Advanced Numerical Analysis
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Lecture – 06
Introduction of Inner Product Spaces

So today, we are going to start looking at another grand generalization from school geometry or 3-dimensional geometry. This is inner product spaces or Hilbert spaces. As I said the nice property of 3 dimensional world or the geometry that we study equivalent geometry that we study is orthogonality between 2 vectors. If we have perpendicular vectors we can define models very conveniently. We can define vectors very conveniently.

So, there are many advantages of Pythagoras theorem and we would like it to hold in a general space which consist of functions which consists of polynomials and so. So, we have to come up with new structure on a vector space which is equivalent to what we have available in 3 dimensions and then tried to see to it that the properties that are importance in 3 dimensions or in school geometry are also preserved in these newly defined vector spaces.

So now we impose one more structure. See till now we started by defining norms, but just length or norm was not enough, it was helpful in defining generalizing the ideas of convergence, convergence to a limit and so on, but we need something more. We need angles. So remember 2 things, what I want to generalize? I want to generalize

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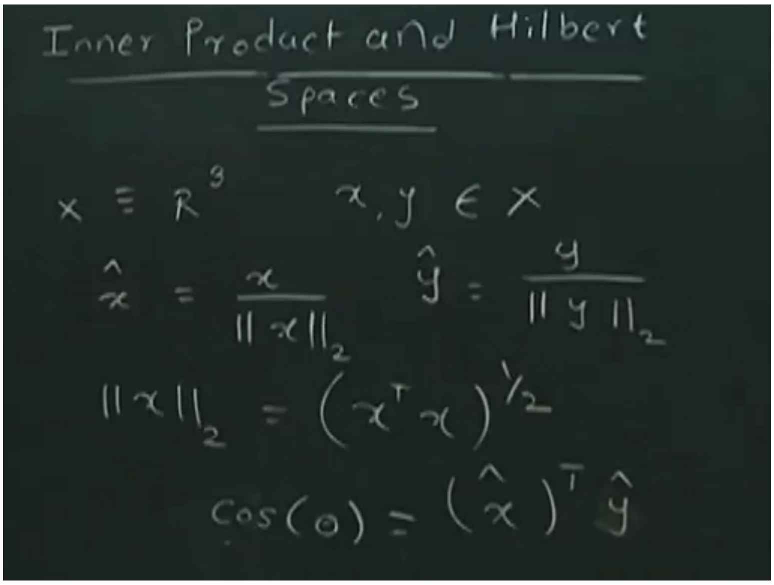

The so called dot product if my x is nothing by R3 then if I am given any 2 vectors x and y that belong to x then what I do is I construct unit vector as x/2 norm of x and y cap which is also unit vector y/2 norm of y so this is 2 norm. x 2 norm is x transpose x raise to half. We construct 2 vectors which are unit vectors in direction of x and y.

And then if I want to find out angle between x and y I have to take a dot product between so fundamental result that we have is cos theta = x cap transpose y cap. A fundamental result that I have is x cap transpose y cap = cos theta angle between them. I want this particular idea to be generalized in a vector space. Now what we know from trigonometry is that cos theta is always bounded between + or - 1.

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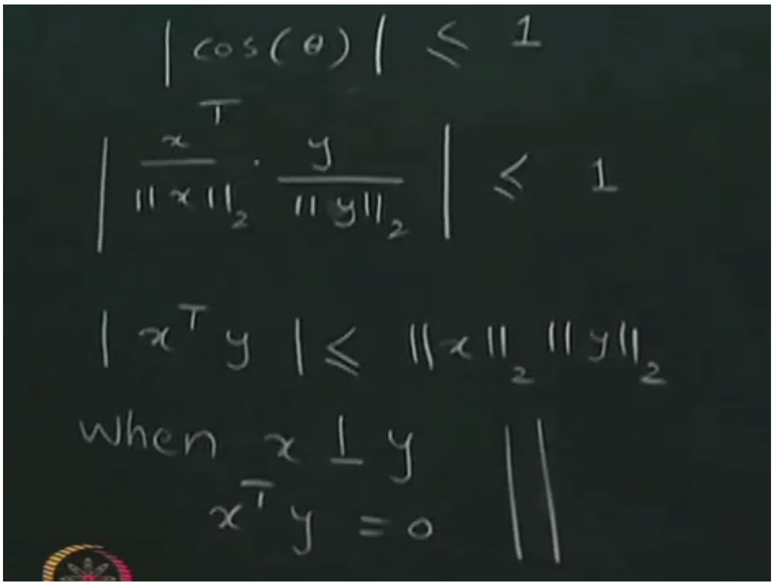

So another way of stating this equality is to say that mod cos theta is <= 1 or this also means that in 3-dimensions x/norm x. This is another way of writing the same inequality. Cos theta is always < 1. So x transpose so this is a scalar and this is also a scalar. So I can write this as mod of x transpose y is always <= and then what was important property of that we said we want to have is orthogonality.

