\omega s+\omega/_{1+\tau^2\omega^2}}{s^2+\omega^2}$ as term 2. Rewriting the first term by dividing by τ on both numerator and denominator,

$$\frac{\omega\tau^2/1+\tau^2\omega^2}{\tau s+1} = \frac{\tau\omega}{(1+\tau^2\omega^2)(S+1/\tau)}$$

So if you want to take the Laplace inverse of 1, that will be,

$$L^{-1}\left[\frac{\tau\omega}{(1+\tau^2\omega^2)(S+{}^{1}/_{\tau})}\right] = \frac{\tau\omega}{(1+\tau^2\omega^2)}e^{-t/\tau}$$

Now let us look at the second term.

The second term is $\frac{-\tau\omega s+\omega/_{1+\tau^2\omega^2}}{s^2+\omega^2}$. So we will try to write it down as sines and cosines which we can write as,

$$\frac{-\tau\omega s+\omega/_{1+\tau^2\omega^2}}{s^2+\omega^2} = \frac{1}{1+\tau^2\omega^2}\left[\frac{\omega}{s^2+\omega^2} - \frac{\tau\omega s}{s^2+\omega^2}\right]$$

We know,

$$L^{-1}\left[\frac{\omega}{s^2+\omega^2}\right] = \sin(\omega t)$$

$$L^{-1}\left[\frac{s}{s^2+\omega^2}\right] = \cos(\omega t)$$

therefore the Laplace inverse of the second term is,

$$L^{-1}\left[\frac{-\tau\omega s+\omega/_{1+\tau^2\omega^2}}{s^2+\omega^2}\right] = \frac{1}{1+\tau^2\omega^2}[\sin(\omega t) - \omega\tau * \cos(\omega t)]$$

Now we will try to simplify this term or condense this term. So we are working on Laplace inverse of term 2, and we will try to rearrange this as,

$$\frac{1}{1+\tau^2\omega^2}[\sin(\omega t) - \omega\tau * \cos(\omega t)] = \frac{1}{\sqrt{1+\tau^2\omega^2}}\left[\sin(\omega t)\frac{1}{\sqrt{1+\tau^2\omega^2}} - \frac{\omega\tau}{\sqrt{1+\tau^2\omega^2}} * \cos(\omega t)\right]$$

So what we have done is we have split this $\frac{1}{1+\tau^2\omega^2}$ as multiples of root, square root terms so that essentially the expression remains the same. But what we have gotten are these two new terms which we are going to write as,

$$\frac{1}{\sqrt{1+\tau^2\omega^2}} = \cos\varphi \quad \text{and} \quad \frac{\omega\tau}{\sqrt{1+\tau^2\omega^2}} = \sin(\varphi)$$

where,

$$sin^2\varphi + cos^2\varphi = \frac{1+\tau^2\omega^2}{1+\tau^2\omega^2} = 1$$

So this is a feasible definition of φ0 and the way we can get φ is tan (φ) will be equal to ($\omega\tau$).

$$\varphi = tan^{-1}(\omega\tau)$$

So based on this definition what we get is the Laplace inverse of this second term,

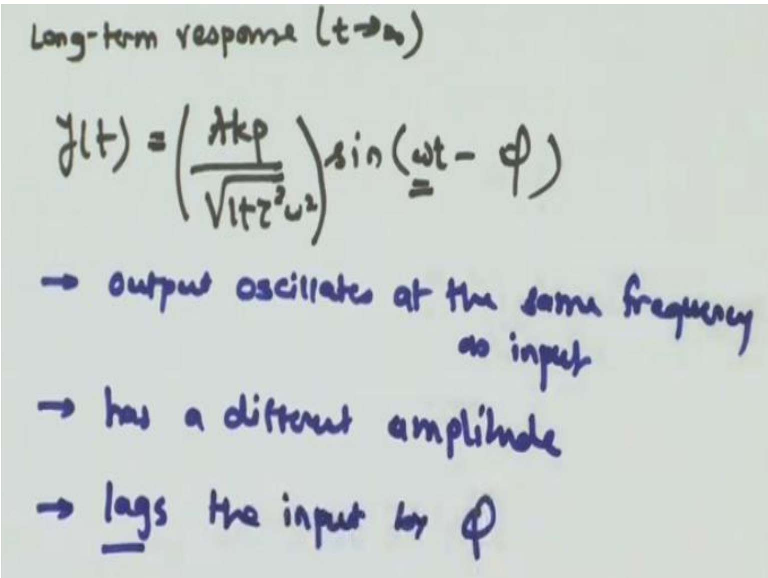
$$L^{-1}\left[\frac{-\tau\omega s+\omega/1+\tau^2\omega^2}{s^2+\omega^2}\right] = \frac{1}{\sqrt{1+\tau^2\omega^2}}[\sin(\omega t)\cos(\varphi) - \cos(\omega t)\sin(\varphi)]$$

