Chemical Process Control

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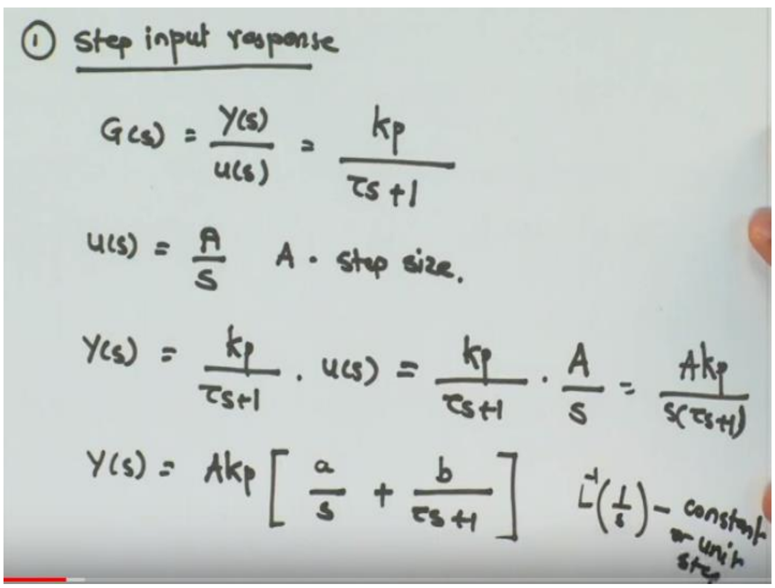
Lecture - 11
Response to Step Input

Let us now see how a first order system reacts to a disturbance. We will look at the two most commonly used disturbances. One is a step disturbance and other is a sinusoidal disturbance. We will try to see the responses of the first order system when there is a first order lag or purely capacitive system. Then we will try to see the significance of the two parameters ‘kp’ and ‘tau’.



The first disturbance we will analyze is the step disturbance or step input. We have a transfer function for our system which is,

$$G(s) = \frac{y(s)}{u(s)}$$

For a first order lag,

$$G(s) = \frac{k_p}{\tau S + 1}$$

We will look at the step response. The Laplace of step input is,

$$u(s) = \frac{A}{s}$$

where A is the step size.

As we have,

$$y(s) = \frac{k_p}{\tau s + 1}\ u(s)$$

So it gives,

$$y(s) = \frac{A}{s}\ \frac{k_p}{(\tau s + 1)}$$

Now in order to get the response in a real-time domain, we will have to invert this Laplace and the method we are going to use is the method of partial fractions. So what does that mean? We will split this transfer function or this Laplace transform into parts for which we know the inverse. In this case, we can write,

$$y(s) = \frac{Ak_p}{s(\tau s + 1)}$$

By applying partial fractions,

$$y(s) = Ak_p\left[\frac{a}{s} + \frac{b}{\tau s + 1}\right]$$

The idea is we have to find this coefficient a and b and then we know the inverse Laplace of 1/s.

$$L^{-1}\left(\frac{1}{s}\right) = 1$$

And

$$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

Once we have converted it into a partial fraction we will be able to get the inverse of the final output. Let us try to find out how we get these small a and b. In order to get any coefficient, we can simply substitute the appropriate value of s.

As we have,

$$y(s) = AK_p\left[\frac{1}{s(\tau s+1)}\right] = AK_p\left[\frac{a(\tau s+1)+bs}{s(\tau s+1)}\right]$$

Let us put s = -1/**τ**. So we will get,

$$\frac{-b}{\tau} = 1 \; and \; b = -\tau$$

Similarly, substitute s = 0.

