Advanced Numerical Analysis
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Lecture – 09
Problem Discretization Using Appropriation Theory

Okay, good morning. So till now, we were looking at concept of vector spaces or fundamental of vector spaces and we revised many concepts, all are very useful in subsequent development, where we will be talking about solving problems numerically. So, we started with concept of a general vector space. We qualified sets which can be denoted as (()) (00:55).

So, the nice things that we like in 3 dimensions have now been made available in any other vector space. You have orthogonality, you have orthogonal functions, you have orthogonal sets, you have orthogonal polynomials and now we are poised to start using them to solve problems numerically. Well, what kind of problems? In my first lecture, I talked about a classification of problems.

And also on the moodle web page, I have uploaded one module, module 0 which talks about different problems in chemical engineering. Now almost, all of you are post graduate students, so I am not really talked about this problem in the board. We (()) (01:59) through assignments or even through when we do some developments.

So, physical problems you already know where you get partial differential equations, where you get set of algebraic equations, nonlinear algebraic equations, ordinary differential equations and so on. So, these things I am assuming that you already know to some extent that is the motivation from the physics or chemical engineering point of view, where do these problems are rise.

My motivation or my next aim is to look at problem transformations. So, first of all what I am going to do is, using the language of vector spaces that we developed till now, I am going to represent different equations that you are familiar with or that you are going to solve as a part of this course and then I am going to show you that actually, we cannot solve them in most of the cases exactly, analytically.

We have to construct numerical solutions, but these numerical solutions are not in the same space as that of the original problem. So, the spaces associated with the original problem and the spaces associated with the solutions are different. So, intrinsically we are finding an approximation to the truth and you should be aware of this reality. So by second module, which we will almost we about 12 to 14 lectures will be devoted to problem discretization.

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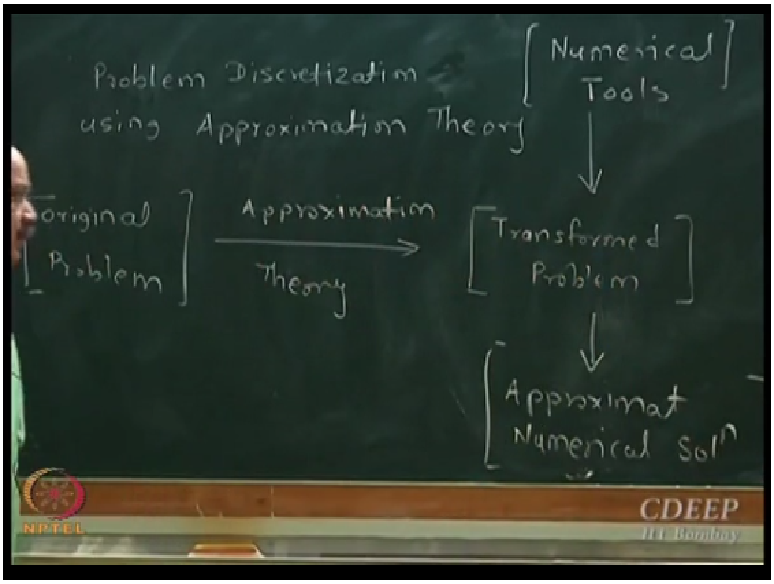

Now, approximation theory is a branch of applied mathematics and we obviously cannot do a justice to this in few lectures. We are just going to see only some part of it, not everything whatever is relevant to this course, but between the problem which you are defined from physics, your real world problem and the solution stands this approximation theory. In most of the cases, we cannot solve the problems analytically.

We have to resort to some kind of numerical approximations and so what is involved, what is conceptually happening when you go from the original problem to the transform problem. First of all, can we represent all these problems? Now that we have defined this new concept of space, can I generalize my idea of you know how a problem is in general defined in applied mathematics.

Can I say that all these problems are and some says the same problems, can I classify these problems? What kind of problems that can arise in applied mathematics or in numerical analysis for solving engineering problem? So, let us look at this classification and then, so we will get an overview bird's eye view and then we will start looking at specific solutions. So, everything that we developed till now, you will start seeing its applications as we go along in the next 10 to 14 lectures, okay.

So, first of all you should know that you have an original problem here and then you use approximation theory and then you get a transform problem and this transform problem is something that we solve using different numerical techniques, okay. So, we applied different numerical techniques or numerical tools I would say, so you use different numerical tools and then use all an approximate numerical solution.

So this is very, very important and if you get a bird's eye view of what is really happening in approximation theory, then given original problem, you will be able to think of a transformation. There is no unique way, what you will realize when you do these transformations. I will take the same problem and show you how it can be transformed in multiple ways, okay. I will start with the boundary value problem.

I will show you that this you know transformation A, transformation B, finite difference method, orthogonal collocation method. There is a different way of transforming the problem and you know finite element method and so on. So same problem can be approximated in different ways, well each one of them has advantages and disadvantages. There is one way which is unique in such that you know text case of all the difficulties.

There is something to be given and something to be taken. So, it is always a given take. So, you get a transform problem and then you use the known numerical tools like algebraic equations solving, nonlinear algebraic equation solving and get approximate solutions. Then, well as a chemical engineer or the physicist, you should go back and see whether the physical solution that makes sense here.

