Heat Transfer
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Lecture – 12
2D Plane wall

So, how do we characterize conduction process and heat transfer rate in 2 Dimensional system, as the rule of top natural systems the tough part is going from 1 D to 2 D going from 2 dimension to 3 dimensions is just a matter of routine repetitive procedure. For instance what you will see is that you start seeing multiple eigenvalue problem then you actually go from 2 to 3 D. We will not discuss 3 D systems in this class I would like to bring your notice that 3 D system is just an extension of what you would see in a 2 D.

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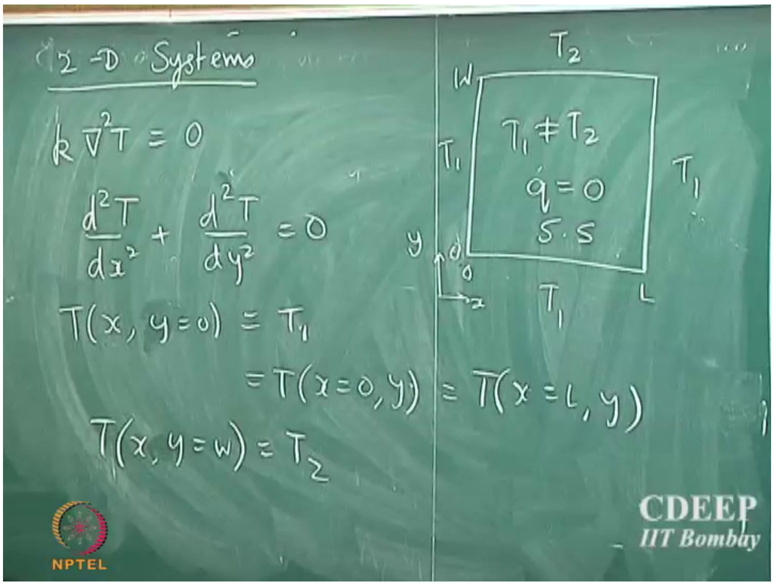

Let us take a simple slab a 2 D slab x direction and this is y direction, you need to coordinate systems and let us say the length in the x direction is L and the width let us say is W and in principle if it is 3 D there will be a third dimension which is outside the board and let us say that we define we maintain 3 sites at the constant temperature in 1 and let us say the fourth side is maintained at the temperature T 2. If I assume that T1 is not equal to T 2 and if I assume that q dot equal to 0 that is the net rate of heat generation per unit volume is 0.

I mean in principle you could include that also, but let us say to start with the net rate of heat generation inside the slab is 0 and if I assume that it is a study taken right. So, what are the processes which are involved here, what are the processing only conduction right, in a general system you have conduction and heat generation.

So, here you have only conduction which is involved, the model is very simple, k del square T equal to 0 right we said it is study state, accumulation is 0 and if I open up the (Refer Time:02:23) here. It will be d square T by d x square plus d square T by d y square this that is the model, what are the boundary condition very easy as specified temperatures, there are 4 temperatures in 4 boundaries and that is going to be the boundary condition.

So, T, I am going to use 2 coordinates as x and y and x, y equal to 0 the temperature is T 1 and that is also equal to T of x, 0, y, also equal to T of x, L which means that at x equal to L any y the temperature is T 1 and similarly in these 2 surfaces and the fourth boundary condition is any x and y equal to w the temperature is T 2, how do we solve this.

Student: (Refer Time: 03:35).

What about x direction.

Student: So, one (Refer Time: 03:44).

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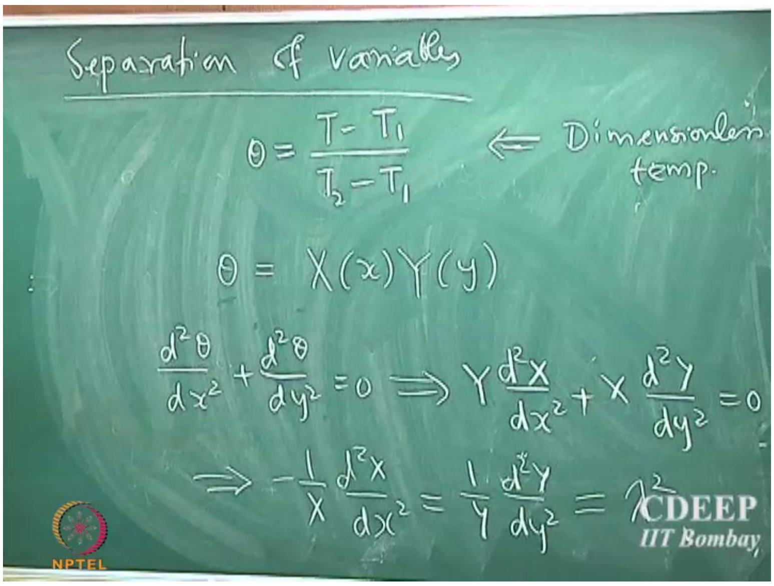

Well not quite so it is slightly different to use what is called Separation of variable.

