Heat Transfer

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Lecture – 15
Transient analyses: Semi-infinite case

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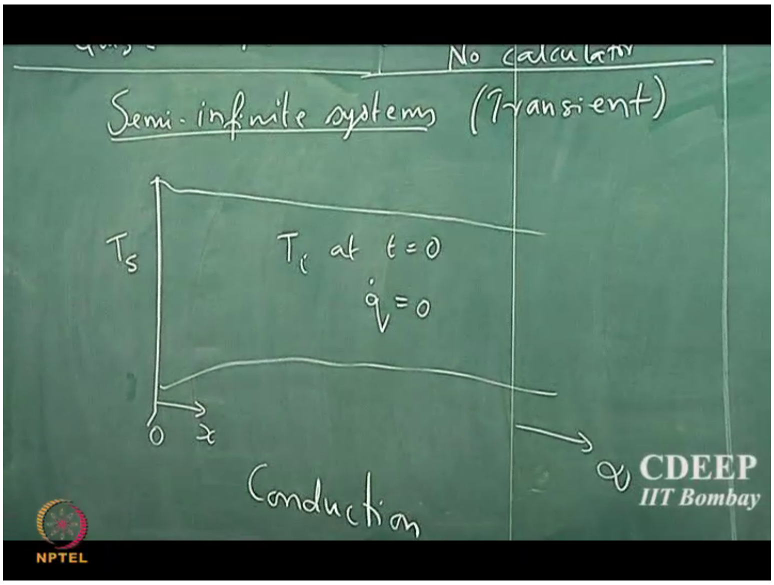

All right so, what we are going to see today is semi-infinite systems. So, all along the we had looked at boundary value systems, where you have specific temperatures or specific fluxes etcetera which is specified in either of the boundaries. We had looked at semi finite system very briefly when we thought about when we discussed the extended surfaces.

So, we are going to look at conduction in semi-infinite system, it is a much more detail today. So, let us say we consider a surface. So, let us say we consider a surface which actually extends all the way up to infinity in one direction. So, this is x, and this is x equal to 0. And it goes all the way up to infinity.

So, we are going to look at how to characterize the transient conduction in semi-infinite system. So, that is what, we are going to see we are going to look transient, a transients in semi infinitely. We did look at it with look at the steady state when we actually discussed the extended showpieces. So, let us see what happens when the transient actually what kind of role that these time axis place in these kinds of system.

So, supposing we assume that the temperature of the surface is Ts, and if we assume that the initial temperature. So, the temperature of the semi-infinite system is at let us say Ti that is the initial temperature, and it is not an unrealistic. So, if you have a very, very long thing for example, if you want to look at the conduction in a lake for example. You have one surface which is maintained at a certain temperature, and lake is a pretty big lake. So, you could in principle assume that it is a semi-infinite systems. In some of the swimming pools, the heating is also done in a certain way.

So, for example, not I think it is not the case in our swimming pool in IIT, but there are other swimming pools where the temperature of the water is always maintained. So, you have heaters placed at different locations, and the temperature of the water is maintained just about 25-degree c. So, it is very important to understand how the conduction process occurs. Of course, there will be conduct convection when somebody is swimming, but let us say when nobody is swimming then water is not displaced. And so, it is primarily conduction which is actually controlling the temperature transport, I mean heat transport process in that kind of a system.

So, one could assume that to be a semi-infinite system if it is a very, very long swimming pool. Depending upon where the heating elements are placed, one could assume in different cases. So, semi-infinite system all right. So, what are all the transport processes which are occurring here? Yeah, what are the transport processes? It is just conduction. So, supposing if I assume that the heat generation is 0. And I maintain that one end of the surface is maintained at a certain temperature surface temperature Ts. And the only process which is occurring, only transport process is conduction. Conduction is the only transport process.

So, what is the model?