There are so many issues when you do from here to here. For example, the transform problem could be ill condition. The solution that you get may not be good solution because of you know ill conditioning problems and so on. So, just because I have a very good computer and I have a good you know powerful program with me to solve it, does not mean that my solution which I get is finally correct.

There could be problems on the way, there could be problems here, there could be problems here, okay. So, again for some time you will see generic things and then, we will start getting into specifics. My first task is to define a transformation, so what is the transformation? Transformation is any operation or any rule that text and element from a space x and transform it into an element in space y, okay.

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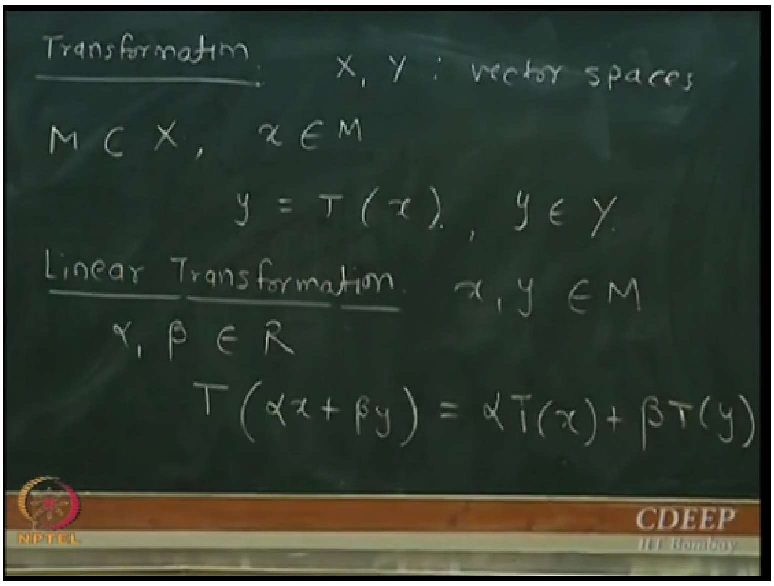

So, I have here 2 vector spaces. I have vector space x, x and y are vector spaces and if some subset of x, well I am still writing generic things, but as we go along all these ideas will become fall in place will become clear. What it means in concrete terms and terms of. So, transformation is a rule that associates with every x that belongs to m and element y which is, okay.

I have not written the words, but the bare definition. If I am given 2 vector spaces x and y and I take a subset m belonging to x, okay and then, if I pick any element x, then a transformation is a rule that gives me element y from space Y, okay. Before I move on to give you examples of set transformations, well you are aware of some transformations and I will show you that everything that we need to solve as a part of this course, actually can be put into the generic framework, okay.

Before I move on, I am going to define 2 concepts, one concept is a linear transformation, okay. So, linear transformation is a special class of transformations, not all transformations are linear, okay. Linear transformations, if I give you x and y that belong to set m, we are still talking about the same set m, subset of x and so on and we have defined this transformation, okay.

And if I pick up any 2 scalars, alpha, beta, say belong 2 set of real numbers are the field on which you are working with let us for the time being that real numbers, okay. So, if I can write transformation of alpha x + beta y = alpha times Tx + beta times Ty. If I can do this, if a transformation allows you to write transformation of this combine vector, alpha x + beta y as alpha times transformation of x + beta times transformation of y, okay.

Then, such a transformation would be a linear transformation. Give you a simplest example, let x be Rn and y be Rm, okay. Y = Ax, where A is a m cross n matrix, is this a linear transformation? Just applied the definition. I take A times alpha x, so let me take 2 vectors say v and u that belong to x. I am taking 2 vectors, v and u belong to x, okay. What will happen to alpha u + beta v times A.

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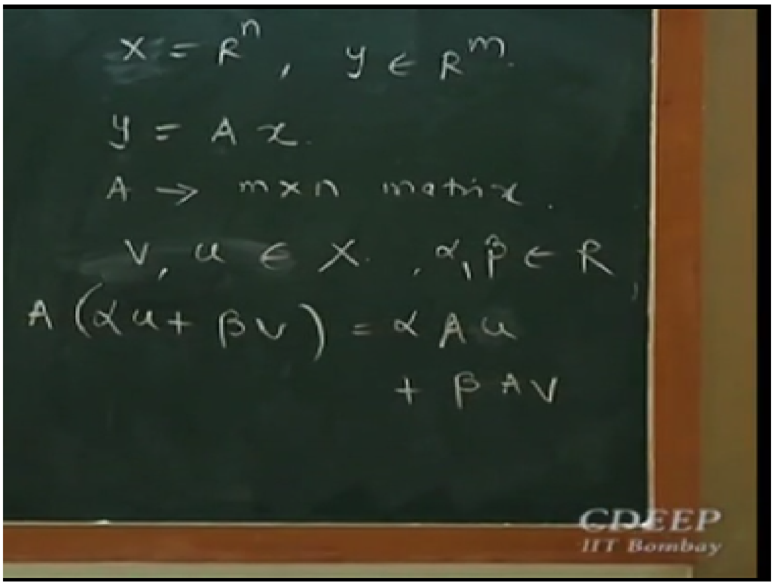

I am constructing a new vector, alpha u + beta v, alpha and beta are 2 scalars, okay. Now, this = alpha times Au + beta times Av, right. Is this a linear transformation? Just applied these definitions. See this is the transformation, why? It is obtained when A operates on, so what is the operator here T transformation is A, A operates on x. Give any element x, A will operate on that and give you an element.

