Advanced Numerical Analysis
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Lecture - 07
Cauchy Schwaz Inequality and Orthogonal Sets

Okay. Good morning. So, inequality called as Caushy Schwaz Inequality.

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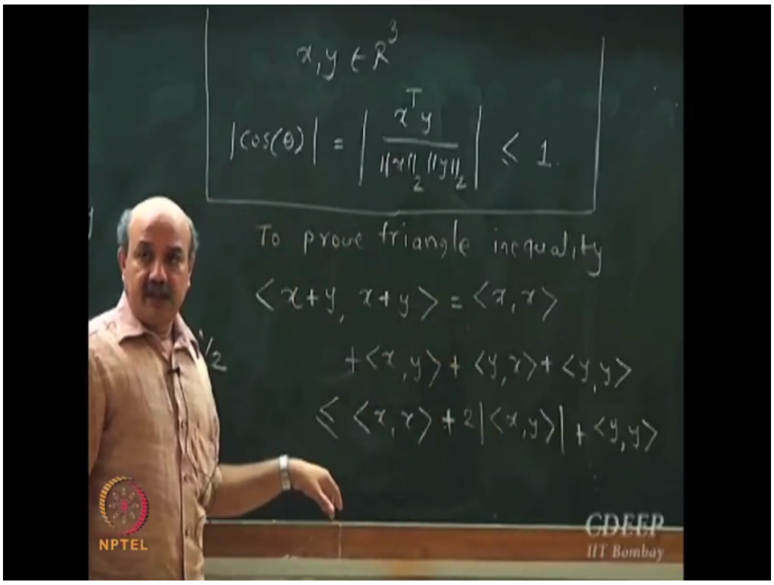

Now, Caushy Schwaz Inequality states that, if I am given in a product space x and I take any 2 elements say x and y, that belong to x, then absolute value of inner product between x and y is always <=. So, we proved this fundamental equality and I said that this was nothing but generalization of the result that mod of cos theta in 3 dimensions, we know this result, this result, which we know from 3 dimensions, Caushy Schwaz Inequality was a generalization of this particular results from 3 dimensions.

In this particular case, x and y are 2 vectors that belong to R3. And here, so, when any 2 vectors x and y that belongs to 3 dimensional vector space, we know this result and this particular result is a generalization in any inner product space okay. Now, the reason why we wanted to work on this particular inequality was 2fold. One was well, we want to reach the concept of angle, inner product space in a general space and at the same time, we also want to prove triangle inequality, in an inner product space.

Why do I want to prove triangle inequality? I want to define a norm using inner product okay. So, that is why I want to prove the triangle inequality. So, how do I prove triangle inequality using this particular result. So, let us move towards that. So, this result is separate, this result is just for you reference. This 3 dimensional result is only for your reference. So, I am continuing again with our inner product space x. I am going to take any 2 vectors x and y.

So, and I want to find out inner product of x+y with itself. So, this would be if I just expand this, this would be inner product x+x inner product y+y inner product x+ y inner product y right. Now, inner product, a number it could be a positive number or negative number, inner product they would not be positive always, norm is positive. Inner product can be positive or negative. Cos theta, in this case, cos theta can be positive or negative.

So, I am going to just replace this particular equality with an inequality. So, this is <= what is x inner product x? Always positive, not a problem okay. So, x inner product x+2 times absolute of x inner product y+y inner product y. Do you agree with me? Absolute value of this, this could be a positive or negative number. If this is a negative number, absolute value is always greater than this value okay.

So, I am just replacing this equality with this inequality fine. Even if these are complex numbers, still this particular inequality will hold okay.

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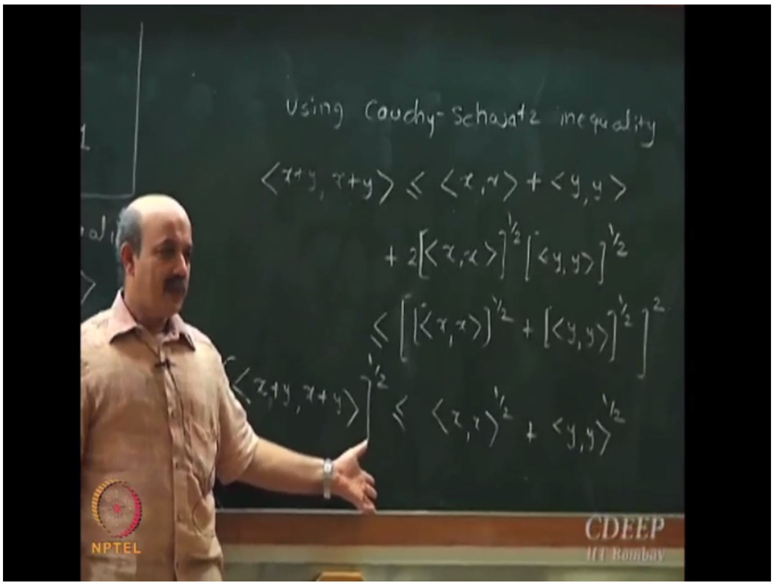

See, now I am going to use Caushy Schwaz Inequality. This absolute value is < x inner product x raised to 1/2 * y inner product y raise to half okay. I am going to use this inequality here. So, this will give me x+y, x+y inner product is <= y inner product y raised to half okay. Is that fine? Just using Caushy Schwaz Inequality I get this. So, this is <= actually this quantity is <= I am continuing. This is a square now.

We can see, this is a square okay. But, what is left hand side? So, I can write that, can I say this? All are positive numbers; this is a positive number. Inner product of a vector with itself is a positive number. I can take a square root okay. I have expressed the right hand side of the square. So, I can take a square root. What is this? This is triangle inequality.

