Introduction to Evolutionary Dynamics
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Lecture – 05
Logistic Growth Models – 2

Hi, and welcome to the next lecture in the course Evolutionary Dynamics. And today we will continue our discussion on where we had of where we had left of last time discussion of modelling reproduction processes in a growing bacterial population. So, where we had stopped last time was we had derive the equation which we called the logistic equation which took into account the constraints which are imposed by an environment in resources being limited and hence constraining the populations up to which bacteria can grow. And just to recap the equation that we had that we have derived which is called the logistic equation.

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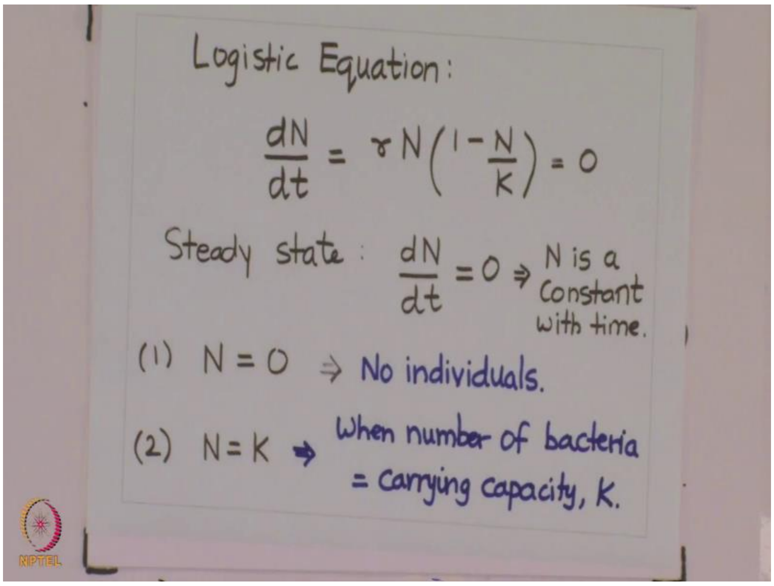

Can be written as dN by dt; where, N is the number of bacteria is equal to r times n, where r is the rate of the growth when bacteria are grown in an environment where there are no constraints times the factor which we had introduced to capture that is the capture the fact that resources become constraint is 1 minus N by K; and K is a variable which we called the carrying capacity of the environment and is only dependent on the makeup of the environment and it represents what is the maximum number of individuals that the environment can support taking into account the constraints that come into being because of resources being limited in the requirement.

And now, coming back to that equation and continuing from where we had left off last time, we will analyse we will start by analysing what are the steady states of the system what are the steady states associated with the equation that the system that this equation represents. So, steady state of a system will be that with value of N where dN by dt is equal to 0, which means N is no longer changing with time which implies N is a constant with time and to analyse what are the values of N at which this equation our logistic equation gives dN by dt equal to 0.

We simply equate dN by dt equal to this expression equal to 0 and what we get is that this condition is met for two values of N first is when N is itself equal to 0 and the second one is when N is equal to K. The first condition N equal to 0 makes this whole expression 0 because this N is equal to 0. So, we have a positive factor times 0 equals 0, and the second steady state N is equal to K makes this factor inside the bracket equal to 0. So, we have we have r times N times 1 minus 1 which is 0 equal to 0.

And what do these we will next continue and understand what do these two steady states physically mean. The first one N equal to 0 implies that there are no individuals; the first one N equal to 0 implies that there are no individuals in the population. Now if you think about it this is a steady state which means dN by dt is equal to 0 and N does not change with time. And this makes intuitive sense because if there are no bacteria in the tube then you do not expect the number of bacteria in the tube to change. N is equal to 0 to begin with; that means, no bacteria that you the culture media that we have talking about is free of bacteria and that will remain to be the case because there is no bacteria to initiate the process of growth. So, this is a steady state.