$$= \frac{1}{\sqrt{1+\tau^2\omega^2}}\sin(\omega t - \varphi)$$

The two terms can now be combined,

$$y(t) = Ak_p\left[\frac{\tau\omega}{(1+\tau^2\omega^2)}e^{-t/\tau} + \frac{1}{\sqrt{1+\tau^2\omega^2}}\sin(\omega t - \varphi)\right]$$

So this is the final response of a first order system to a sinusoidal input, and you can see that it has two parts to this response. First part is an exponentially decaying function. So as time t goes to infinity, this part will go away, and there is a repeating or a sinusoidal part to the solution. So this is the sinusoidal part which will remain irrespective of whether we look at a very long term response.



So if we look at the response of this system after the initial exponential decay is over, so the long term response which is mathematically, time t goes to infinity, what we have is,

$$y(t) = \frac{Ak_p}{\sqrt{1 + \tau^2\omega^2}} \sin(\omega t - \varphi)$$

So you can see that if we look at the response after some time when the exponential has decayed, the output also oscillates similar to an input with the same frequencies. We can see that the omega remains the same. The output has a certain amplitude which is different than the input, and then output oscillates with the same frequency, but it lags the input by a phase angle of φ. So if I write down the characteristic of this response, you can say that the output oscillates at the same frequency as an input, has a different amplitude and it lags the input by a phase angle of φ. Now let us try to look at how this amplitude comes up or what is the significance of this amplitude. So we will define something known as an amplitude ratio.

So it is the ratio of the amplitude of output to the amplitude of input which is,

$$Amplitude\ ratio = \frac{\frac{Ak_p}{\sqrt{1 + \tau^2\omega^2}}}{A} = \frac{k_p}{\sqrt{1 + \tau^2\omega^2}}$$

So we can see that the output amplitude depends directly on the kp. So the larger is the gain of the system we will see a larger impact on the amplitude ratio. But we can also see that it is inversely proportional to the frequency. As the frequency of the input increases what we are going to see is that the amplitude ratio substantially decreases, that is one of the reasons why such systems are sometimes used to reject very fast disturbances. So we will try to see the impact of this particular phenomena through a simulation.

Earlier days simple surge tanks were used to kind of decouple different parts of a process so that a disturbance in one unit does not get propagated to the subsequent downstream unit. So we will consider this example.

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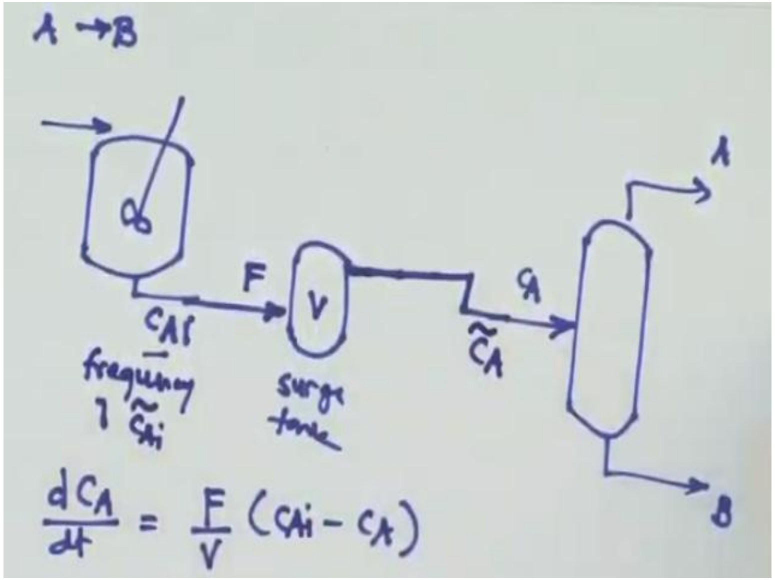

So let us say you have a reactor which is going to convert a reaction component A to B. So at the output of the reactor you will have unreacted A as well as some product B. Typically it would eventually go to a separation unit, let us say a distillation column which is going to separate this A and B. Now typically when this reactor operates, there will be changes. The reactor would not operate exactly at the same concentration, and there might be some fluctuations in the outlet at the reactor.