It will take an element from n dimensional space will give you an m dimensional vector, okay. So, this is the linear transformation, right. Now, tell me whether this is the linear transformations. So next one, well my x is still Rn, my y is Rm, okay. I am defining a transformation y = Ax + b, where b is a constant vector. Just do it, is this a linear transformation?

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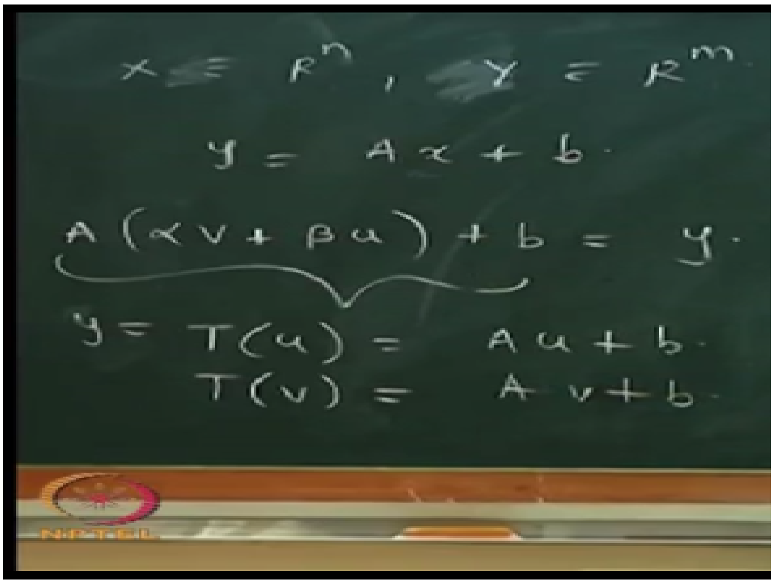

"Professor - student conversation starts" Who said yes sir. Why? (()) (15:52) +b. "**Professor - student conversation ends"** So, if you actually take a vector, if I take alpha v + beta u. I would write when I am operating on this vector, I would write like this, is not it, y =, okay. Now, this part cannot be split as sum of 2 transformations. I cannot write this, I cannot write this as what will be Tu?

Tu, y = transformation of u will be Au + b, okay. What will be transformation of b, Av + b, okay. So for this particular transformation, I cannot split, I cannot write T of alpha x + beta y =, well one mistake which I have made here in notation, let me correct it here, well let us not take this x and y, let us take this u and v. Otherwise, you will get confuse between this y and y here, okay.

So u and v here, let us write u and v here. So u and v are 2 vectors for m, not x and y because we are using y to denote the element in the range space. What is range space? Which is the domain space? X is the domain space and m is the domain and y is the range space. So now, let us u and v are the elements from m and y we are using to denote the range space. So, this definition stands corrected.

So what actually would appear to be a linear transformation is not really a linear transformation. You cannot satisfy the basic condition of linearity for this particular transformation, okay. Now, everything that is not a linear transformation, any transformation that is not a linear transformation, is a nonlinear transformation, the simple definition, okay.

Whichever transformation does not obey this simple law, when you apply operator T on alpha u + beta v, you should be able to write alpha times T operating on u + beta times T operating on v, it is not a linear transformation, okay. So, let me give you some more examples of the transformations and I just want you to think that understand that these new examples that we are going to talk about are not different from.

So, which is the equation which is probably most familiar to you? The most familiar equation that you have from your undergraduate or right from your 12th standard or 11th standard is this or may be now it is introduced in 9th and 10th standard, right. Solving linear algebraic equations, so well after sometime you realize that it is nothing but a matrix equation. So, you are thought about a matrix.

So, what is the matrix? Matrix is an operator, operating on x giving you an element y in the range space. What I want to show you is that after we have generalized the concepts of vector spaces, almost every problem that we encounter in engineering can be viewed in the same manner, it is not different, okay. It is an operator operating on a vector giving you another vector in the range space, okay.

So now let me moved to the third example, so one example I give you was Ax + b, my third example here is, so my x here is C1 set of continuously differentiable functions on interval a, b. set of once differential functions on interval a and b. So, independent variable t belongs to a, b. So, yt = d/dt of xt, okay. I am taking a vector from set of continuously once differentiable functions operating this operator d/dt on it.