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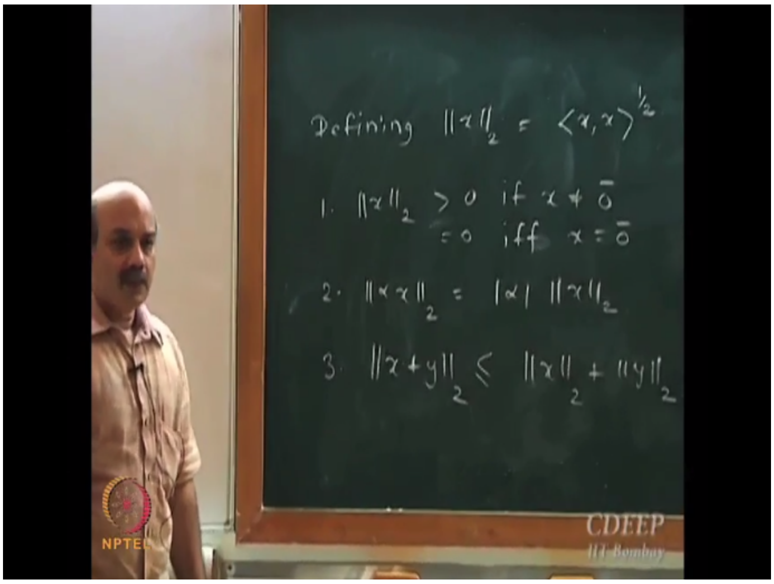

See, if I define now, if I define a norm okay, which is like this, then, I have all 3 axioms satisfied. What is the first axiom? So, we saw this, the first axiom is norm of x2 is < 0 if x is not = 0 vector and = 0, if and only if x=0 vector okay. Second, we saw what really held was alpha times x 2 = mod alpha norm x 2 alright. And what is the third triangle inequality, which norm we have just now proved okay. So, my third result is norm x+y2 is <= norm x 2+norm y.

That is the result which I have proved just now right. So, inner product defines norm very nice. Inner product, this norm is defined using an inner product or induced by an inner product. So, can we draw now. Now that, it is a norm, it is a length measure okay. Can I extract something more out of Caushy Schwaz Inequality? So, what was my Caushy Schwaz Inequality? Let me see whether I can draw some more mileage out of Caushy Schwaz Inequality.

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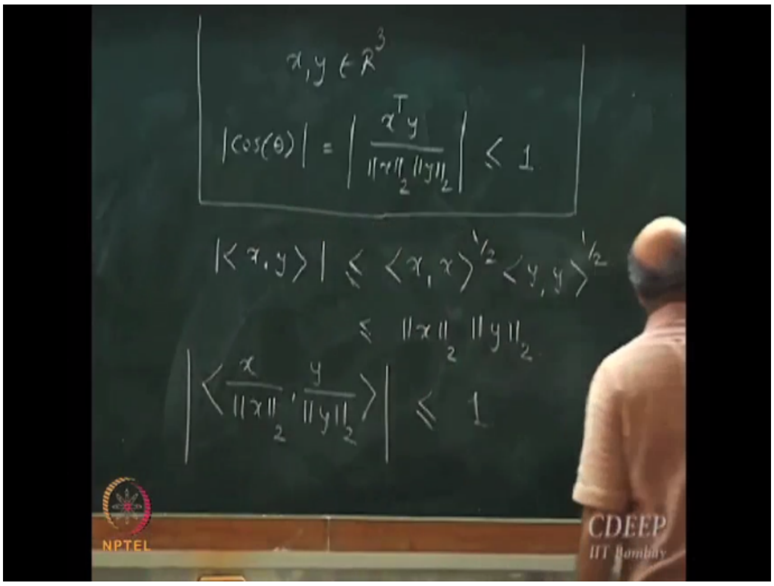

My Caushy Schwaz Inequality was x,y <= right, which is nothing but norm x2*norm y2. Everyone with me on this? (()) (09:27) This is a positive number, this multiplication of these 2 positive numbers is > this positive number. So, only you can use one way. So, when you add a higher number on the right hand side, you get inequality. If you wanted to use minus of something then, probably it is different. But, there is only one way to use it.

When you derive triangle inequality from Caushy Schwaz Inequality, I do not see any other way you could use it. No, we are not assuming. I proved Caushy Schwaz Inequality. I do not know whether you were present in yesterday's lecture. So, we proved it, no assumptions. Caushy Schwaz Inequality, we have proved by logical arguments okay. And now, I am trying to see whether I can get some more insights through it. So, is this fine?

Now, we have defined a 2 norm okay. Now, what is the norm ultimately? It is a positive number, these 2 are positive numbers right okay. I can take positive numbers inside absolute value, not an issue right. So, I will not be wrong if I say absolute of x fine. Is this okay. Just compare. I wanted to have x transpose y divided by 2 norm of x 2 norm of y. take 2 unit vectors in 3 dimensions, take inner product, what do you get? Cos theta okay.

Take 2 unit vectors in 3 dimensions and their inner product will give you cos theta. Exactly that is what I have arrived at. In any inner product space, same inequality, no difference okay. So, I am going to say now well, let me define an angle. So, now let me define an angle in any inner product space okay how do I define an angle?