These may be because of some changes in the feed condition or in the heating or cooling of the reactor, or it may be the catalyst deactivation degrade over time. So it is quite possible that this outlet concentration from the reactor may not always come out to be exactly the same as what is desired. However, this distillation column would be designed to separate the component coming at that particular composition.

If the reactor composition changes and if the same stream goes directly to the distillation column then the distillation column will get affected by the changes in the purity of the reactor. In order to avoid that and decouple this reactor as well as a distillation column, simple surge tanks were introduced. So a surge tank is nothing but a simple tank which will kind of act as a mixer. So the output coming from the reactor let us call it as C_{Ai} will go to the surge tank, it will mix with whatever amount which was present earlier in the surge tank and you will take out the mixed material out of it and pass it to the distillation column. So it is nothing but a simple mixer and if you try to write down the dynamics for this what you would get is the way this concentration changes dC_A/dt,

$$\frac{dC_A}{dt} = \frac{F}{V}(C_{Ai} - C_A)$$

and flow rate F and volume V we are assuming to be constant, what we will get is a first order system.

So let us simulate and try to see what is the effect of frequency of C_{Ai} change on the change at the output of this particular system. So we have simulated this system.

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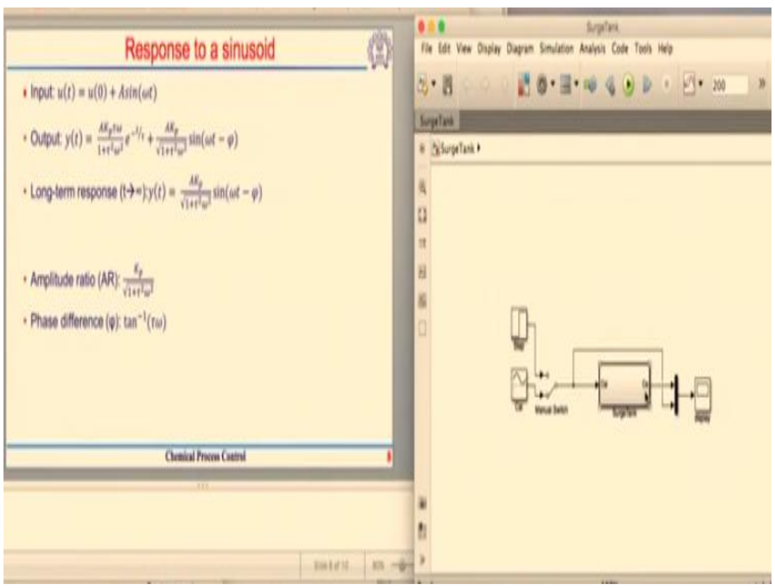

And what you can see is we are going to pass some sinusoid into the system as C_{Ai}, and we will try to monitor what do you get at the output. Now let me make it clear that at steady state C_{Ai} will be equal to C_A. So whatever is the concentration on an average that is going to go into the surge tank, the average composition coming out of the surge tank is going to be the same. So material balance will always get satisfied.

But we will just try to see how what is the effect of oscillations coming from the reactor and how do they get translated to the distillation column. So let us start with a very low frequency. We will start with 0.001 radians per second, and we will try to simulate what is the output of the surge tank.

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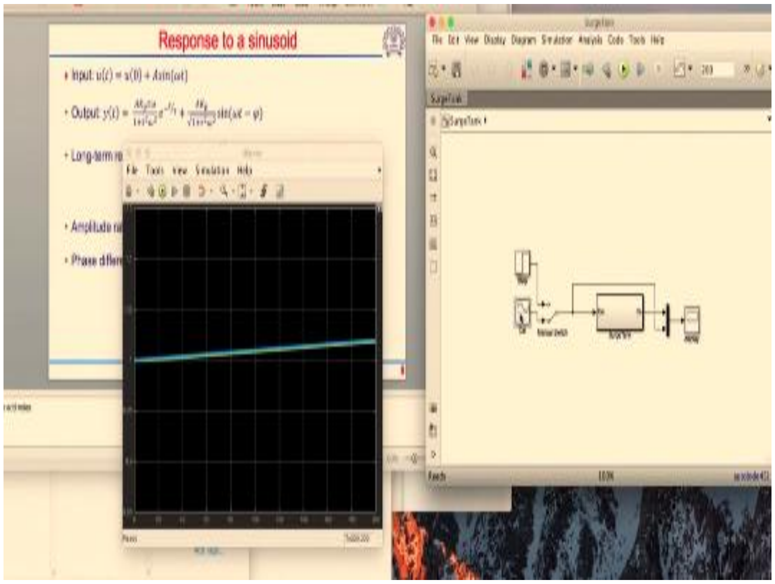

What we will see is, there are in fact two lines. So the input and output are matching. So what you feed in as the input, same thing you are getting as the output. Now as you start increasing the frequency of oscillation, we will start noticing the effect of the response of sinusoid to a first order process.