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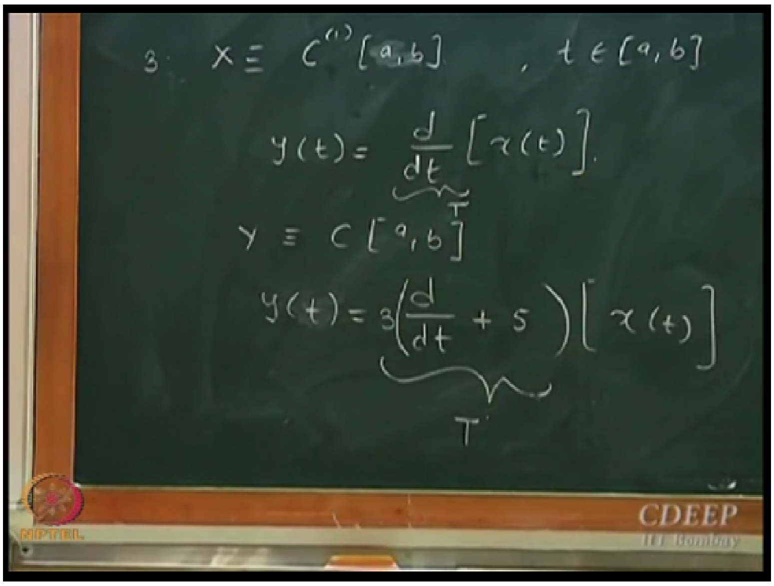

This is my operator t, it gives me another vector y, which is from the set of continuous functions, the resultant need not be differentiable, okay. I am getting set of continuous functions, okay. So this is my space x, this is my space y, this is the operator t which operates on a vector, this equation is fundamentally not different from y = Ax, A is the matrix which operates on a vector x gives me a vector y. No difference.

Here, I am getting a continuous function when operated by a differentiation operator on, you know a vector, what is this vector? This is a differentiable function or defined on some interval, interval could be whatever. It could be 0 to 1, it could be 0 to infinity, it could be -infinity to infinity depending upon their application of your choice, the interval could be different.

But conceptually, y = Ax and this equation are not different, they are one and the same and so is the case, if my operator is little bit different, if I write, this is third another example. Yt will be 3. This is another operation, this is another operator of t, 3 times d/dt + 5, so actually you will get 3dx/dt + 15x. This is an operator operating on this vector giving me another vector, okay. No difference.

It is an operator exactly in the same sense as, okay. Now here, we are going from n dimension to m dimension, m could be smaller, n could be larger depending upon what kind of matrix we are talking about when you deal with matrix equations. Let me give you another example of transformation. In fact, when you start understanding this language of vector spaces, very, very powerful language developed in the beginning of 20th century or end of 19th century.

Everything you know falls logically into the place. You will start seeing you know one unified structure in all the applied mathematics that you have been studied, okay. Now, let me look at this next example. You know my x, so a, b is some interval and I am taking set of all integrable functions on the interval and then I am defining a transformation alpha = a to b, what is this? You are familiar with this, definite integral.

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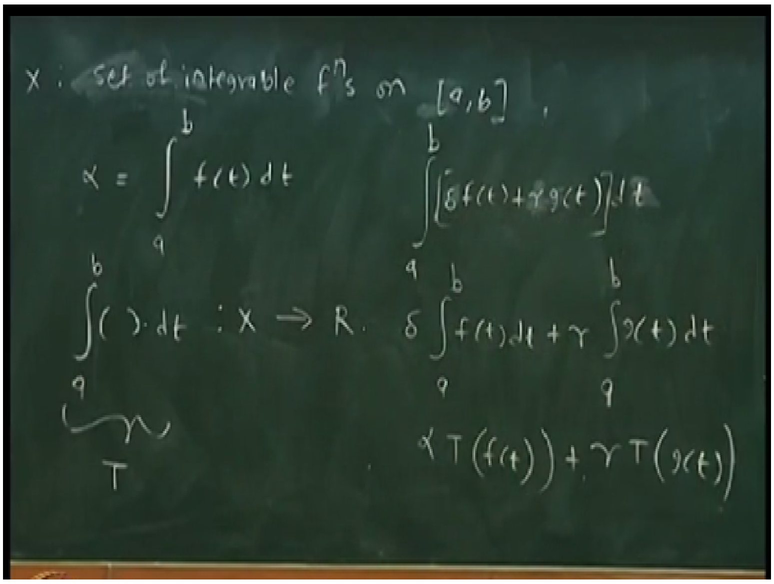

Area under a curve, so what is the range space? R, right. So this is an operator, so I would say that this operator a to b, something dt, this is my operator T, it takes a vector from the space and maps it onto R, okay. So, it is a function from let us called this whatever, this x which is defined here, okay, x to R. This operator T operates on this vector, okay gives me a one scalar number, okay.

So this is from, see just like in this case, you can have an operator A, which is from higher dimensions to lower dimensions, okay. You can have operator A which is only where y is a scalar, you can have that right. You can have the operator which will give you as a scalar value in R. So, likewise this is an operator which takes a function from this infinite dimensional vector space and gives you scalar value in R.

This is a transformation, is it a linear transformation? So what will happen, if I do aft + bgt, no, no, not a, b. We are taking a, b here know, we will taking delta and gamma are the 2 scalars and this goes from a to b, so I can write this as delta times ab ftdt + gamma times integral over ab gtdt, okay. So, this is delta tft + gamma tgt, everyone let me on this, so this is the linear transformation.

I can write this as a linear combination of 2 vectors in the product or in the range space, okay. This solution + this solution a linearly add up, so linear transformation. In the same sense, this is the linear transformation, try to understand this, okay. No difference between what you have done here and what you have done here. All these are linear transformations.