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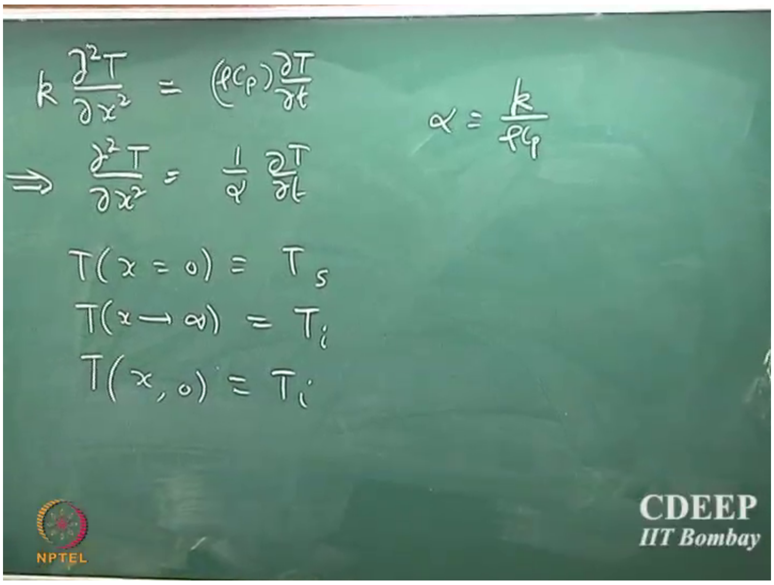

So, the model is a k d square del square T by del x square equal to rho Cp in to that is the model. So, we could rewrite this dou x square equal to 1 by alpha where alpha is k by rho Cp the thermal diffusivity of the semi-infinite system that we are considering. What are the boundary conditions? T at x is equal to 0, equal to T s. And T at x tending to infinity is given by Ti, it is not 0.

So, because it is infinite system, you could expect that the amount of heat that is transferred from one end of the surface is not penetrated sufficiently enough to the other end of the system that we are looking at. Therefore, the at x equal to infinity you would expect that the temperature is maintained at the same initial temperature of that particular series. And we could write initial condition. Now this Ti. Is that cleared everyone? Any questions on the so far?

So, this cannot be translated into a eigen value problem that we have been using all these days, right in the last 7 lectures. Whatever eigen value problem method we have used, it will not work here. Because one of the boundary is actually infinity. So, it is not a clearly bounded problem. Therefore, we need to use the a slightly different methods. So, which is called the similarity transformation method, or similarity variable method.

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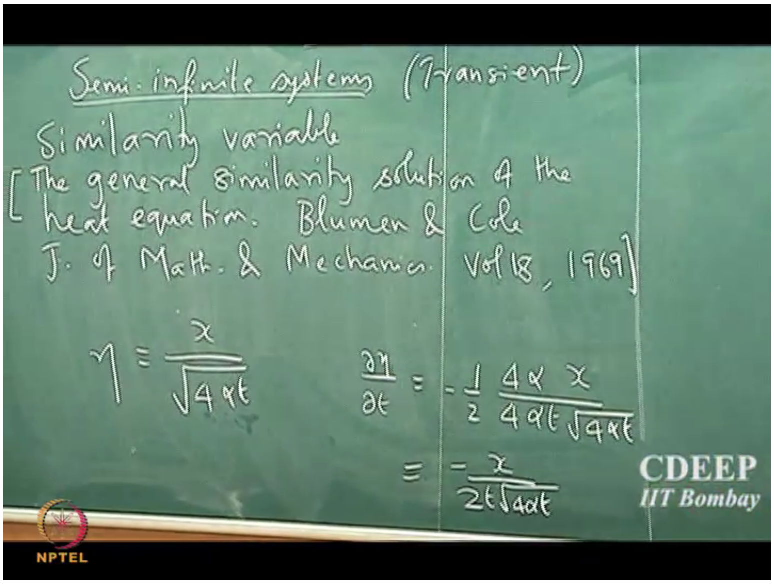

In fact, I am not going to give you what are the details of the similarity.

So, those who are interested in the math, you should actually read this paper. So, the quite a very well presented general similarity solution of the equation. And the authors are blumen and cole, and it is in the journal of math and mechanics. So, volume 18 published in 1969. So, that sort of provides a very general framework of how similarity solution has to be obtained for these kinds of heat equations in many different coordinate system. It is a very, very general method on how to use similarity variable for different time.