In that case,

$$a = 1$$

Now we have,

$$y(s) = AK_p\left[\frac{1}{s} - \frac{\tau}{\tau s+1}\right]$$

Rearranging,

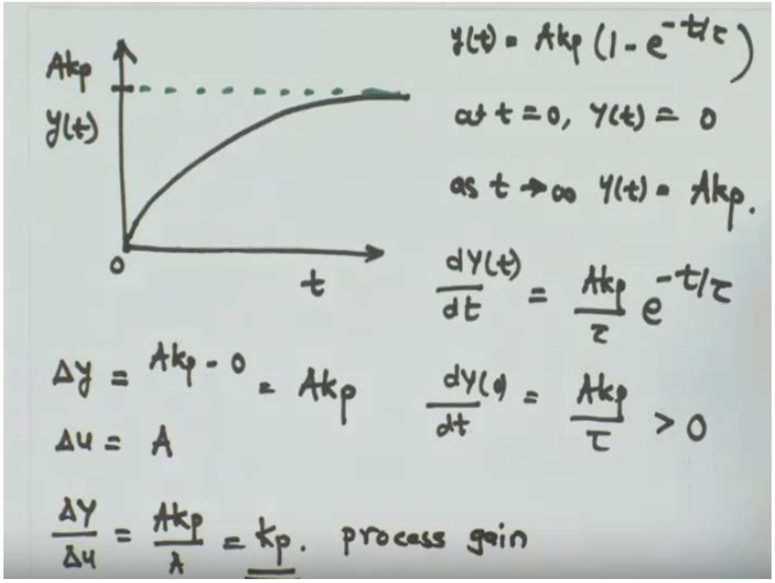
$$y(s) = AK_p\left[\frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}\right]$$

If we take inverse Laplace, we will get-

$$y(t) = AK_p\left[1 - e^{\frac{-t}{\tau}}\right]$$

So, this is the response of a first order system to step change of the size A.

Let us now try to analyze how does this solution look like and what is the importance of these two parameters kp and tau. If we plot the response of y(t) versus time, the plot we are going to get,



We know,

$$y(t) = AK_p\left[1 - e^{\frac{-t}{\tau}}\right]$$

So when t = 0,

$y(t) = 0$

So, the response will start at the origin. Now let us find the final value of this response. Let us consider t tending to infinity,

As $t \to \infty$, $$y(t) = Ak_p$$

If the dotted line in Figure represents ‘A kp’, then it shows that after a very long time, the response would somehow reach or approach this particular line. Now let us look at how does the response change as a function of time. We will calculate the derivative of the response which we can get as,

$$\frac{dy(t)}{dt} = \frac{Ak_p}{\tau}\, e^{\frac{-t}{\tau}}$$

If we look at the derivate at time t = 0, it will be,

$$\frac{Ak_p}{\tau} > 0$$

So your response will have a positive slope at time t = 0. The response is going to kick off at time t = 0 and if we simulate or plot the actual values of this function, the response would look as shown in the above figure which is the response of a first order system to a unit step change.

We will now look at what is the total change in output is the change from the start value to the end value i.e.

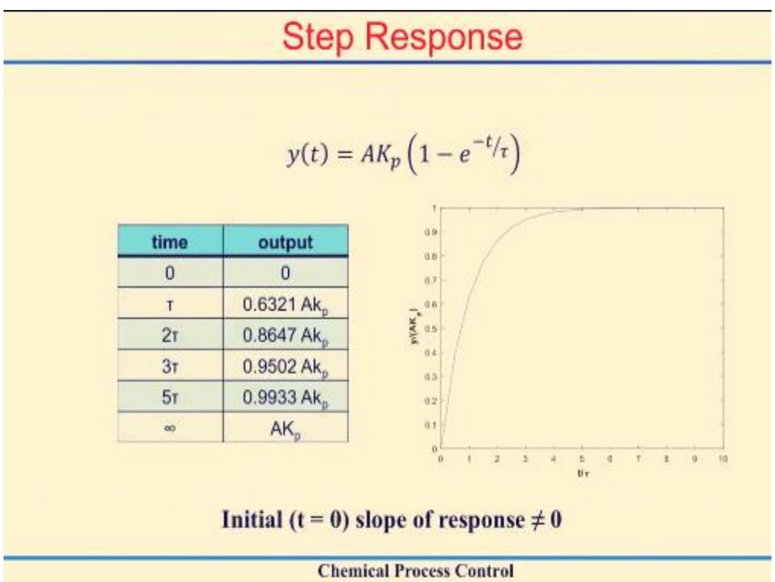
$$\Delta y = \; Ak_p - 0 = \; Ak_p$$

This change in the output was caused by a change in input of the value A.