The other steady state N is equal to K represents the case when number of bacteria equals the carrying capacity of the environment which means we are defining an environment whose carrying capacity is K; that means, K is the maximum number of individuals that the environment can support and if we have a situation where the bacterial population has gone up to the numbers where the number of individuals is now K, the environment is no longer capable of supporting any more growth because it is carrying capacity is K.

And hence bacterial growth holds and the number of bacteria in that environment is limited to K. So, these are intuitively these two situations intuitively make sense that if you are operating in a case where N is equal to 0 no bacteria or the other case where N is equal to K we are operating at number of bacteria equal to carrying capacity of the environment, we should not see any change in numbers associated with the bacterium.

So, these math these two steady states which we arrived at mathematically from the logistics equation, also make biological science in the context that we are talking about of bacterial growth in a test tube. But what we want to do now is take us take one step further and analyse the dynamical properties of these two steady states and in particular we are interested in a concept called stability of these two steady states. So, to understand that let us develop our equation a little bit further.

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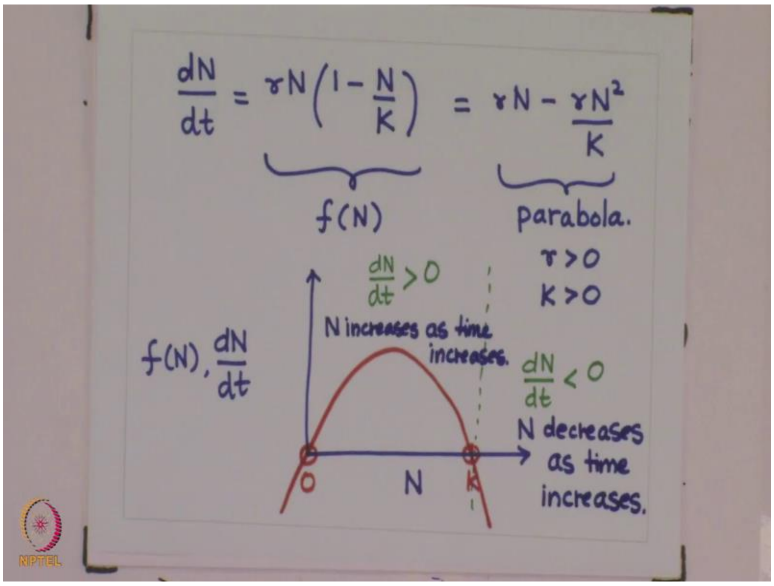

So, our logistic equation is dN by dt is equal to r N times 1 minus N by K. We can see that the right hand side is a function of the variable, and two constants which are r and K. R is a constant which is associated with the specie of the bacteria that we are dealing with because it represents the rate at which that specie grows, and K is associate with the carrying with the environment in which this experiment is taking place because it represents what is the maximum number of individuals that the environment can support.

So, what we are going to do is plot trying to a sketch of what happens to the quantity dN by d t; in other words f N as we change N that is the graph that we are going to try and plot, and will we will see what sort of insights does this graph give us. So, what do we know about this? We know that when N is equal to 0, f N is equal to 0. So, we know that when at N equal to 0, f N is equal to 0. We also know from our steady state analysis that at N equal to K, f N dN by dt is again equal to 0.

So, when we plug N equal to K, we know that f N is again equal to 0. So, we want to plot dN by dt as a N is changed, and we know the two starting points that N equal to 0 and N equal to K give me the value of f N as 0. Now to plot this graph we can simplify open the brackets associated with this equation and write this as r N minus r N square by K which represents a parabola. Because of the system that I am trying to model I know that this biological setup has only makes make sense only if r is greater than 0 which is the growth rate, because negative growth rate would not make any sense or and K which has to be equal to 0, the carrying capacity of the environment also has to be greater than 0, negative carrying capacity again does not make sense make any biological sense in the context that we are defining this equation.

So, for positive values of r and positive values of K, this is a parabola and I leave it to you as an exercise to check that this is a parabola which opens downloads. In particular for any value any positive value of r and K if I were to plot this curve, I would get this curve looking something like this right, this is a parabolic curve which opens downloads. Now what this graph tells me is that the values of dN by dt are more than 0 between 0 and K for any value of N between 0 and K the value of dN by dt is greater than 0 for any value of N which is greater than K, the value of dN by dt is less than 0 and for any value of N which is less than 0 dN by dt is again less than 0.