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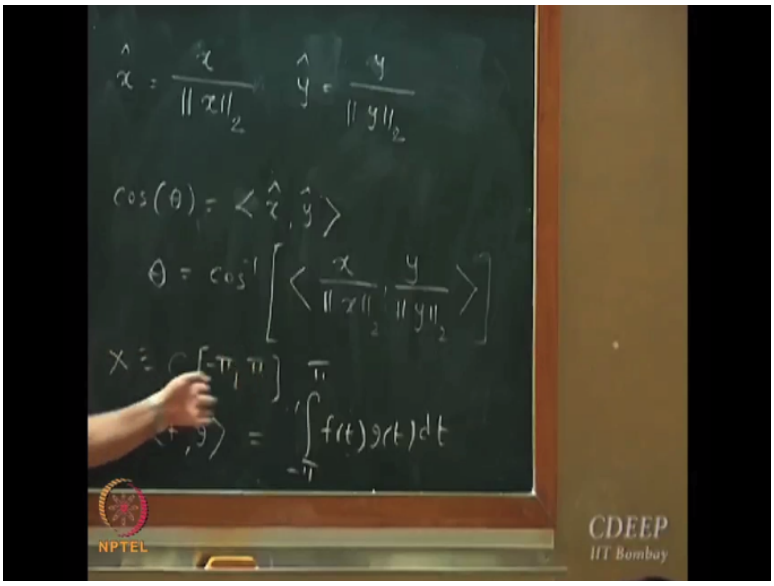

So, let me define 2 unit vectors x divided by 2 norm okay. Let me define another unit vector y cap, which is y divided by 2 norm of y and then cos theta okay. Or theta = cos inverse of inner product of x. is that okay, is this fine? So, inner product, generalization of concept of dot product to inner product allowed me to prove a very, very important result from 3 dimensions into a any general inner product space.

So, we could define angle between 2 vectors. Now, inner product space could be any set of objects like we had set of continuous functions over an interval okay. And when you study your undergraduate, you come across many such functions. They are called orthogonal functions. They are called orthogonal polynomials. Why are they called orthogonal? What is the basis? Okay it is basically you are talking because there is an underlying inner product space.

The inner product defined on it. And that inner product okay allows us to define concept of angle between 2 vectors. Vectors as in elements of the vector space, they could be continuous functions. That is why, you know we know all those results when you look at 4ier series. See, if you take this inner product space, set of continuous functions over -pi to pi okay, then we are told when you study 4ier series, that let me define the inner product here, inner product is defined between any F and G, 2 functions as integral -pi to pi ft gt dt.

Let me take this inner product space okay and let me define this particular product.

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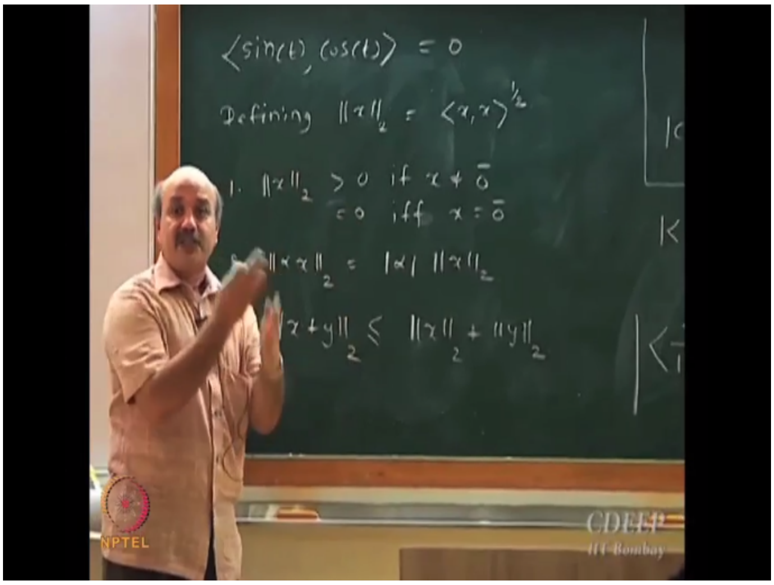

Well, what we are told is that, inner product of sin t cos t is 0, because they are orthogonal okay. When you hit up on this consult first time that 2 functions why are they orthogonal in what sense? Because, when you think of orthogonality, you are trained to think in terms of 3 dimensions i, j, k and so on okay. But, what you should realize is that is in the same sense i, j, k 3 unit vectors in 3 dimensions are orthogonal.

These 2 vectors are orthogonal in that in a product space, set of continuous functions over -pi to pi or if you change 0 to 2 pi. So, this is interesting. So, this allows us to talk about orthogonality of functions. Orthogonality of general vectors, in any vector space, in any inner product space. So far so good. So, we have defined angle, obvious thing that comes next is orthogonality okay.

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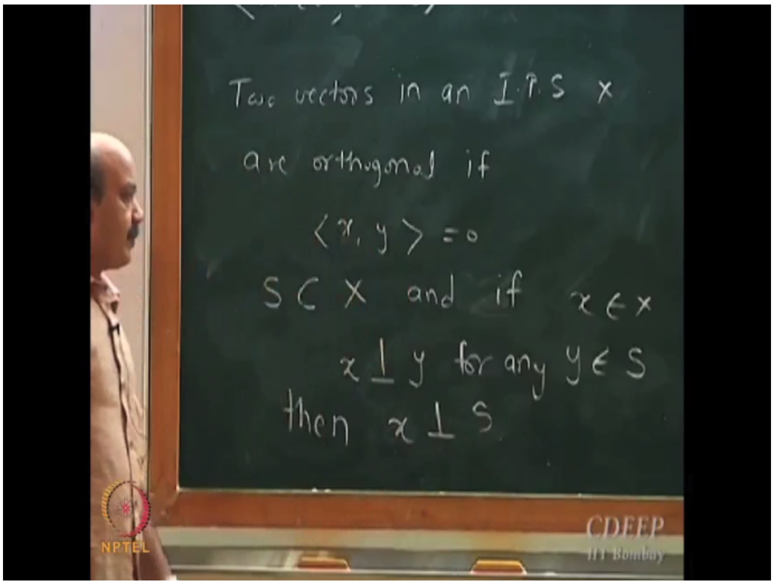

We say that 2 vectors are orthogonal, inner product is 0 simple. If the inner product is 0, then these 2 vectors x and y, any arbitrary vectors x and y for which inner product is 0 in a product space, they are orthogonal vectors okay. What are the other concepts that you need when you start orthogonality? Well, one thing we talk about is that a vector is perpendicular to a plane right. We have to use an ocean of vector being perpendicular to a plane or a set.