In fact, it also briefly tells you about how to find the similarity variables. So, this is as this is not a math course I am not going to discuss in detail as to how to get the similarity variable. But I will assume that this is the similarity variable. And in fact, there is one could use an intuitive argument as to why that should be the similarity variable.

So, for example, we said that when x tends to infinity. The solution is maintained the temperature at that location is maintained at the initial temperature itself, which is the same temperature at T is equal to 0. So, somehow the process which is occurring in the x direction, and time has to be related. Because in some location the temperature turns out to be the same temperature as the initial condition. So, this is the reason for trying attempting to relate the processes which are occurring in x dimension, and the time coordinates. For a moment you could assume time coordinate as a similar coordinate system as xyz. So, they have to be related to each other, because the boundary conditions specifies so.

So, if you look at this general similarity method if you follow it, what you will get is that the similarity variable is eta which is given by x by square root of 4 into alpha T. What is alpha? It is thermal diffusivity, right. So, it tells you the extent of diffusion per unit time. So, when you multiply by time, what you get is the square of the distance that the energy has traveled in that coordinate system, right. And square root of that will tell you what is the approximate distance free energy has traveled because of the conduction process.

So, that is the reason for having a scaling of x by square root of 4 alpha T. So, this 4 actually comes from the scaling argument which you will actually see in this particular paper if you read it carefully. So, really the rational is that alpha characterizes the diffusion thermal diffusion, and alpha and square root of alpha T in some sense it tells you what is the characteristic length of diffusion. So, square root of alpha T is the characteristic length of diffusion. So, note that there is no clear length scale right. In fact, that was the reason why we were not able to use the eigen value problem method.

There is really no clear length scale in that direction. So, one could define a length scale as square root of alpha T; which is in some sense dimensionally is equal to the thermal diffusion length. And so, we use that as a scaling factor for the x direction. And that is how this similarity variable is defined. So now, this 4; obviously, I told you that it comes from the general method if you use it properly a 4 will come out as a scaling factor. So, this is the transformation variable that we will use.

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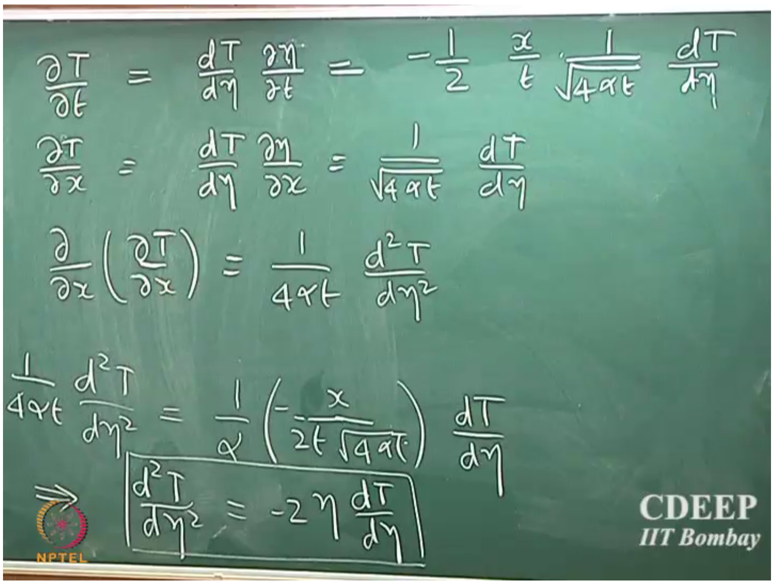

So now we need to define dT by dT we have to transform all the differentials in that particular similarity variable. So, we can use chain rule, the dT by d eta into d eta by dT. That is the total variable. What is dou eta by dou T? You can find from here. What is it? Minus dou eta by dou T is minus 1 by 2 4 alpha into x divided by 4 alpha T square root of 4 alpha T, right? So, that will be minus x by 2 T square root of 4 alpha T. That is the that is the first derivative.