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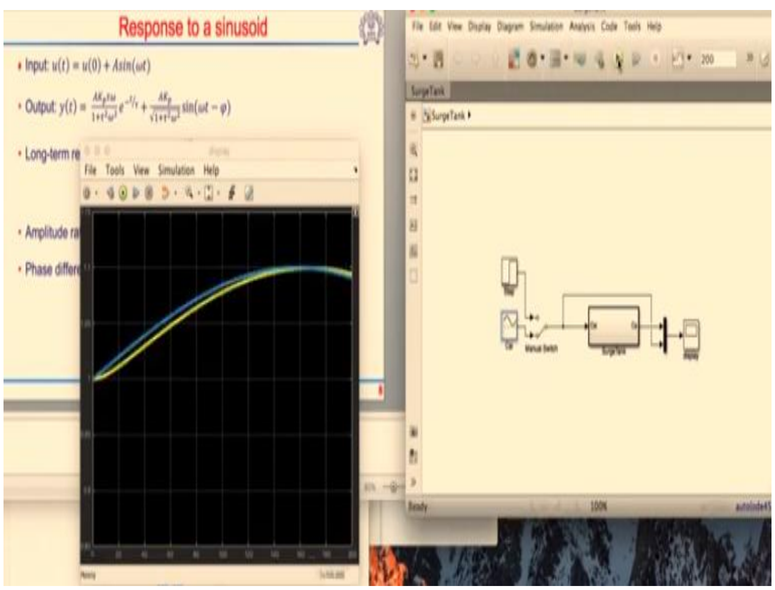

So see there is a difference between what is input and what is output.

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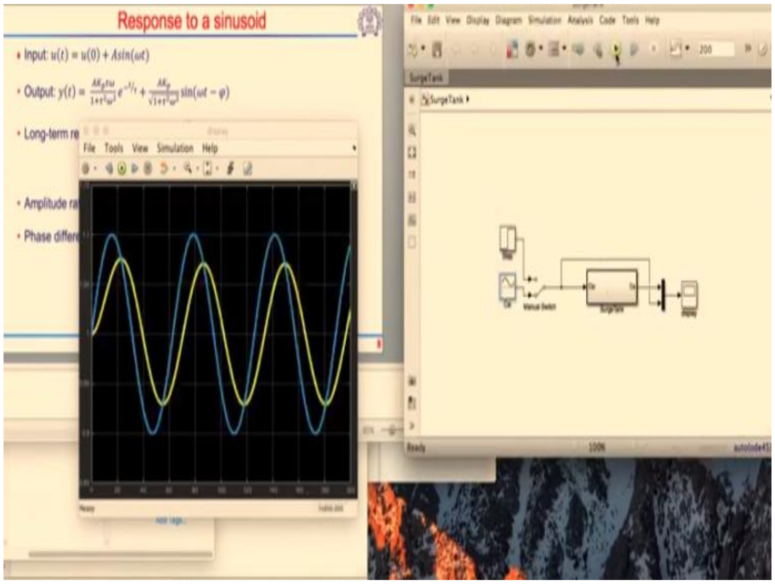

So you can see that now the output is in yellow and the input is in blue. So you can see that the input oscillates at an amplitude of 0.1. However, the amplitude of the output is less than 0.1. So you can see that the effect of oscillation has dampened out. So if the reactor is oscillating between 0.9 and 1.1, the concentration changes seen by distillation are definitely smaller than that. So it is kind of suppressing the oscillation, and the effect will get even pronounced if we go to an even higher frequency let us say 1 radian per second.