Let us say, so this, if I have this plane okay, I can say that, this plane okay, this particular vector is perpendicular to every vector in this plane right, every vector in this set. I could talk about a entire plane, I could talk about this limited set and this particular vector will be perpendicular to all the vectors in this set okay. So, if you have a subset as which is subset of inner product space s and a vector x that belongs to inner product space is such that x is perpendicular to y for any y, that belongs to s, then we say that x vector is perpendicular to s.

If s is some subset of inner product space okay. And I take any arbitrary vector x in the original space. This is a subset, this is a vector, if this vector is perpendicular to every vector, that belong to the set, then the vector is perpendicular to the entire set. All these concepts will require more and more once we progress okay. So, what result that I wanted to generalize, what was the best result in geometry that you keep using all the time? Pythagoras theorem okay.

Can you prove Pythagoras theorem? What is Pythagoras theorem? What is the statement of Pythagoras theorem? In 3 dimensions, let us look at 3 dimensions, what is the statement? How will you state Pythagoras theorem in 3 dimensions? If you are given any 2 vectors x and y okay.

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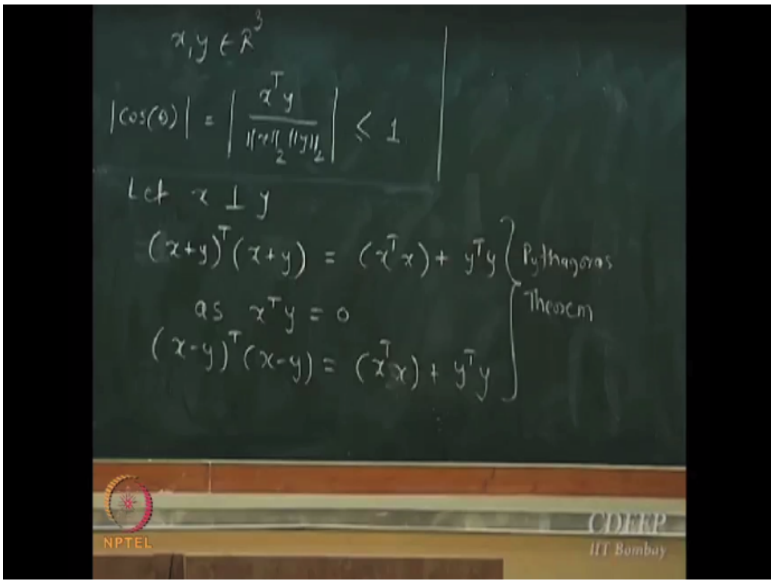

I am given any 2 vectors x and y in 3 dimensions and let x be perpendicular to y. How will you state Pythagoras theorem? (()) (18:33) x and y are 2 vectors, what will form an triangle? x+y will form the triangle, x-y also can form a triangle, both will hold. Then so you want to say that, can I say this is = x transpose x + y transpose y. as he is rightly pointing out this could be said even for x-y okay.

So, x-y transpose x-y will also give you x transpose x+y transpose y. So, this is my Pythagoras theorem in 3 dimensions. Do you agree with me, has anyone has doubt here? What is x+y? if I take 2 vectors in 3 dimensions? Just try to visualize, unless you visualize, you will not get it.

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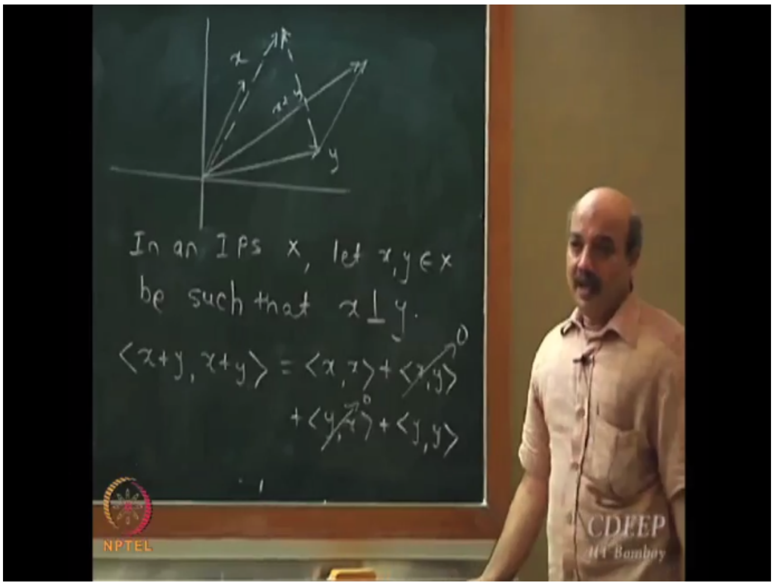

Let us say, this is x vector and this is y vector. What is x+y? Parallelogram law in fact, now I would expect the Parallelogram law to hold in here. Do you remember Parallelogram law? Everything will hold. I mean you are just generalizing. This is my x+y vector right and this is x-y vector okay. So, now if x and y are perpendicular okay, then what we are saying is square of, so we are looking at a scenario where x and y are like this and x+y is actually this right.