So, that will be minus 1 by 2 x by T into 1 by what about 2 x? Similarly, we could write put dT by d eta dou eta by dou x. That will be dou eta by dou x is 1 by root 4 alpha T. So, that will be 1 by square root of 4 alpha T into dT by d eta. And we need to write the second derivative d by dx of dT by d x. So, that will be 1 by 4 alpha T into d square T by d eta square.

So now we have found the derivatives in terms of our new variables. So now, we can substitute that in the model equation. There will be 1 by 4 alpha T into d square T by d eta square, that is equal to 1 by alpha into minus 2 x by; oops, x minus x by 2 T into square root of 4 alpha T to d T by d eta. So now, we can do little bit of algebra, cancel a like terms. So, it will be d eta by d square is minus 2 eta into d T by d eta.

So, that will be the that will be the model equation in terms of the new variable. So, what has happened? Because we have related the processes the conduction process which is occurring in x direction. And the time that it takes for the diffusion to occur, we have actually reduced the number of variables or the number of dimensions if you use time as one of the dimensions. So, everything we have now expressed in terms of a o d e we have reduced the p d e into o d e by using a similarity variable. What are the boundary condition? T at x equal to 0 comma anytime T s or does it translate into our new variable? X equal to 0 is eta equal to 0, right.

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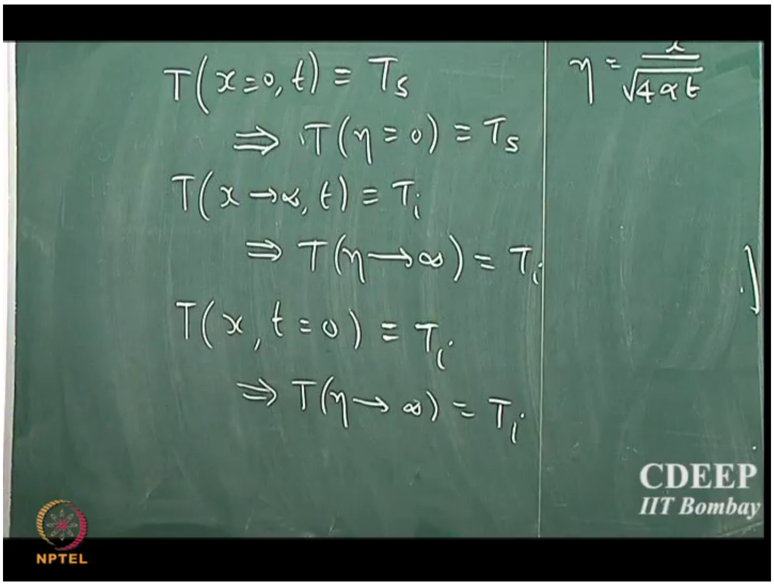

So, remember the transformation variables x pi square root of 4 alpha T.

So, T at eta equal to 0 is T s what about this boundary condition. So, this says T at as eta goes to infinity will be Ti. What about the initial condition? It is the same, because when you put T 0 eta goes to infinity right. So, that will be so, it is a second order ordinary differential equation. So, we really need only 2 boundary conditions. Because of the similarity variable that we have used, 2 boundary conditions that is required they have crashed into one boundary condition for the new problem. So, this is the correct problem. So, the way to solve this is any suggestions on how to solve this equation take d T by d eta on the denominator and

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Integrate. So, we can slightly rewrite this equational d by d eta d T by d eta equal to minus 2 eta into d T by d eta.