If you knock of one by one then you will see that the eigenvalue problem that you will get you will actually not be a closed form eigenvalue problem, it is better to assume upfront that the separation of variable means that the x and y dependence of these 2 x and y dependence of this equation are independent of each other. Is that a correct assumption, why is that that separation of variable, who works for this does anyone know. So, whenever you have a linear problem, linear problems they have this property that you will be able to decompose the linear problem into it is multiplied linear components.

It linear in both x and linear in both y x and y direction therefore, the solution has to be a have product of 2 functions where each of these functions are dependent only on that coordinate system. The way to do that is you make a substitution lot like the way we did before, we said theta is T minus 1 divided by T 2 minus T1, now you use a dimension less quality.

So, note that theta is a dimension less quantity you have a ratio of temperature difference with another temperature difference. This is the first time you start seeing at dimension less temperature in this course I mean you could still use the original change of variables that we did, but this is just makes your life a little easy in terms of the solution and the solution comes out in a come more elegant fashion.

Now we assume that theta equal to some capital X which is a function of only x coefficient another function Y which is function of y coefficient, I substitute this equation, it is the original model. It will be d square theta by d x square plus d square theta by d y square equal to 0, now, if I substitute the separation of variables, representing the solution as a product of 2 different independent functions is what is called as separation of variables. I can write this as Y to d square X by d x square plus X into d square Y by d y square equal to 0, all I have done is I have substituted theta and then just evaluated the derivatives.

Now, I can rewrite this as minus 1 by X d square X by d x square equal to 1 by Y d square Y by d y square I can rewrite the expression like this . So, the left hand side if you look at the expression here the left hand side is only a function of x direction and the right hand side is only a function of y direction. So, what should be each of these quantities if you have 2 functions each of these are function of different coordinates and if they are equal then they should be it should be a constant. If you assume that this is a constant, if I put lambda square as the constant because these are completely independent functions and only way by which 2 independent functions can be equal to each other is when these 2 functions are a constant and that constant is lambda square.

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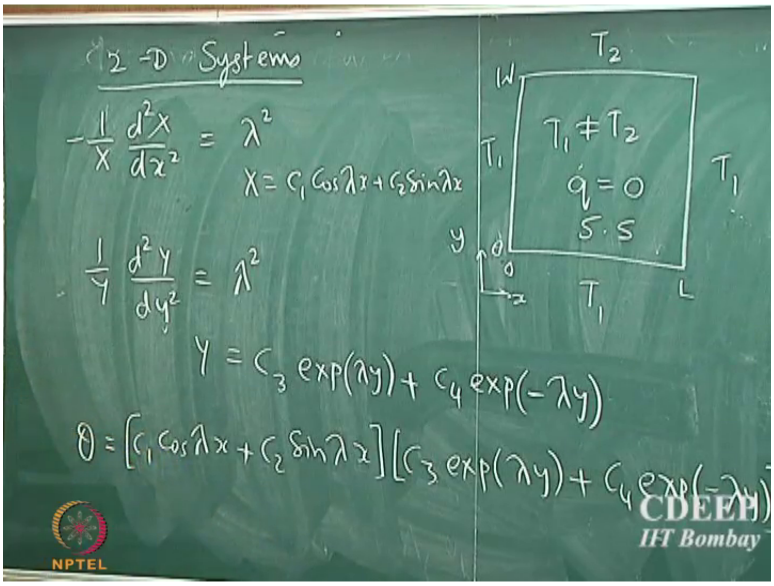

I can now break this down into 2 independent eigenvalue problems, minus 1 by X d square X by d x square equal to lambda square 1 by Y d square Y by d y square equal to lambda square I can write this down into 2 eigenvalue problems what is the general solution of the first problem general solution.

Student: Cosine sin.

Cosine sin, note that the eigenvalues are now imaginary, X will be C1 lambda x plus C 2 sin lambda x and what about the general solutions of Y it is exponential, will be C 3 exponential of lambda y plus C 4 exponential of minus lambda y . So, theta is now equal to C1 cos lambda x plus C 2 sin lambda x multiplied by C 3 exponential lambda y plus C 4 exponential minus lambda y, that is the solution. This is an excellent example of where some of the concepts that you learnt in your map courses are directly applied in during system. The boundary conditions are theta at x, y equal to 0 is what is the boundary condition.