Now, because we are talking populations and N is the number of individual's, value of N less than 0 does not make any physical sense. So, we went to ignore this part of the graph and only focus on positive values of N. So, let us see. So, so far what has this graph told us that we can partition this into two sections, one is where N is between 0 and K and in this region dN by dt takes on a value which is greater than 0 because this graph dN by dt is above this horizontal line which represents y equal to 0 at all values between 0 and K. And the other section that is been divided that has been divided here is for all values of N which are greater than K, which represents the scenario that dN by dt is less than 0.

So, what does this tell us what this tells us is the fact that when N is greater than 0 N is between 0 and K or we are lying in this window dN by dt is positive; and dN by dt being positive implies that N increases as time increases that is the physical meaning which is associated with dN by dt being greater than 0. On the other hand dN by dt less than 0 implies that N decreases as time increases. So, again let us redraw this and see what these two facts tell us.

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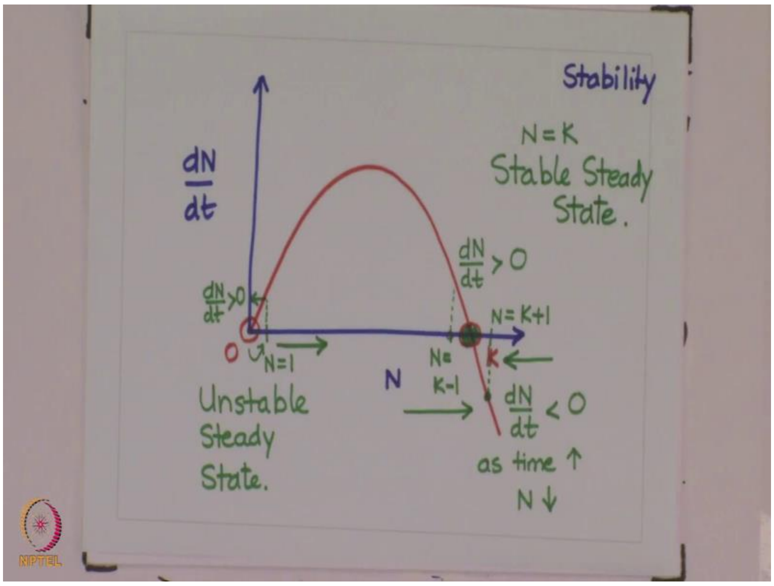

Now, by two steady states associated with the system are N equal to 0 and N is equal to K; and now we are going to define a concept which is called stability of a steady state. What does stability mean? By stability we mean that if we have a dynamical system which is operating at a particular steady state in this case we have to steady states N is equal to 0 and N is equal to K and these values represents the number of individuals at that particular steady state. N is equal to 0 represents that there is no bacteria N is equal to K represents that the number of bacteria is equal to the carrying capacity of the environment.

Now, we know that these are steady states, but we do not know if these are stable steady states or unstable steady states, which brings us to this concept of stability and we say that a steady state is stable if upon introduction of our disturbance to the environment; the system comes back to that steady state. On the other hand, if we apply a disturbance to the environment and the system runs away from that steady state that steady state is said to be an unstable steady state, on the other hand on application of disturbance if the system comes back to that steady state that is called a stable steady state.

So, want to comment on the stability of these two steady states that we have in our system. So, how do we look at that? The way we look at that is first let us first look at the first steady state which is N equals 0, in this case we want to comment on the stability. So, we apply a disturbance which means we are operating at we are operating at a situation where there are no bacteria we are operating at N equal to 0 and we know that at N equal to 0 dN by dt is also equal to 0. Now if we apply a disturbance and add one bacteria to the system; that means, we move from N equal to 0 to an equal to 1.