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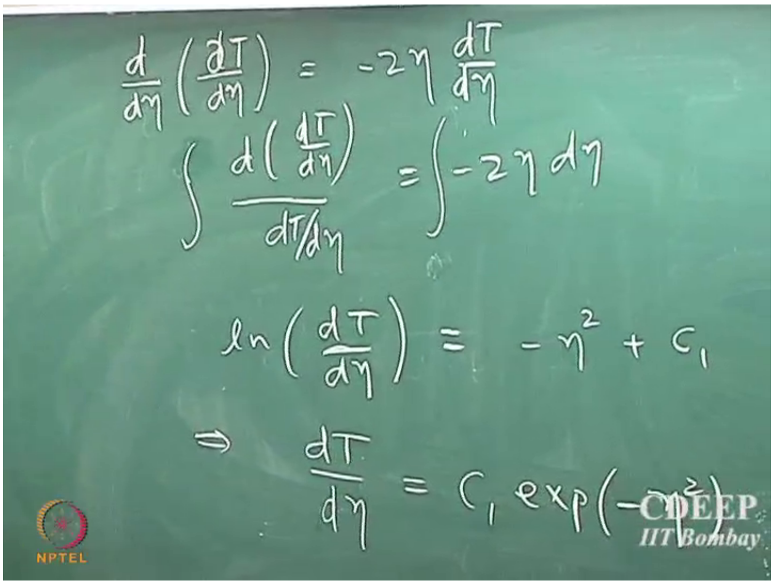

So, that is the model equation. And integrate this equation, integral d of d T by d eta. So, that is the model equation. And so, that will be log d T by d eta equal to minus eta square plus a constant c 1. So, we can rewrite this as d T by d eta equal to c 1 exponential minus eta square. That we just assume that the constant is some logarithmic form. And so, we can simply write this as c 1 into exponential of minus eta square. So, therefore, temperature T is 0 to eta c 1 exponential minus eta square plus some other constant c 2.

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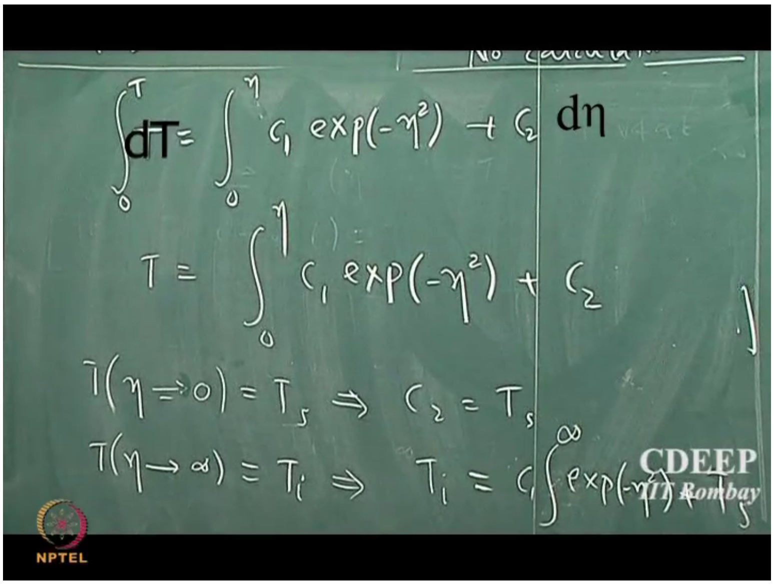

So, that will be T equal to when eta goes to infinity eta goes to 0.

So, we said T at eta equal to 0 is T s right. So, we substitute this boundary condition, what happens to this integral when eta goes to 0? It is 0 right. So, therefore, from here we find that c 2 equal to T s. And then you have eta going to infinity that is Ti. So, that will be Ti equal to integral 0 to infinity exponential c 1.

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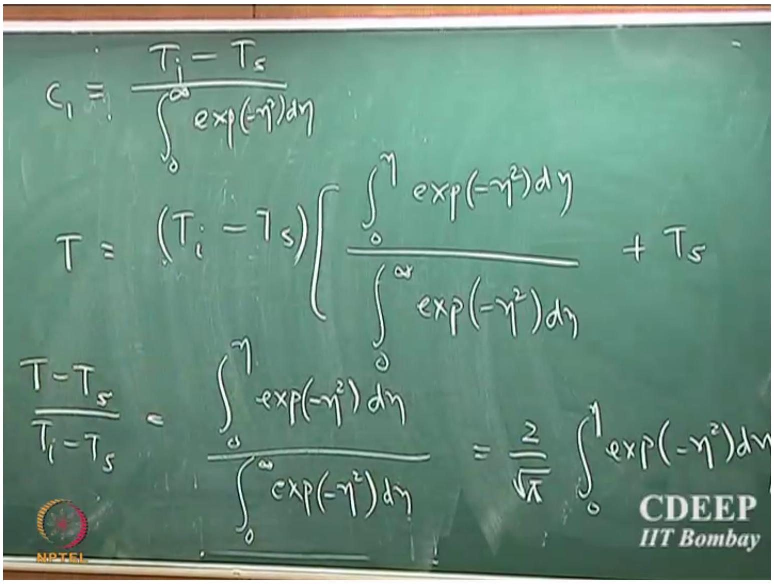