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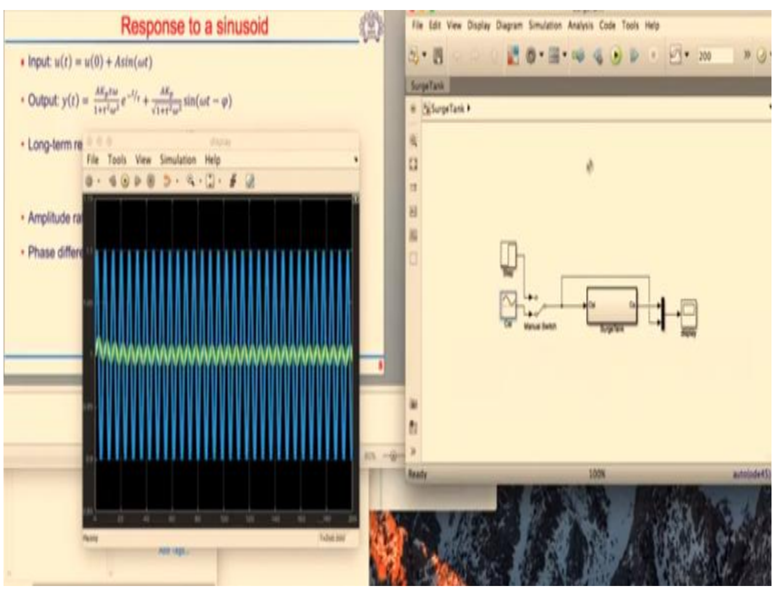

And we will see that even though input remains the same, the reactor outlet is oscillating between 0.9 and 1.1, but you can see that what distillation is going to see are much smaller variation amplitudes. So in a way what we are seeing is that on an average you can still see that the performance is the same. So on an average still the composition from the reactor is one and the composition going to the distillation column is 1. But you see that the variation has diminished significantly because of the use of this particular tank.

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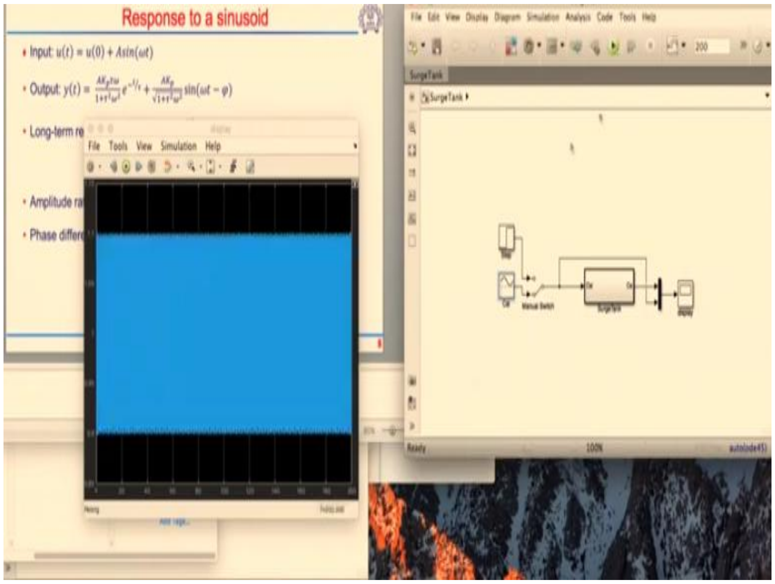

And if you say if we go to the frequency of 10 you will not be actually able to distinguish and if you reduce the time so we can see that there is hardly any variation in the output of the surge tank which is going to the distillation column.

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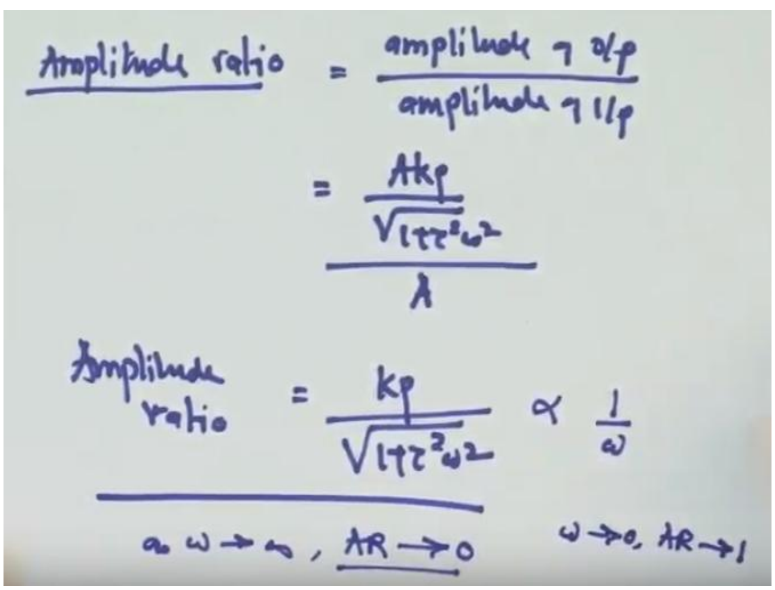

So all that is happening because our result was that the ratio of amplitude between the output and the input is inversely proportional to the frequency or inversely proportional to omega. So very high-frequency oscillation, the amplitude ratio will go to 0. Where $\omega \rightarrow \infty$, amplitude ratio will tend to 0, and that is what causes the suppression of these oscillations. So this tank somehow surges the oscillations, high-frequency oscillations coming from the reactor.