What we are interested in is from N equal to 1 does the system come back to N equal to 0 f N or move away from N equal to 0. If it comes back to N equal to 0 that means, our steady state that we are interested in talking about here is stable and if it moves away from N equal to 0; that means, this is an unstable steady state. So, how do we comment on that? All you have all we have to do at this point is introduce this disturbance. So, from N equal to 0 wave introduced the disturbance, and just look at the sign of dN by dt weather dN by dt at this point is positive or negative.

In this case at an equal to 1 dN by dt is positive and then by dt is positive means that as t increases N also increases which means as I go forward in time as t increases and also increases because dN by dt is positive, and hence my population moves towards the right. So, what this implies is that on introduction of a disturbance at N equal to 1 the population of this bacterial species that we are looking at runs away from the steady state that we are talking about. In terms of stability what that means, that the steady that the steady state associated with the value of N equal to 0 is unstable because if we move from N equal to 0 to N equal to 1 the tendency of the bacterial population is to continuously increase from there and move away from the steady state associated that we were talking about which is N equal to 0.

Hence N equal to 0 represents an unstable steady state of the system, this biologically makes a lot of sense because if we have a tube which does not have any bacteria it will continue to remain at an equal to 0 for as long as there is no contamination, but as soon as you have one bacteria introduced into the tube that bacteria is going to divide and further the numbers are going to increase and the population numbers the variable N is going to move away from N equals to 0 f N at a very rapid phase.

Hence our first steady state N equal to 0 is an unstable steady state, what about the second one? So, let us look at the second steady state which is N equal to K, now N equal to K we can introduce our disturbance in either of the two directions; we could introduce a deviation by adding an individual or subtracting an individual. Let us first subtract an individual and see where the system takes us. So, from N equal to K we apply a disturbance where N is equal to K minus 1. Again we are interested in tracking the movement of the system up on introduction of this disturbance and all we have to do to understand that is look at the sign associated with the variable dN by dt at this particular value.

So, at N equal to K minus 1 the sign of dN by dt is again positive; dN by dt it can be clearly seen from this graph is a positive quantity. So, at this value dN by dt is greater than 0 and again what; that means, what dN by dt greater than 0 means is that as t increases N also increases with increasing time the variable N the number of bacteria in the species increases and hence it the graph will move towards the right. So, dN by dt greater than 0 which means as t increases N increases and hence if we were to bring the system by introducing the disturbance to N equal to K minus 1 as time progresses the system moves to the right.

What that means, is that upon an upon introduction of a disturbance which brought the system to the left, we are again moving towards the original state of the system N equal to K. So, this steady state is stable as far as the left hand side is concerned; what happens if the disturbance that was introduced to the system was on the other side. So, if we introduce a disturbance now where N is equal to K plus 1, again all we have to do is look at the sign associated with dN by dt and in this case we can clearly see that at value N equal to K plus 1 dN by dt now has a negative sign. So, dN by dt has a negative sign which means as time increases N decreases.

Now, what; that means, is if the disturbed state of the system is equal to N is equal to K plus 1 and now if I allow that time to move forward and I monitoring the system N will decrease and hence the system will move towards lower N. So, this tells us about the complete story of the stability of this point, if you were to introduce a disturbance which was to the left of the steady state the system move towards the right which is back towards a steady state we started with or if we introduce the disturbance which move the system to the right of the steady state the system now moves to the left which is again towards the steady state that we started with. Hence N equal to K because irrespective of which disturbance we apply the system wants to move back to N equal to K this steady state N is equal to K is called a stable steady state.

In the context of bacterial growth let us try and understand these to steady states with again going back to the example that we had discussed in last class, growth of bacteria in a test tube. Imagine a imagine a test tube with liquid media in it, but no bacteria in it; that means, we are operating at a steady state N equal to 0. Now for as long as we do not introduce any bacteria in to the system the population size remains at N equal to 0; that means, we are operating at the steady state N equal to 0, but as soon as we introduce one single bacterium into the system, that bacterium is going to start replicating and take over the entire population the entire liquid media and the numbers of the bacteria in the test tube are going to keep on going up.