So, from here we find c 1 is T 1 minus T s divided by integral 0 to infinity of I called it Ti. Ti minus T into s integral that should be a d eta here should be a d eta. So, the solution is Ti minus T s into integral 0 to eta exponential of minus eta square d eta divided by plus T s. So, we can just rewrite this as T minus T s divided by Ti minus T s is integral divided by.

So, what is this numerator called as? Has anyone seen that expression before? It is called error function. So, it is a fairly well characterized function. And 0 to infinity integral as has already been calculated. So, the root pi by 2 very good. So, that will be 2 by root pi into integral 0 to eta exponential of minus eta square b eta. So, it is called the error function, and it is a fairly well characterized function. And all the properties of the error function is known.

So, therefore, we should be able to calculate the actual temperature distribution, which is the objective of the problem that that problem at hand right. They want always quantify what is the temperature distribution. Any questions on this so far? So, this whole thing is called the error function, those who do not know that. So, this is called the error function of eta. Any questions?

So, as we observed in the last several lectures, we said that what the what we want to really find is what is the heat transport rate, that is what we want to find.

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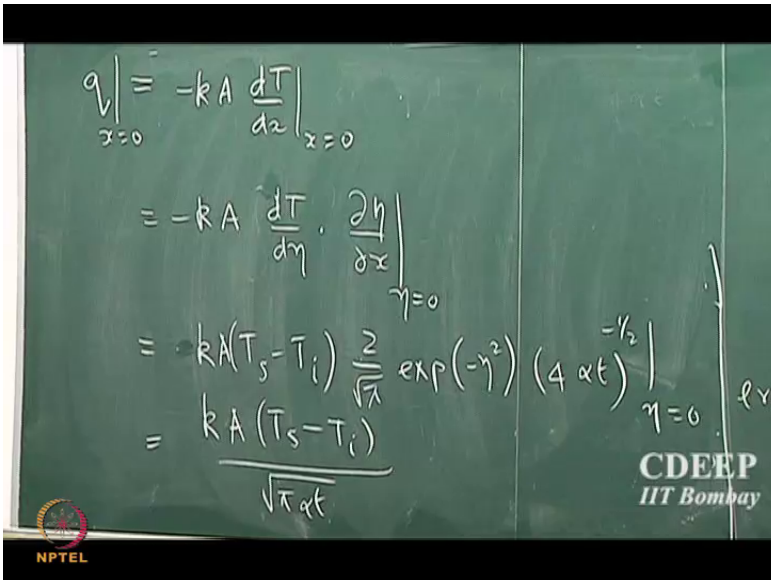

So, here when we say heat transport rate, what is meaningful is the heat transport rate at x equal to 0. We want to know how much heat is being transported at the you know the boundary where the temperature is maintained at T s. So, that is what we want to find. So, that is given by minus A into whatever is the area into d T by d x at x equal to 0. So, once again we can use chain rule. And we say this is minus A A into d T by d eta into dou eta by dou x.

And x equal to 0 is eta equal to 0. And so, that is what we want to find. And so, this will boil down to minus k T s minus Ti into 2 by root pi exponential minus eta square into 4 alpha T to the power of minus half evaluated at eta is equal to 0. So, that will be oops forgot a here; that will be k into minus sign is already taken care of. So, that will be k into A into T s minus Ti divided by root of pi alpha T. So, that will be the heat transfer rate. That is the rate at which heat is being transferred from the leading surface to the system.