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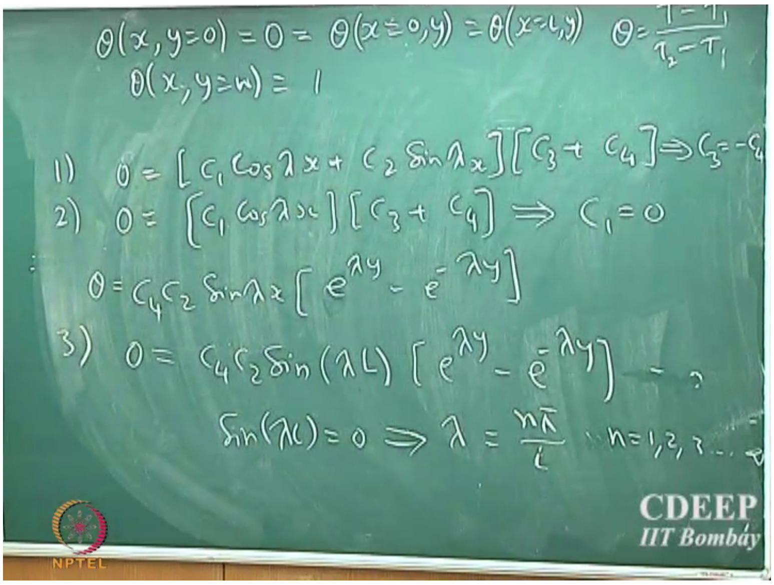

So, note that theta is defined as T minus T1 by T 2 minus T1, this is 0 and that is also equal to theta at x equal to 0, y and that is also equal to theta of x equal to l, y and theta at x equal to x, y equal to w that will be equal to 1. In fact, it is for this reason why we chose that type of a scale. So, your boundary condition essentially becomes 0 and one and it helps in solving the problem, if I impose the first boundary condition that will be 0 equal to, any x and y equal to 0.

If I set y equal to 0, then it will be C 1 cos lambda x plus C 2 sin lambda x multiplied by C 3 plus C 4 and if I put the second boundary condition that x equal to 0, we have 0 equal to C 1 cos lambda x multiplied by C 3 plus C4. So, from these 2 you can easily read out, cos lambda x is the eigenfunction of the eigenvalue problem and that cannot be trivial which mean it cannot be 0. Therefore, this imposes that C 1 equal to 0 and this imposes that C 3 equal to minus C 4 that is what, these 2 boundary condition will impose, theta equal to C 2 sin lambda x into I can pull C 4 out hereo, e to the power of lambda y minus e power minus lambda y.

Student: (Refer Time: 12:28).

Yeah.

Student: (Refer Time: 12:32).

No not necessarily not necessarily.

So, you will come to the third boundary condition.

Student: (Refer Time: 12:43).

It will all from 3, all it is that is fine you can put the third one here you could not matter actually how you put them it would not matter you will get the same solution . The third boundary condition you can impose and say that the C 4 C 2 sin lambda L into e power lambda y minus e power minus lambda y equal to 0, what is the solution of this problem

Student: (Refer Time: 13:16).

Sin lambda L equal to

Student: (Refer Time: 13:22).

Lambda equal to

Student: (Refer Time: 13:24).

Lambda equal to 2 n pi by L is that the solution

Student: (Refer Time: 13:29).

The reason is if C 4 or C 2 if neither of them are 0 then the solution theta is always equal to 0, you know that theta is not 0, therefore, sin lambda L is the one that has to be 0, sin lambda L equal to 0 means lambda equal to n pi by L what is the value of n.

Student: (Refer Time: 12:32).

Integer any integer

Student: (Refer Time14:01).

Except

Student: (Refer Time14:03).

Except 0 why.

Student: (Refer Time14:05).

You get a trivial solution if you plug in lambda equal to 0, if you plug in n equal to 0 then lambda is 0 and sin 0 is 0, what you get is a trivial solution where the temperature everywhere in the slab is 0, but we know that is not the case. Therefore, the permissible value for n is going from 1, 2, 3 up to infinity, those are the permitted values of n that can be applied here.

Now, comes to the fourth boundary condition.