So that is why these are known as surge tanks, and you had seen that if the frequency is very small if the frequency goes to 0, then the amplitude ratio goes to 1. Especially for this particular case, in general, it will go to k_p and what you will see that all the very low-frequency oscillations which typically are some planned changes in the reactor they would directly get carried to the distillation.

So the whole purpose of this surge tank is that if you make any planned changes into your reactor, those changes will directly get transferred to the distillation column. However, if there are some unplanned changes, some fluctuations in the reactor, they would get suppressed by this surge tank, and this is kind of acting as a filter. So it is the filter which will reject anything which is of high frequency but allow anything which is of low frequency. That is why these first order processes are also known as low pass filters.

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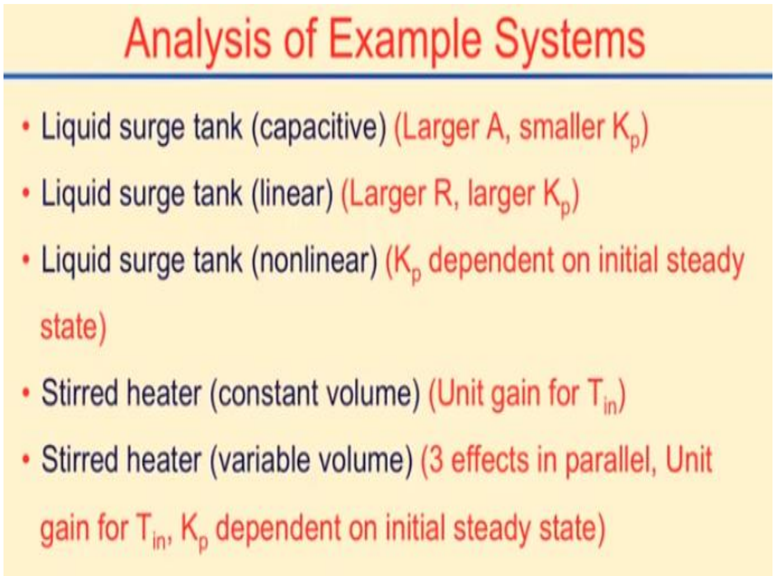

So let us revisit the examples which we have considered. The first example was our surge tank which was a purely capacitive process, and there was only one parameter k_p which was the gain, and that is equal to the area of the tank. So the implication of that is if it was 1/A so the larger the area of the tank, smaller is the k_p and accordingly what we had seen from the response is that a step input would give a linear response.

So the larger the area, smaller will be the gain, and it will take a longer time to overflow. The next example was the linear surge tank and there also the gain was equal to R which is the resistance of the valve. So the implication of that is if the valve is very restrictive, it is not going to allow the flow to go out of the tank easily then a small change in the input flow will cause a much higher rise in the height. So that is the implication of having a larger k_p.

When we looked at a nonlinear tank where the outlet flow rate was proportional to square root of the height, we linearized the system, and we made an assumption while linearizing that the linearization approximation works as long as we are not moving too far away from the steady state around which we linearize. And what we got was the k_p dependent on the state at which we linearized.

So the implication of that is if we try to make predictions out of this linearized model, they will be very good when we are close to the steady state. However, we move away from the steady state this prediction will start to show some deviations. So again we can see this from a simulation.