So if linear transformations have some nice properties, they not only hold for finite dimensional vector spaces, you can translate them into any other vector space. That is the power of using the vector space approach, okay. So far so good, let us move on to the problems that we encounter more frequently, one is boundary value problem, can I represent it in the new language of spaces.

Can I view it as a transformation from vector space to another vector space, how will I represent this? Let we will do a little bit more work, we are to go back to the definition of product spaces, but still you can do it. You can show, it is a transformation from particular type of one domain space to another range space, okay. So this particular Ax = b, y = Ax, I am going to wipe it out, I am going to just leave it.

So this is the reference, we know this very well and I just want to map everything to this particular. What about ODE IVP, ordinary differential equation initial value problem. Can I represent it as a transformation, okay? Well, my ODE IVP is given by say dx/dt = f of xtt, where f is some, this kind of equation we encounter say solving the batch reactor problem or batch distillation problem.

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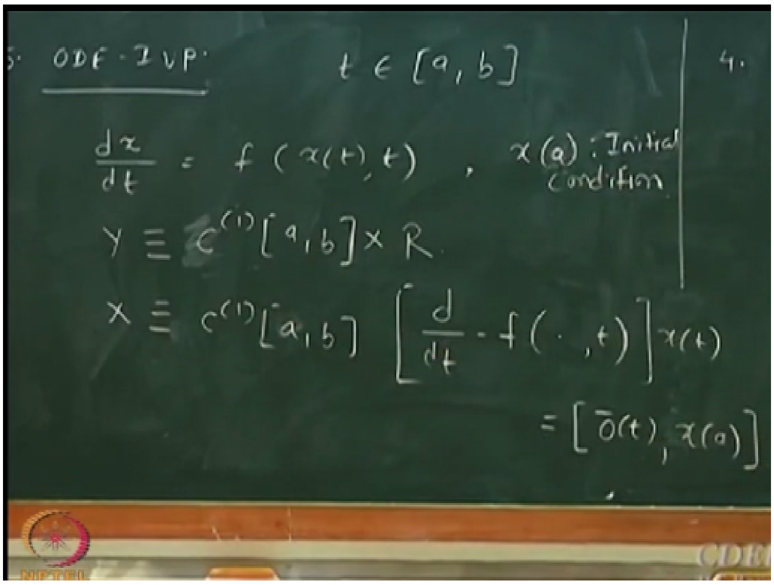

So, you will have equations which are of this type, dx/dt = f of xt where x states and then t would come because you are giving some input policy you like cooling policy or hitting policy and so on. So, this function of t will appear, so this kind of equations, we very, very frequently have to use. Let us look right now one dimensional equation, so one dimensional means with the only one straight.

Let us not complicate the life by getting into n differential equations and n unknowns, which will be anyway hitting into later, but right now, for representation purpose, let us look at, so what should x belongs to the solution x, where should it belongs to, C1. **"Professor - student conversation starts"** She says C1, do you agree? What is C1? Set of once differentiable functions. "**Professor - student conversation ends".**

So the solution x should belong to the set of once differentiable functions. Is it clear, it has to otherwise you know the equation is not defined, okay. So my solution x, so my y should be, now this independent variable t, it belongs to some set a, b, okay. so, y should be, but this problem it comes up we are normally asked to solve this problem. Solve from initial condition at xa, right, okay.

We are normally asked to solve this problem for some specified initial condition xa at point a, let us say that this is the initial point of the interval, if it is time 0 to infinity, at time 0, you specify what is the initial condition and then you want to solve this problem from 0 to t infinity, okay. So, you are typically given, so that is why it becomes initial value problems, ordinary differential equations initial value problem, okay.

So, the solution actually belongs to the product space R, you just think about it where R appears because of the specified initial condition. The solution should satisfy 2 things, it should be once differentiable function, initial value should be = R or initial value should be = xa, xa is some number from real life. You are talking of real differentiable equations, real valued functions, we are getting into complex, okay, real valued functions.

So the first value, initial value is a real number, okay. So the range space is nothing but set of once differentiable functions. The product space form by this and R. R comes from here, okay and what about x, so domain is, first I must check out here whether y also has to be once differentiable. Yeah, I think this description is correct, so my x is a set of once differentiable functions.

My y is a set which is once differentiable and cross R, so this is the map from, so this transformation, so what is this transformation, how will I represent this, so I would represent this as d/dt - f operating on xt, see this xt operates dx/dt, f operates on x, you will get f of xt, okay. f is the operator, which operates on vector x, okay and what should be the right hand side?

Right hand side should be 0 vector and the initial condition, if you ask me to represent this as a transformation in the same sense as y = ax, I would write it like this, okay. This operator, what is this operator? Dx/dt - f of xt, okay. Given any vector, this operator operates on x, okay. When I take this on the left hand side, what is on the right hand side, but it is not 0, it is 0 vector, okay.

It is 0 vector in this space, set of continuously differentiable functions and what should be the solution, what is the requirement that the solution should have for the solution that initial condition should be xa, okay. So, this is my vector in the cross space here, okay. So, this operator operating on x should give me 0 vector + the initial value of x should be, this is not x, do not confuse this 0 to be x. This is not x.