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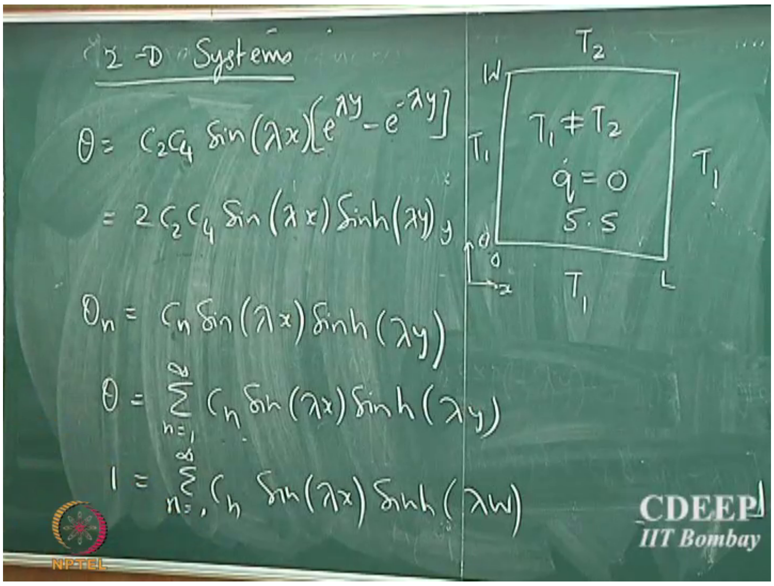

So, theta equal to C 2 C 4 into sin lambda x into e power lambda y minus e power minus lambda y, I can now replace this as sin hyperbolic, sin hyperbolic is just e power lambda y minus e power minus lambda y divided by q.

This is 2 time C 2 C 4 sin lambda x into sin hyperbolic of lambda y and as lambda goes from anywhere between 1 to infinity, you will get different values of constant for everywhere of n. Therefore, theta n will be C n sin lambda x into sin hyperbolic of lambda y, that is the value of theta for any n, but it is a linear problem and all of these are the likely solution therefore, theta is nothing, but sum of n going from 1 to infinity C n sin lambda x into sin hyperbolic of lambda y how do we find C n.

Student: (Refer Time16:09).

By plugging in fourth boundary condition by normalizing, which one.

Student: (Refer Time16:16).

No it is actually both, when you plug in the fourth boundary condition where you say y equal to w 1 equal to sum n equal to 1 to infinity C n sin lambda x into sin hyperbolic lambda w. So, you still have to find C n, how you use the property of the eigenfunctions. So, all the eigenfunctions for individual values of n or orthogonal to each other, sin pi by l is orthogonal to sin 2 pi by L and both these are orthogonal to sin 3 pi by L.

So, if you use the orthogonality condition.

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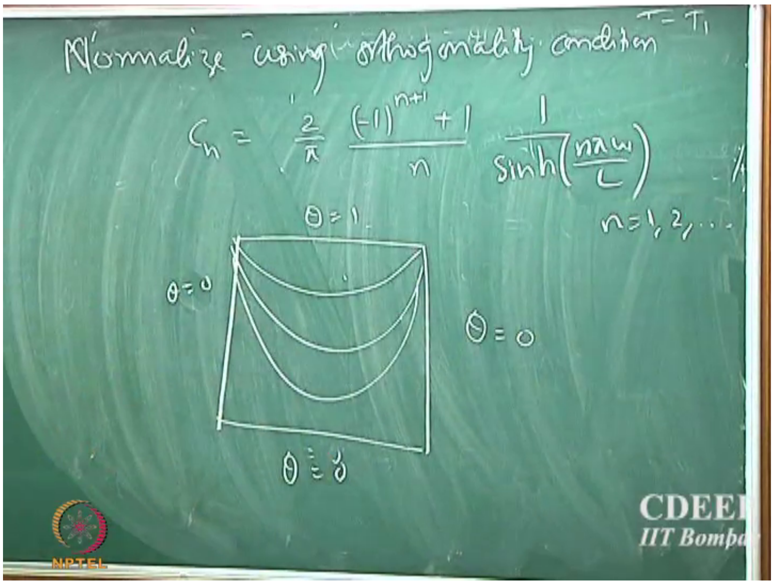

So, normalize using orthogonality condition, you will find that C n is given by 2 by pi minus 1 to the power of n plus 1, plus 1 divided by n, 1 by it is 1 by phi hyperbolic n pi w by L, n going from 1, 2, 3 etcetera. That is the value of the n and these are the isotherms that you will get these are the constant temperatures lines that we will find on the slab.