But the numbers will not keep on going up forever as we saw in the last two letters, but will be constrained by the environment that this growth is taking place in, and that constraint is represented by the variable K which is the carrying capacity in our model and hence the population growth will eventually stop at when at a situation where bacterial numbers are equal to K, which represents the stable steady state associated with our system.

So, that is this analysis of stability of the system; let us also analyse the stability of the simpler system that we had talked about what we had started with where we had not taken into account resource limitation and see what that tells us.

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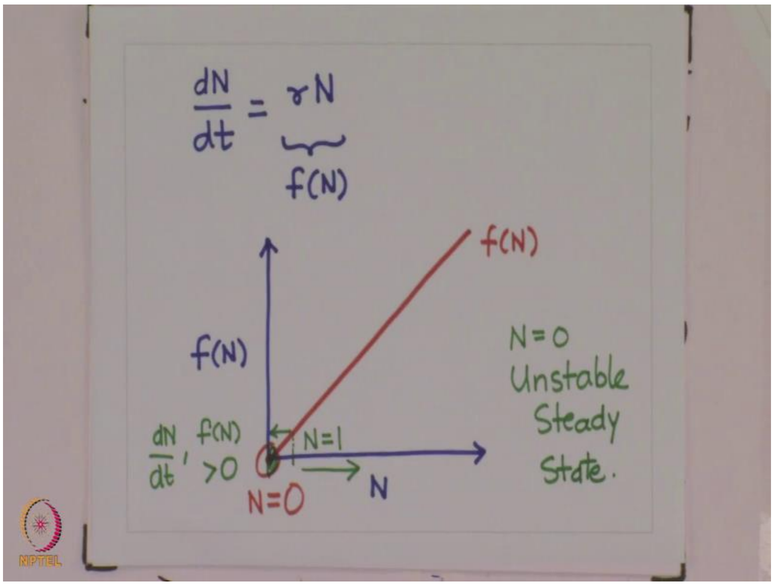

So, if you remember from the last lecture that the differential calculus model that we started with, where resource limitation was not taken into account was dN by dt equals r N. And now the f N the right hand side is simply a linear function r N and all I am going to do now is plot N against f of N, this is just a straight line at N equal to 0, f N is equal to 0 and this increases linearly as N increases. So, this is f N as you can see the steady states are those points where f N curve meets the f N equal to 0 the x axis line, and in this graph there is only one point where x axis and the curve f N meet which is this point N equal to 0.

Again because we are talking populations, we are only interested in positive values of N negative number of individuals of a specie does not make any biological sense and hence we are only going to introduce a deviation which is to the right of the system. So, if we if you are operating at N equal to 0, and we introduce a disturbance such that N is equal to 1 we want to see whether the system wants to move back or move forward in time. So, we introduce this deviation N equal to 1 and now to decide whether the system is moving to the left or the right all we have to do is look at the sign associated with f N at N equal to 1 which in this case is positive. So, f N or dN by dt at N equal to 1 are both greater than 0; dN by dt greater than 0 means that the system will move towards the right as t increases N will also increase and hence the system moves towards a right f N and as we saw in our last lecture the number of bacteria in the population keeps on increasing towards infinity and it never halts and because this f N curve never meets the x axis, this movement towards the right never stop and the bacterial population moves toward infinity.

What; that means that the only state steady state associated with the model that we had developed last time is N equal to 0 which is an unstable steady state? So, in this lecture what we have talked about is the concept of stability of dynamical systems, dynamical systems are often very complicated and getting an analytical solution for these differential equations is often not possible. By using this approach we can simply analyse the dynamics by commenting on the quality of the steady states whether they are stable or unstable in nature in the term in the sense that up on introduction of a disturbance do systems want to come back to these steady state or do systems want to run away from these steady states and these have biological implications and this math can be related very nicely to how we understand biological systems to perform.

So, let us stop here and we will continue this discussion in the next lecture.

Thank you.