See when a transform, okay let us look at it, okay. See this is a d/dt - f, this whole thing when it operates on xt, it gives me dx/dt - f of xtt, this is what I get, okay, but when I have taken this f on the left hand side, okay, what is on the right hand side? This = 0, but 0 means 0 where? This is 0 for the entire interval. This is not one-point right, see this is the vector, xt is the vector defined from a to b, okay.

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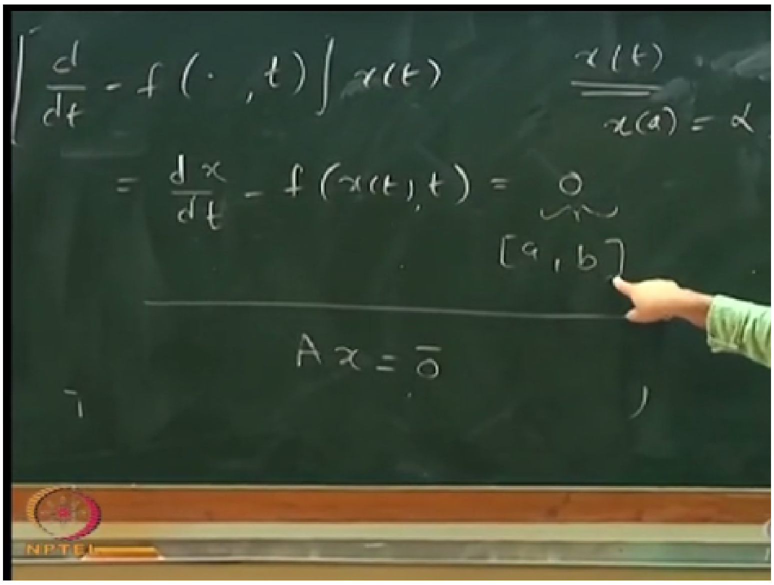

So if the solution satisfied this equation, I should get 0 everywhere on the interval a to b. So this is like solving you know operator operating on the vector gives me 0. We have solved these kind of problems, right. We have solved Ax = 0. We solved these kind of problems, Ax = 0, right. When columns of A are linearly dependent, you can get a solution we saw that, we took an example, we got nonzero solutions, okay.

So, those of you who have been by now introduced about null space, range space, you would no that the solution belongs to the null space and so on. So this is the problem which is similar to that. Operator operating on x gives me 0 vector, so that is why here I am writing 0 vector and x is addition should have initial value to be = xa. So, this is the representation of, so same thing is happening like this problem, okay.

See we solve for Ax = b, where b is the specific vector, right. This is the specific vector here 0. We solve for Ax = b, okay. In fact, I am trying to solve for Ax = 0, okay. The parallel of that is this, operator operating on x gives me 0 vector, okay. Yeah, **"Professor - student conversation starts"** (()) (40:09) because my solution should have initial condition = this, I want that vector as a solution.

This is the way of representation, I want that vector as a solution for which initial value should be this and the right hand side should be all 0 vector in the equation. I take a vector, see I want the solution xt, such that okay, let us take, I think this would be easier. Xa = some alpha, given that xa = alpha, okay. So, I will write here alpha. I want a vector xt such that which is solution of this problem such that xa = alpha, condition number one, okay.

Initial condition should be satisfied by the solution, right. This is the condition number one, what is the second condition, if I operate by this operator on this x, I should get 0 vector, I should get 0 everywhere on interval a and b, why because xt is defined on interval a to b, okay. "**Professor - student conversation ends".** This is the representation of the problem that. Let us move onto boundary value problem.

Just think about it may not sink because you know, we are suddenly you are being asked to change the gear from 3 dimensions to some infinite dimensional spaces and operator from any space to any space, it may take some time to sink, think about this, okay. Think about what I am saying. Now, consider boundary value problem. So a typical boundary value problem consists of a differential equation that holds in the domain say 0 to 1, okay.

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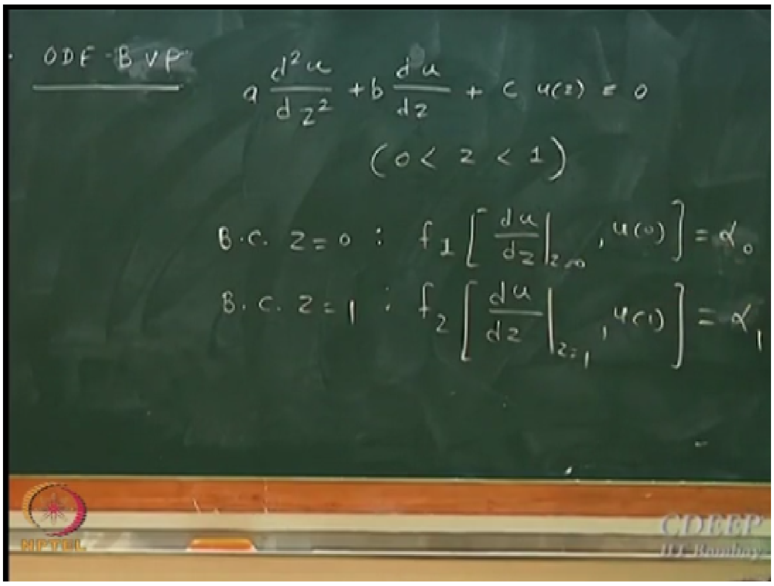

Z going from, the independent variable going from 0 to 1, 1 is dimension less length, let us say and there are 2 boundary condition, one at the z = 0, other one at z = 1, okay. So, some boundary condition f2 du/dz at z = 1, u1 = 0, okay and you make an attempt to write what are the spaces here. So, this operator operating on u should give you 0, not at one value, where it should give you 0 everywhere between 0 to 1, excluding the 2 boundary points, okay.