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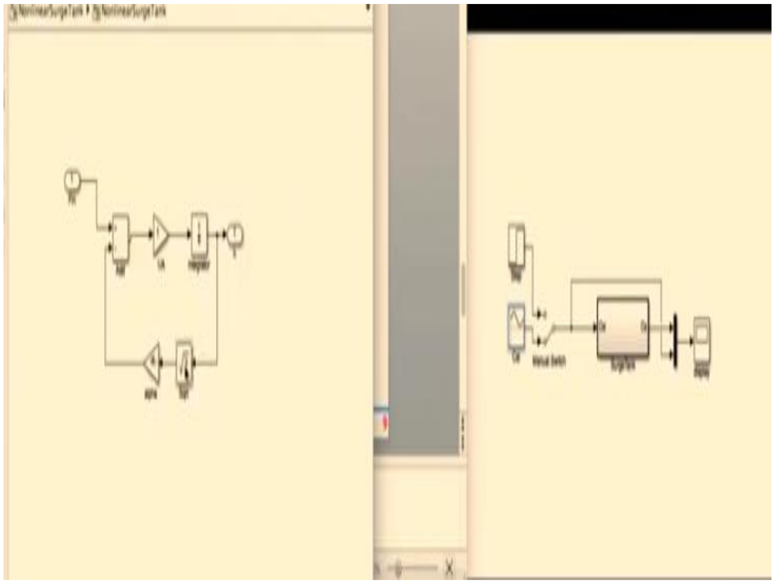

So here I have a simulation which compares the actual tank dynamics which are proportional, where the outlet flow rate is proportional to the square root of the height.

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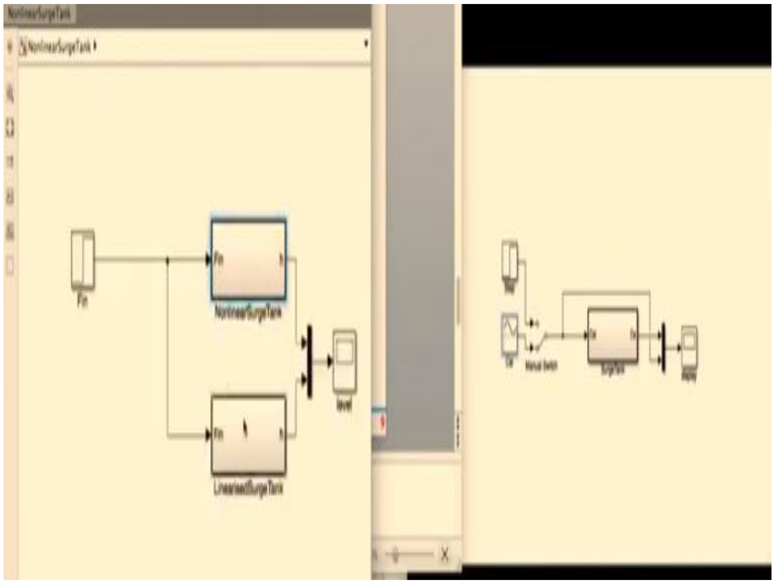

And here is the linearized version of that around the steady state and we gave a same step change to the input, and we tried to compare how do the two outputs look like.

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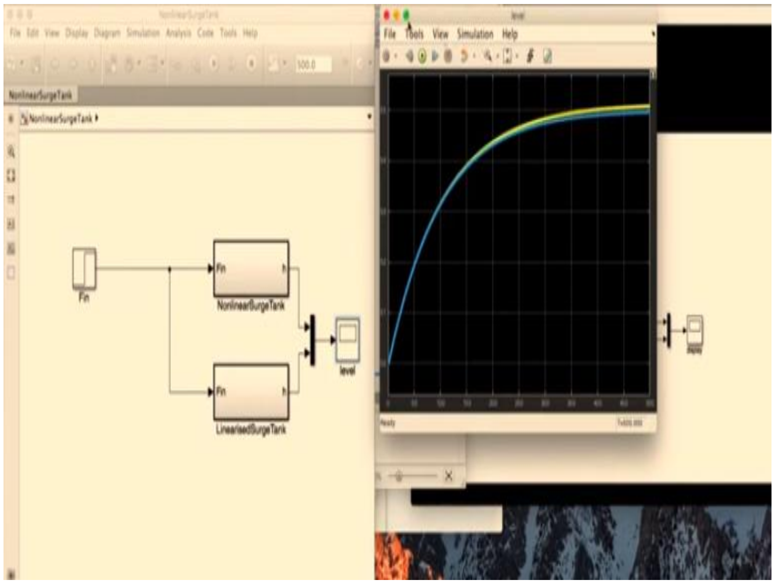

So what we can see that there are two lines here, one from the exact response, nonlinear response and the other one is coming from the linearized response.