So, square of length here is this square + this square that is all I am stating here okay. Is this fine? So now, all that I need to generalize this in inner product space is to use the concept that, if 2 vectors in inner product space are orthogonal, then their inner product is 0. I just start with the same thing so, in an inner product space x, let x and y belonging to x, well I am writing everything in this cryptic language because it is faster to write otherwise, and you will get used to it after some time.

X is perpendicular to y right. I pickup 2 element x and y in a inner product space, which are perpendicular. All that I have to do to prove Pythagoras theorem is to start with x+y, this is = norm x, this is = x inner product x+x inner product y+y inner product x+y. This is 0, this is 0, x and y are perpendicular, inner product is 0 okay. What follows is the classic Pythagoras theorem generalized to any inner product space, a grand generalization of ideas.

Same ideas what you should not forget is the ideas of geometry which we are using from your school, 3 dimensional vector space, which you are used to in your college, same thing is being generalized in different spaces okay. So, if your geometrical ideas in 3 dimensions are clear, you will understand what is happening here. You cannot visualize what exactly this means in a function space, visualization is not so easy.

I do not know whether it is possible or in n dimensions or but, geometrically it is the same thing, what is happening here when you have 2 perpendicular vectors and writing orthogonal Pythagoras theorem, geometrically it is not at all different. That is important to understand okay, same geometrical concepts are this. We are not able to visualize this okay. So, what next, how did orthogonal vectors help you in 3 dimensions, what were they used for?

Can somebody show light? Standard basis, orthogonal basis, very very useful right. We use orthogonal basis all the time. So you had 3 unit vectors, which are orthogonal in fact, you chose them orthonormal. What is orthonormal? Unit vectors. Orthonormal their magnitude was 1 okay. So, orthonormal vectors helps us to define any arbitrary vectors in terms of its components along certain directions right.

So, we write a vector x component along I direction and y component along j direction. This is the first time when you start looking at coordinate geometry. This is how you start representing a vector right. So, we need now exactly the same thing, we need to generalize a set of orthonormal basis vectors in any inner product space and then we should be able to express a given vector in terms of a orthogonal basis right. Because orthogonal basis has many, many advantages in computations as compared to non-orthogonal basis okay.

See, in 3 dimensions, how many ways you can construct a basis? What is a basis in 3 dimensions? So, for example, 3 independent vectors okay.

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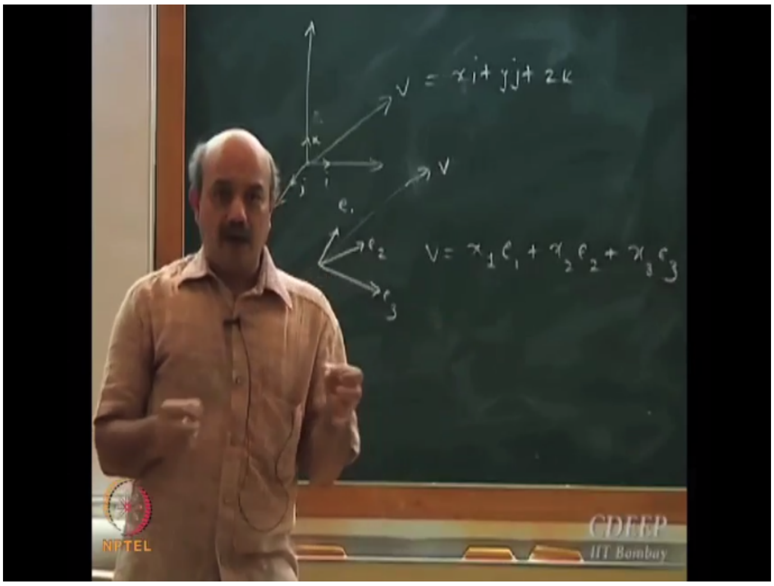

So, in 3 dimensions just like this, let us say k, this is unit vector here say k, i and j, just like these 3 unit vectors form a basis okay I can take some 3 other vectors say e1, e2, e3 as long as these 3 vectors are linearly independent, they can form a basis. There are infinite ways of defining a basis in 3 dimensions. Given that there are infinite ways of defining, if I give you some arbitrary vectors in 3 dimensions, say this vector v okay. I can write vector v as v = day x1e1+x2e2+x3e3, where e1, e2, e3 are 3 basis vectors.

I am perfectly allowed to do this okay. Yet we prefer to work with, so same vector v, we find it convenient to write in terms of some component xi, yj+zk and so on. So, we prefer this basis over this basis. We have this basis over this basis okay. So, likewise are there some special basis which are more useful when you do computations, it turns out that there are okay. Now, for example I have a continuous function which is the polynomial okay.

Can I define an orthogonal basis for a set of polynomials, then I can express a polynomial something like this using orthogonal components. I can express a function using orthogonal components along certain orthogonal polynomial directions. So, I am going to generalize this idea. I do not like this when I work in 3 dimensions, I prefer this. So, I need orthogonal basis when I go to an inner product space.