Now 2 boundary points you know, you can actually specify, okay, this instead of giving 0, we will put this as alpha 0 and alpha 1, so I could specify some conditions here, boundary conditions, but derivative + some operator operating on this. First derivative and u0 gives me alpha 0 and this gives me alpha 1. So these are 2 conditions, okay. So, what is the underlying domain here?

"Professor - student conversation starts" Twice differentiable function on 0 to 1. Interval is 0 to 1, okay. "**Professor - student conversation ends".** So, my domain space here is C2 0 to 1 and my range space here is why is this R and R coming? 2 boundary conditions, the solution should satisfy this boundary condition at initial point, this boundary condition at z = 1, okay and this is the map.

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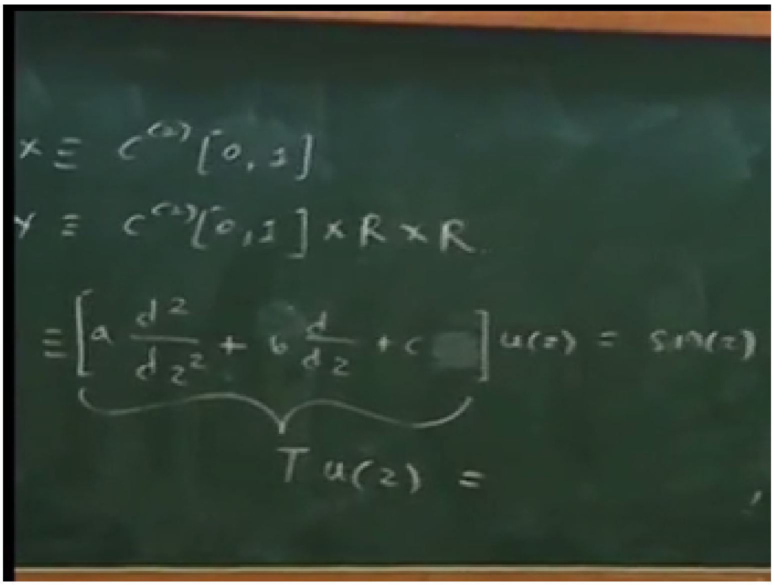

So, what is the map? So what is my transformation? My t corresponds to d2/dz square a times + b, one minute, here we are not taking any a, b, so b times d/dz + c times whatever it operates on, so this operates on uz, so this is my t operating on uz, okay. Here it gives me 0 vector, here I am asking to find out solution for 0 vector on the right hand side. In general, the right hand side here need not be a zero vector.

I can give you this = sine z, for example right. So, then it will be you know this operator t operating on uz giving me sine z with the solution exist or not is the different story, but I can ask you to do this, okay and now, this is same as y = or Ax = b. Conceptually, no difference between Ax = b and you know this operator operating on this vector giving me another vector which is sine z, okay.

I am trying to solve Ax = b, okay. No difference, the same thing. So, once you have this unified view of vector spaces, you can start looking at all these problems, actually the same problem, okay. So now, can I say what is, can I state a generic problem in a applied mathematics that are in or other saying applied engineering mathematics. Well, I will do a generalization here.

And say that any problem, so likewise here in my notes, I have also talked about how do look at a partial differential equation as a transformation from particular space to another space, domain space to range space. You can go through the example here, okay. So what is the problem that we have to solve as a part of applied engineering mathematics. We are given some domain, some subset m which is subset of x and y is range space, okay.

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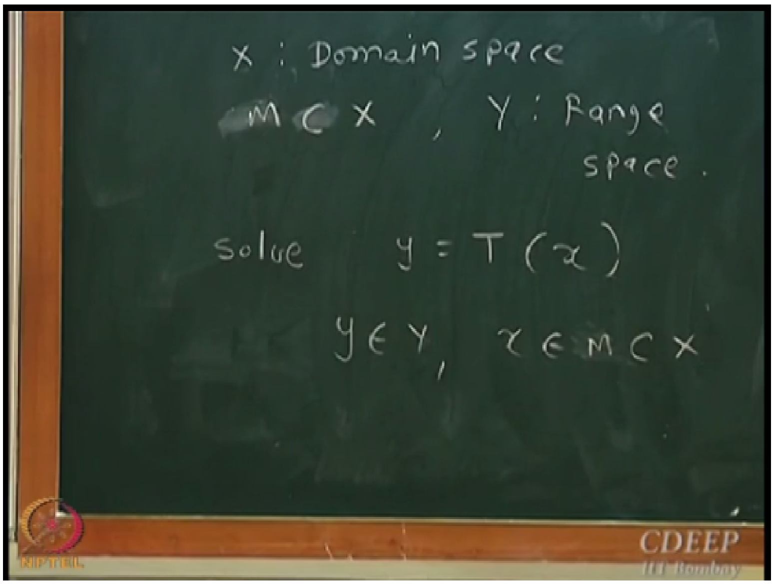

And then what am I suppose to do? I am supposed to solve this equation. Y = transformation of x, okay such that y belongs to y and x belongs to m which is subset of x, okay. So, y belongs to the range space and x belongs to, so this is the problem, this is what we have to do and there are 3 problems that arise from this, okay, which I mean all the problems that we encounter an engineering mathematics can be further classified into 3 problems.