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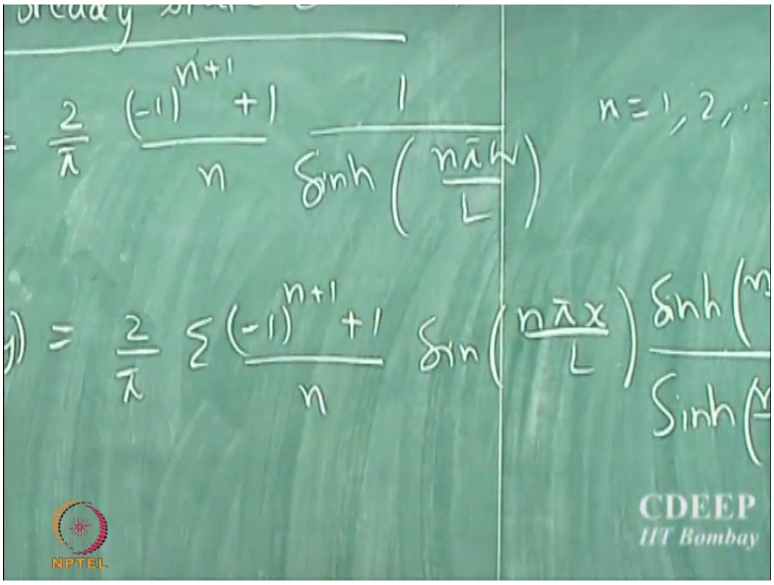

Final expression, the study state 2 D, we said that C n which is the coefficient in front of the series summation expression that is given by 2 by pi and minus 1 to the power of n plus 1 n into 1 by pi sin hyperbolic n pi w by L going from 1, 2, 3 etcetera and the solution is 2 by pi by n, sin n pi x by L sin hyperbolic n pi y by L, sin hyperbolic n pi w by L, that is the solution.

So, let us just quickly this look at what is the temperature profile and discuss a little bit about that before we march forward into the next topic.

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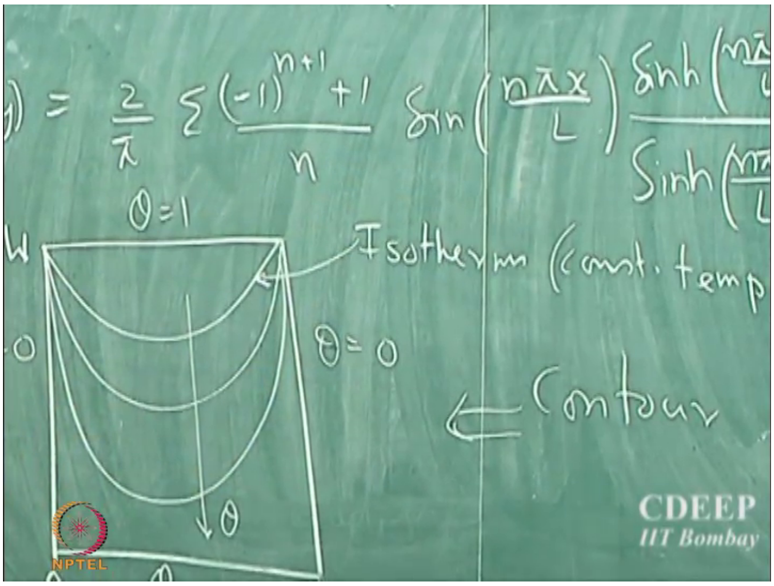

So 0 to L is the length in the x direction and 0 to W is the width in the y direction and according to the definition, that is the temperature specification along the boundaries and the profile would be that, this is the isotherm.

Student: (Refer Time: 20:17).

In principle you could draw 3 dimensional pictures you can draw a 3 dimensional surface of how the temperature is going to look, but the other way to do is what is called the contours, this is a contour drawing and contour is essentially every location you use these are the either the isotherm line or if you have colors you could represent the temperature with colors. This is the constant temperature lines along this slab here and naturally it has to be in the decreasing theta, as theta goes down, you can see this would be like could be like 0.75 theta is 0.5, 0.25 etcetera. So, these are the isotherms or constant temperature lines.

Is that clear to everyone and in fact, it is not hard to plot this and see to in fact, one should do that. If I think you must be introduce to math lab by now your numerical course, you should take and plot this expression and convince yourself that this is the temperature profile that you will get. There are some very simple tools in math lab to actually draw contours and 3 D pictures, whichever way you could actually plot them and you should convince yourself that this is the solution any question so far.

The next aspect of the course is we call lumped capacitance method we have already come across the word lumping, remember a couple of lectures ago I told you that when you want to crash a certain dimension because you want to assume that the gradients are negligible in a certain dimension then you integrate and what you do is called lumping.