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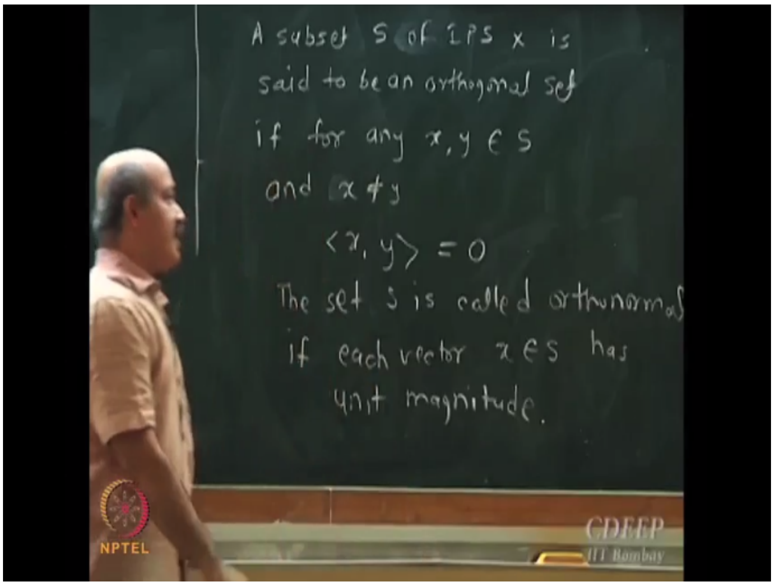

So now I am generalizing a concept of orthogonal set, in this case i, j and k is a set of orthogonal vectors. In fact, they are set of orthonormal vectors right. So, in inner product space, if I give you a set okay and if I pick any 2 elements in that set okay and if the inner product of any 2 elements is 0, then that set is orthogonal set. When we will call it as orthonormal set? If each one of them is a unit vector, then it is a orthonormal set.

So, if each vector has unit magnitude, then we call this set as a orthonormal set okay. Now, do you remember how do you construct, if I give you in 3 dimensions, if I give you 3 vectors, which are not orthogonal. I want to construct an orthogonal set starting from a non-orthogonal set, how do you do it? Does if I say Gram-Schmidt Process, does it ring a bell? No? you do not know what is Gram-Schmidt Process?

Okay, we will study what is Gram-Schmidt Process. So, I like orthogonality because it helps me to represent vectors in a very way. And so if I am given a set which is not orthogonal, I would like to construct a set, which is orthogonal okay. If I am given a set which is not orthogonal, then I can construct an orthogonal basis, which helps me 2 represent vectors okay.

First, I am going to start looking at 3 dimensions, we will generalize to setup polynomials, we will then go to functions, functions space and on. So, okay, let us go back to our 3 dimensional vector space.

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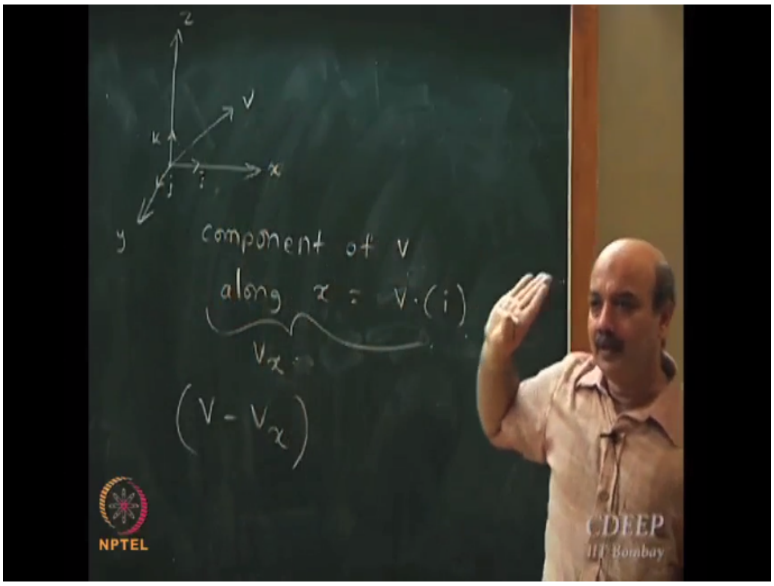

We have this vector v here and so, let us call this x, y and z directions and this our i, j, k are 3 unit vectors okay. If I wanted to compute component of v along x, how do I do it? Dot product with unit vector in that direction right. So, I use the unit vector, we transpose i. So, this vector, this will give me inner product, dot product actually, we should not say v transpose v dot i that is the right. So, dot product will give me x component okay.

Dot product of v with j will give me y component and dot product of okay. So, suppose I find out if I am given vector v okay, I find out the component along x okay. Let us call this vector as vx, what is vx? Vx is component along x. I am going to call it as vx okay. So, what will be v-vx? What will be this vector v-vx? It will be 2 components that are remaining along y and z directions. So, everything that was along x has been removed okay. Now, what remains is only.

So in fact, you would expect that component to lie in which plane? Yz plane right okay. Now, this idea I am going to use to come up with this concept of Gram-Schmidt Process okay. Is this clear what I talked about just now that you find a component along a particular direction, remove it from the original vector, what remains is along the remaining orthogonal components okay. So, this is the very, very important concept.

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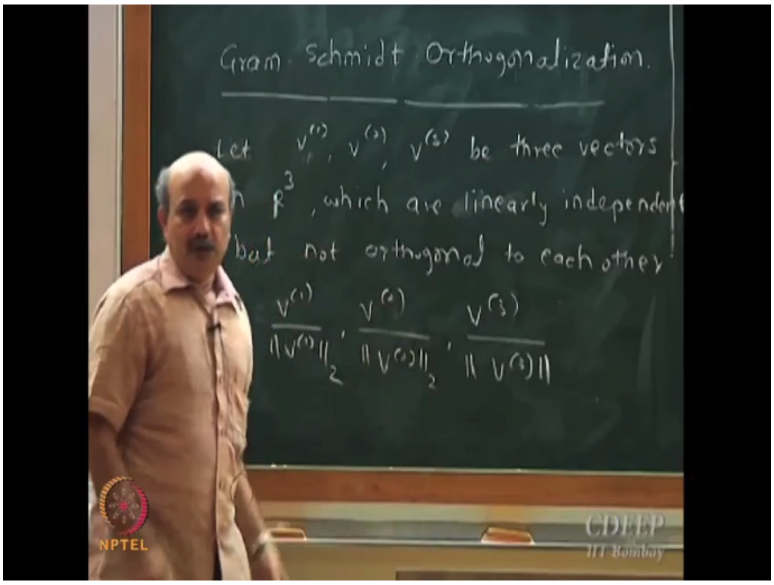