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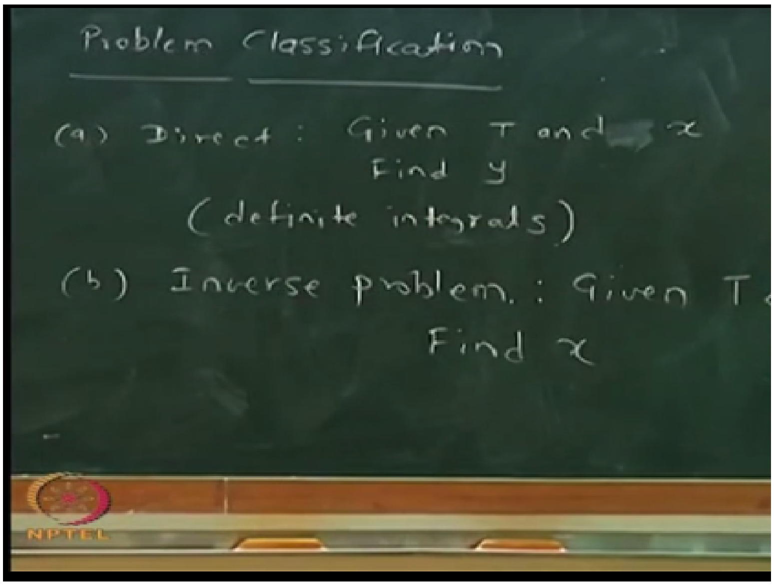

Direct problem, in direct problem operator T is given, x is given, you want to find out y. Given T and x, find y? okay, example definite integrals, okay. Inverse problem, what would be inverse problem? Just think alone, y is given, T is given, you do not know x, usual problem. Ax = b, b is given, A is given, x is not given to you, okay. A differential operator is given, right hand side is given, what is the vector that gives you the right hand side, okay.

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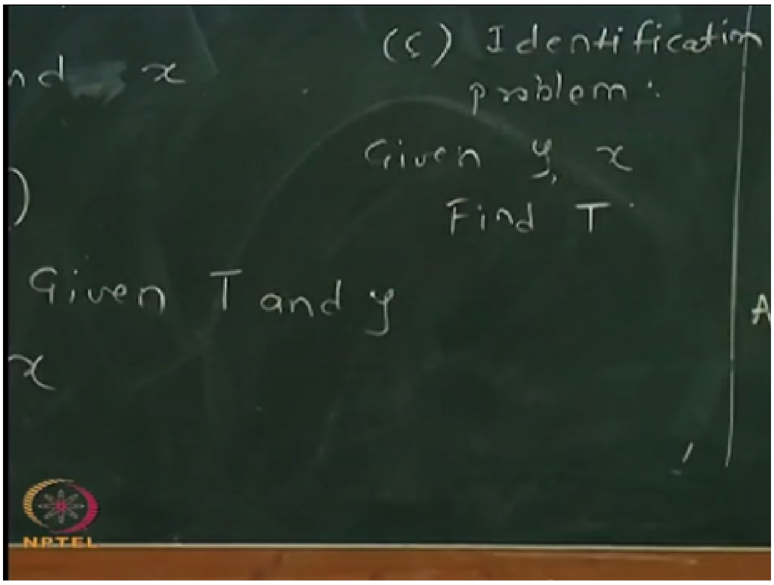

And a third problem, can you think of the third problem, third classification. Find a transformation? Given x, yeah, example fitting correlations, finding out rate equation for in reaction engineering. You are given some measurements, you know you want to find out the module parameters, okay. So the third problem is identification problem, well a deal little bit more about this in our next lecture.

So, the 3 essential problems that, our 3 essential classes of problems that we encounter in engineering mathematics, out of which, I am not going to be so much worried about the direct problems, they are relatively easy to solve. What are difficulty to solve to are, inverse problems, okay, solutions to the so called, solutions of algebraic equations, nonlinear algebraic equations, boundary value problems, partial differential equations, all these will fall into the inverse problems.

We are given a vector in the range space, we are given the operator, we want to find out a vector in the domain space, which satisfies this particular equation, okay and a third problem that we are going to look out little more elaborately that is identification problem that is given y and x, you want to find out the operator t, okay. Given the observed data input given to a plant, output given to a plant, I want to find out a transfer function.

In poses control, if you are remembered, you are finding out a transfer function, finding out a transfer function is nothing but finding out a differential equation that governs the input and output behavior, okay. This is the problem of identification, okay. So, these 2 classes I will once again briefly go over this classification in my next class and then now, we will start you know dealing with the last 2 classes that is inverse problem and identification problem. So, how is approximation theory going to be used here that is what we start from the next class.