If we look at what is the magnification of the effect of u, we can find out,

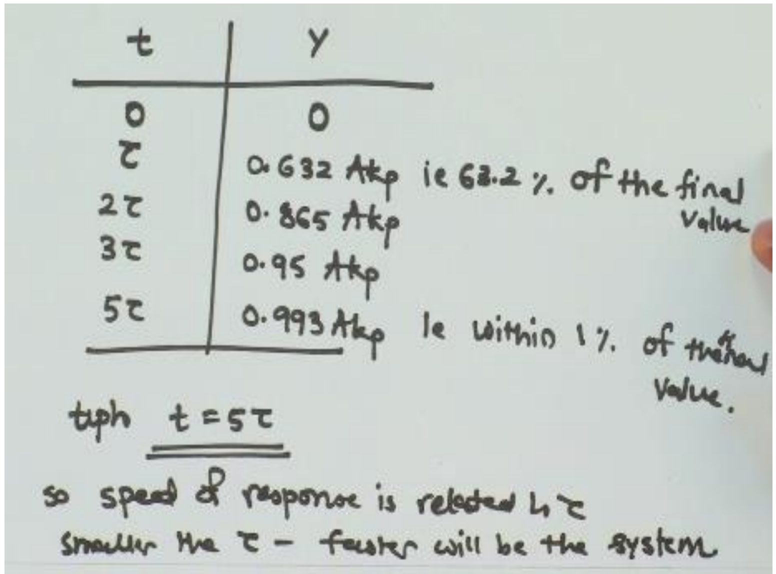
$$\frac{\Delta y}{\Delta u} = \frac{Ak_p}{A} = k_p$$

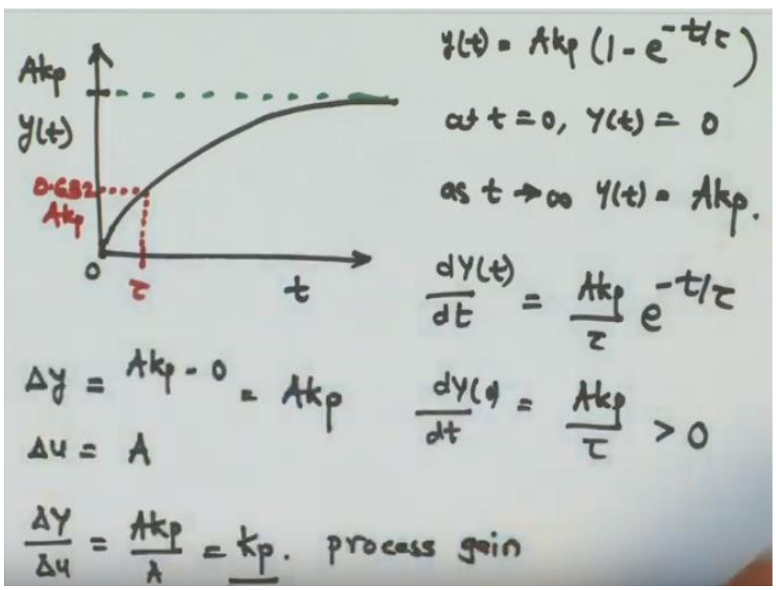
So 'kp' represents how much the effect of input gets magnified when we see that effect in the output and. It sort of acts as a gain between input and output and that is why it is known as a process gain. So, it tells me if the input changes by 5%, then the output will change by 5% times kp.



That is the significance of one of the parameters of the first order system which is the process gain.