Gram-Schmidt orthogonalization can be done only in an inner product space not in any vector space because inner product defines angle, orthogonality and the things that you really need to construct an orthogonal basis okay. Idea of orthogonal basis cannot be thought of in some other arbitrary vector space, where inner product is not defined okay. So, definition of inner product is crucial when it comes to okay. Now, let us start with R3.

So, I am taking 3 vectors v1, v2, v3 which are linearly independent in R3 but not orthogonal okay, they are not orthogonal, they are just okay. I am given 3 vectors in R3 and then I want to construct a set which is orthogonal basis right. I could have constructed a basis from this, which is a non orthogonal basis. This basis would be, one way of constructing a non orthogonal basis will be v1 up on norm v1 right and v2 up on norm v2 and v3 up on norm v3.

I can construct a unit vector; I can construct 3 unit vectors but they are not orthogonal okay. So, I would like to go to orthogonal set from this okay.

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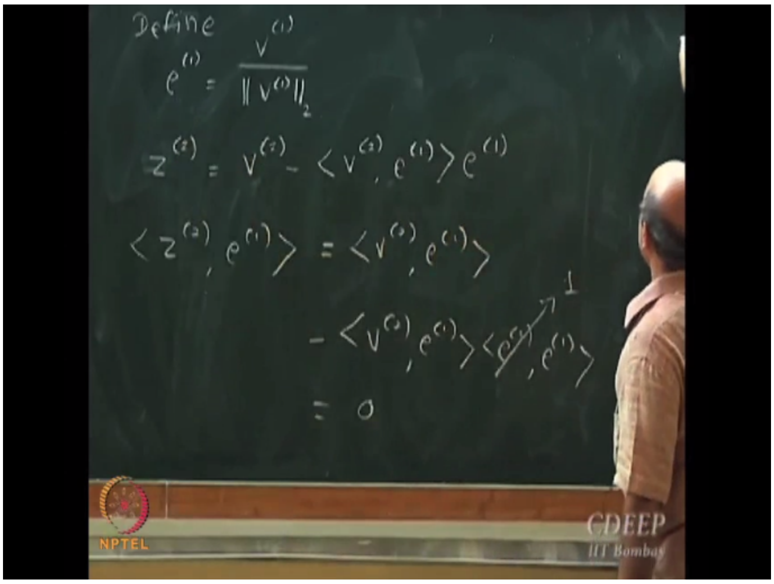

So, let us define vector e1 okay. This vector e1 I am going to define as v1 divided by 2 norm of v1. Is this fine? Okay. Now, I want to construct, this is unit vector. So, I got one-unit vector, I want 3 unit vectors, which are orthogonal. In fact, I would like them to be orthonormal. Then, what I am going to do is, I am going to remove the component, I am going to define a new vector z2 okay, which is v2-component of v2 along e1.

How do I find component of v2 along e1? Dot product times, this is a scalar right, this is the component, this is the scalar component along e1. So, this vector - this will be everything now, that is left, which is not along. So, z2 will have everything, that is not along e1. Is e1 perpendicular to z2? You can just check that z2 dot product e1, what is this? This is v2 dot product e1 - v2e1 dot product e1 or e1 inner product e1.

What is e1 inner product e1? 1. So, this is 1. So, what do you get here? 0 okay. So, I have constructed a vector z2, which is orthogonal to okay.

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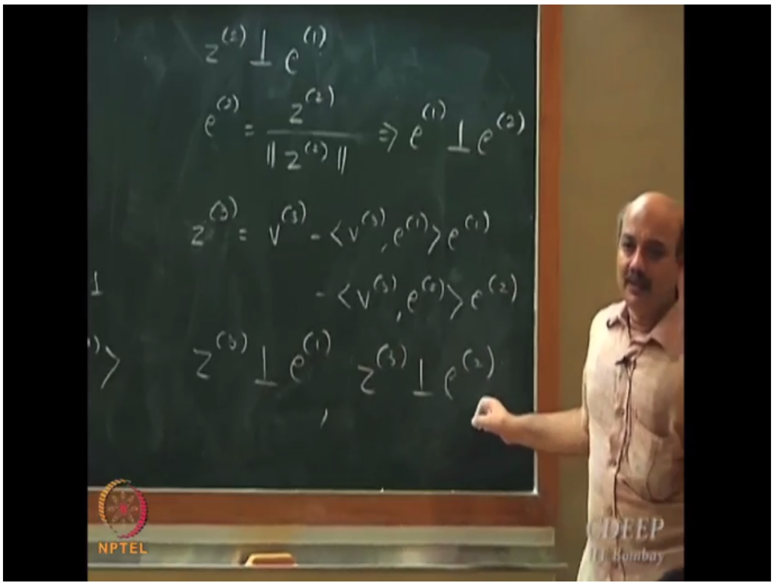

So, z2 is perpendicular to e1. But z2 is the vector, which is not a unit vector. I like unit vectors i, j, k okay. So, how do I get a unit vector? I will take e2, which is z2 divided by norm z2 okay. So, I have 2 vectors. See, this z2 and e2 are aligned along the same direction, magnitudes are different right. So, e2 and e1 are also perpendicular. So, e1 is perpendicular to e2 okay. Is that fine? Now, what next? I want to now construct a third vector, so, v3.