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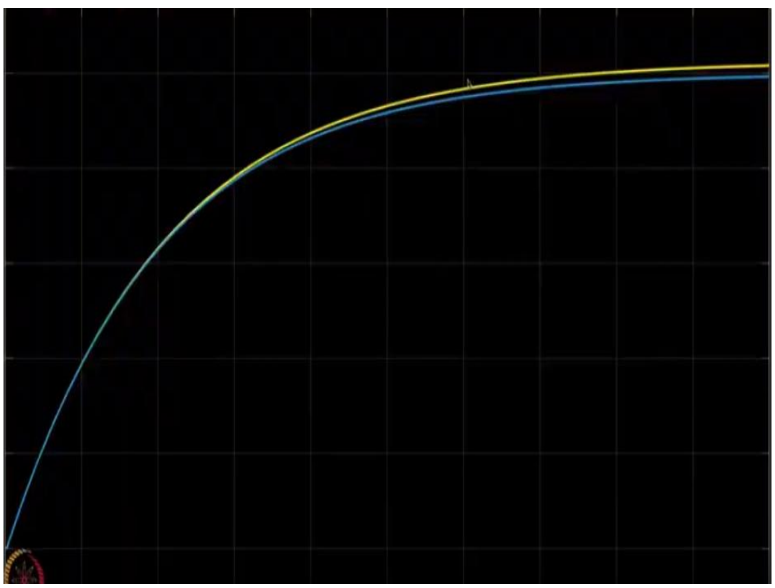

So what we can see that the yellow is the linearized response. So the gain if we multiply the A times k_p this is the predicted final change into the height and however what you can see that what you are getting is different than the actual response. The response starts to deviate from the linearized response because we are moving away from the steady state at which it was designed. In the case of stirred tank heater, our gain was one for input T_{in}.

So what it tells me that if the input, there is an inlet temperature change of 1 degree then the output temperature will also increase by 1 degree. So that is the importance of gain equal to 1. We also saw that surge tank example, that example also has gain equal to 1. So all those changes in the input directly get carried out to the output. And lastly, when we had a stirred tank heater with variable volume, there were three effects in parallel. All of them were the first-order capacity.

One of them was T_{in} which had unit gain and the other gains as we had linearized the system those gains were also proportional to the steady-state gain, as well as the time constant, were dependent on the steady state around which we had linearized the system. So to summarize for this week, what we have seen is first order systems are the simplest of the dynamical systems.

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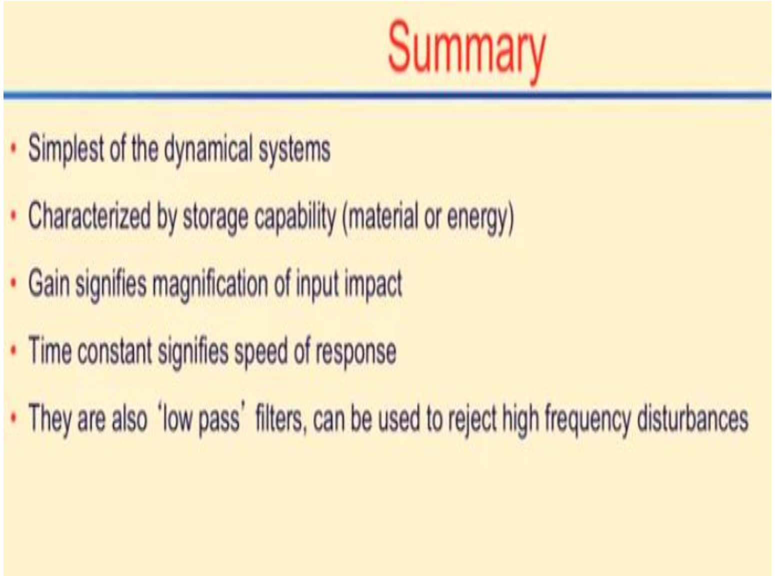

They can be represented by a system which has a capacity to store either mass or energy, especially for chemical engineering systems. They can also have some resistance to this capacity building. If they do not have resistance, we call them as purely capacitive processes. There are two parameters which characterize the first order system. First is the input-output gain which exactly tells what is the magnification which is going to be added to the input change when we look at the output.

And the other is a time constant which kind of gives you the speed of response. If the time constant is very large, the system behaves slowly, and if the time constant is small, the system responds very fast. And lastly through the response of first order system to a sinusoid and coupled with the example which we had seen about surge tank, what we could see is that these first-order processes are kind of act like low pass filter.

If there are input changes at a very low frequency, those changes get directly carried forward to the output. But if the changes at the input are of very high frequency, they will get rejected by the system. And this is a very important phenomenon which gets used when we are trying to decouple different parts of a process. Thank you.