Let us now try to see the response as a function of time and try to capture how the response changes. So we will make a chart of time and the corresponding value of y.





If ‘t’= **τ**, so if we go back to the previous slide, the response we get is,

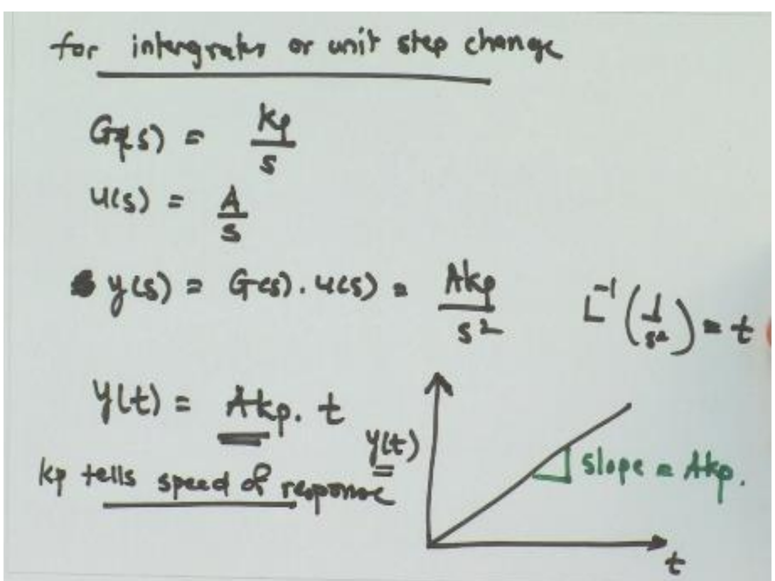
$$y(t) = 0.632\ Ak_p$$

It is the same as the 63.2% of the final value as the final value is ‘A kp’. So within the interval of time [0 **τ**], we have already completed the journey up to 63% of the final response. If we look at a similar value at ‘2 **τ**’, the value we are going to get is 0.865 Ak p. So within ‘2 **τ**’ the system reaches almost more than 85% of the final response. In '3 **τ**' the similar value will come out to be 0.95 'A kp'. And lastly, if we calculate up to '5 **τ**', it will come out to be 0.993 'A kp'. So it is 1% of the final value, within 1% of the final value. So for all practical purposes that are almost equivalent to considering the final value, so we do not need to consider the response up to infinite time.

We typically would consider the response up to t = '5 **τ**' which sort of tells you that the system has reached the final value. It means that the speed of response of a first order system is directly tied to this time constant '**τ**'. If the tau is small then correspondingly '5 **τ**' will also be small and then we will have, the system will reach its final value at a faster rate compared to a system which has higher '**τ**'.

So the speed of response is related to '**τ**', smaller the '**τ**' then faster will be the system. That is the significance of time constant. If we are comparing two first order systems, then we can compare the speed of response by comparing the values of '**τ**' for the two systems.

Now before moving forward, let us also see how a purely capacitive system or an integrator reacts to a unit step change.



We have the process transfer function as,

$$G(s) = \frac{k_p}{s}$$

We will take-

$$u(s) = \frac{A}{s}$$

So we have,

$$y(s) = G(s)\, u(s) = \frac{Ak_p}{s^2}$$

As

$$L^{-1}\left(\frac{1}{s^2}\right) = t,$$

$$y(t) = \mathrm{A}\, k_p\, t$$

The response linearly increases as a function of time and the slope of the response will be ‘A kp’. So, in this case, ‘kp’ somehow tells me about the speed of response.

Larger the value of ‘kp’, larger will be the slope and faster will be the system reaching its ultimate value which in this case is the physical limitation on how much the output can change. Mathematically it will go to infinity, but physically there will always be some limitation or maximum value of y which it can take. For example, if it is a tank, it is the 100% level of the tank which is the maximum value. So, the higher the value of ‘kp’, the faster will be the system response to go to the final or the extreme limit. Thank you.