So, I will construct a vector z3, which is v3 - component along e1. You can very easily check that. Z3 is perpendicular to e1, z3 is perpendicular to e2. Not difficult to check okay. Just take inner products, we will see that z3 is perpendicular to e1, z3 is perpendicular to e2 okay. So, e1, e2, e3 are mutually orthogonal okay. So, how do I create e3 now? Take a unit vector along z3 okay.

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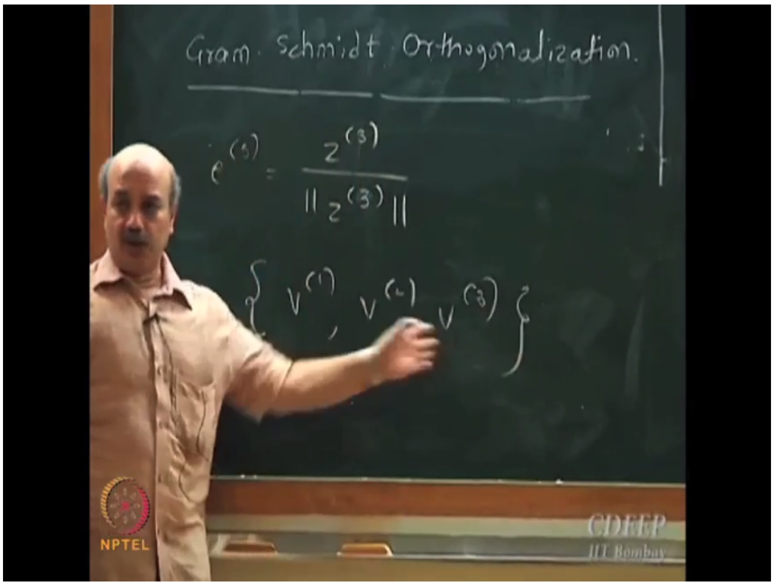

So, we started with a non-orthogonal set and we got an orthogonal set. I can do this, why just in 3 dimensions? You have some doubt? Which one? See, this e3 is a third vector, which I am going to define just by taking unit direction along z3 okay. See, I started with. What did I start with? I start with v1, v2, v3, these are not orthogonal okay. So, from v1, I constructed this e1 vector okay. Then, I removed component along e1 from v2 okay. Whatever was left was perpendicular to e1 okay.

Next, then I defined this z2, I defined a unit vector along z2 okay. Then, I removed component of v3 along e1, v3 along e2 right. Whatever was left was perpendicular to both e1 and e2. We can just check this. See, because e1 and e2 are orthogonal, if you take inner product of e1 with e2 will be 0. And inner product of e2 with e2 will be 1 okay. So, it will just nicely follow. So you started with 3 non orthogonal vectors, finally I got this z3, which is not a unit magnitude vector.

So, I am just making this a unit magnitude vector here okay. So, I can generalize this process in n dimensions, if you are given n vectors in n dimensions okay, how could I construct an orthogonal set?

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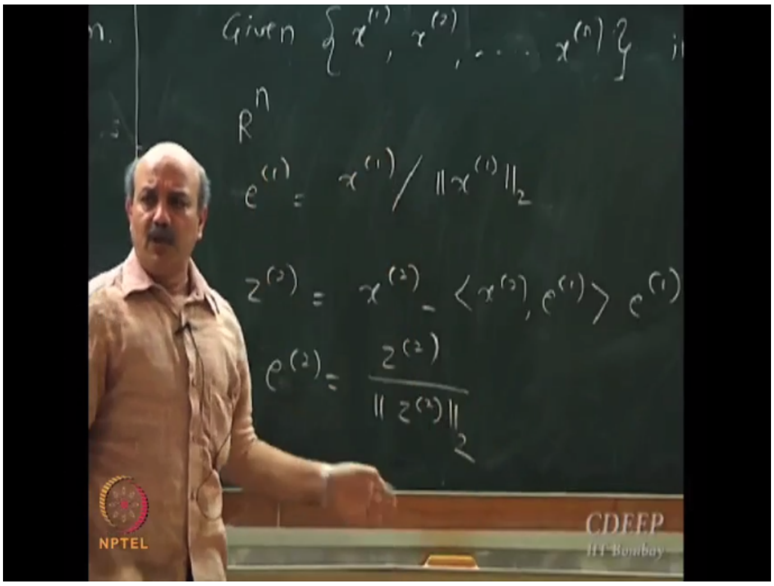

In n dimensional space, how can I go on doing this Gram-Schmidt orthogonalization? So, I could systematically go from 1, 2, 3, 4 and so on. So, this is e1 is x1, then z2=x2 - inner product x2e1e1 and e2= and so on. I just go on methodically doing this same thing okay. Then e3, then e4, then e5, then e6, I can go up to en. So, starting from a non-orthogonal set okay, so, what we will see in the next class is we will take an example in 3 dimensions, construct an orthogonal set.

We will take a set of polynomials, which are not orthogonal, construct set of orthogonal polynomials. If you just follow Gram-Schmidt Process, what will pop out is Legendre polynomials okay. I think you have heard these name Legendre polynomials. And then you must have heard shifted Legendre polynomials. And then you must have heard Bessel's polynomials. All these things will fall into line, if you understand Gram-Schmidt Process okay.

It is some orthogonal set constructed on some inner product space of interest okay and those sets can be constructed simply following the simple idea from 3 dimensions, Gram-Schmidt orthogonalization okay. That is the message okay. So, next class, we will look at examples of this.