So when x is perpendicular to y, x transpose y = 0. This was very, very important for us. We extensively used orthogonality. We use orthogonal basis. For example, the most well known orthogonal basis is i, j, k unit vectors perpendicular along the coordinate directions. So orthogonality is a very, very important property we want it. So I want to now come up with generalization of these results in a vector space that is my aim. So I am going to define a new entity called inner product.

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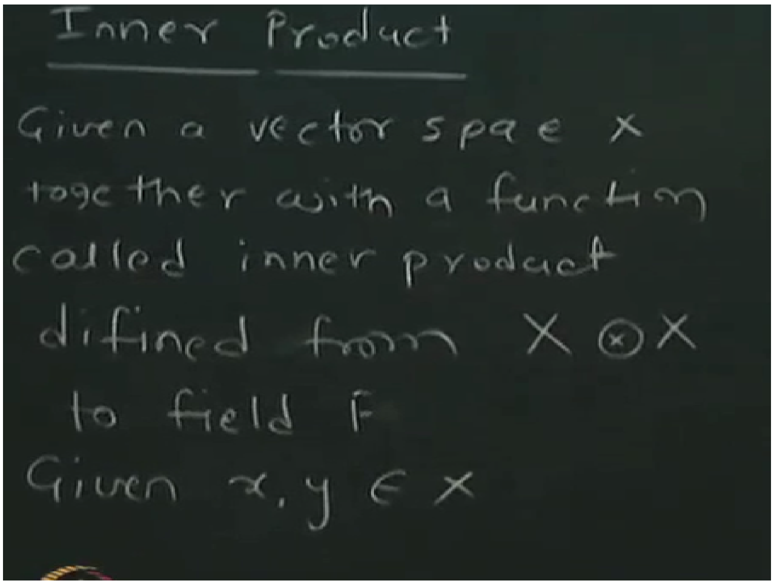

I am given a vector space x together with a function called inner product. So I am given a vector space. I am given a vector space x and a function which is defined on x * x to the field f so given xy that belong to x.

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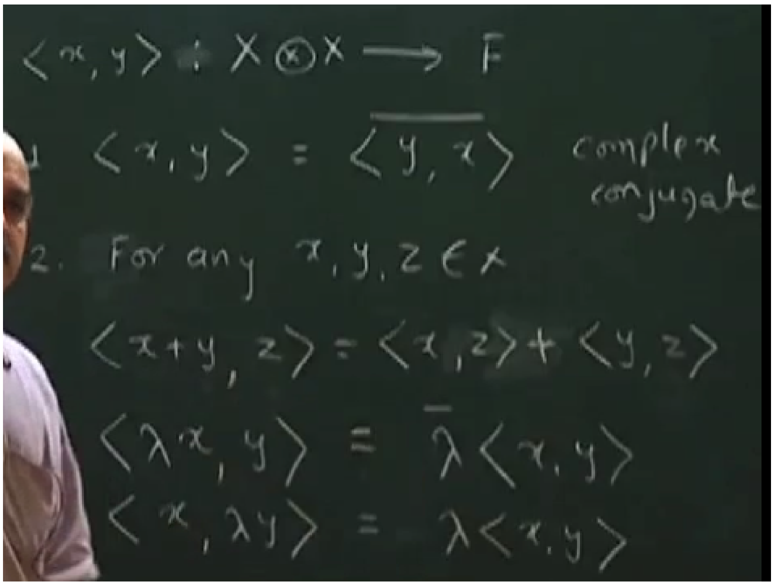

I am going to define inner product x inner product y this is the notation that we are going to use throughout the course x inner product y is defined from x * x to. So what are the axioms that govern this definition? There are 3 axioms the certain properties which are generic to inner product in 3 dimensions which I want to now generalize and come up with a generalized definition which will in a special case would be this dot product which you know in 3 dimensions.

So my first axiom is well when I am working with vector spaces in many situations I have to work with complex valued functions and complex valued vectors so what this says is that if I change the order if I take inner product of x with y then and if I change the order what I get is the complex conjugate. So typically the field that we are going to work with is rrc set of complex numbers.

So well if you are working with real valued vectors or real valued functions then this is very obvious. If I change the order, if I make x transpose or y or y transpose x I am going to get the same value with complex numbers you get a complex conjugate that is important. The second property I want the inner product to observe is that if I given any 3 vectors any x, y and z that belong to x.

So this inner product that we define should distribute over vector addition so if I take x + y and take inner product with z then that is same as adding these 2 inner products. x with z and y with z. that is the second important property of a function to qualify as an inner product. So what is the third important property? The third important property is that if I take a scalar lambda and multiply with x then this is same as lambda bar.

This is same as, but the way it happens with the second element and the first element is different in inner product. If you are working with real numbers, these both results are same because lambda bar = lambda. So if you are working with complex numbers you need to separate these 2 equalities. So if I take the first vector * lambda that will be same thing as multiplying inner product of x and y with lambda bar.

That is complex conjugate of lambda and if I take the second vector multiplied this is y if I take and multiply by lambda and it is same as lambda * so this is another essential property of this is an axiom that defines a function to be inner product. I want to maintain this because we have to generalize a few things. So I want to draw parallel so let this be there for sometime. What is the last axiom?

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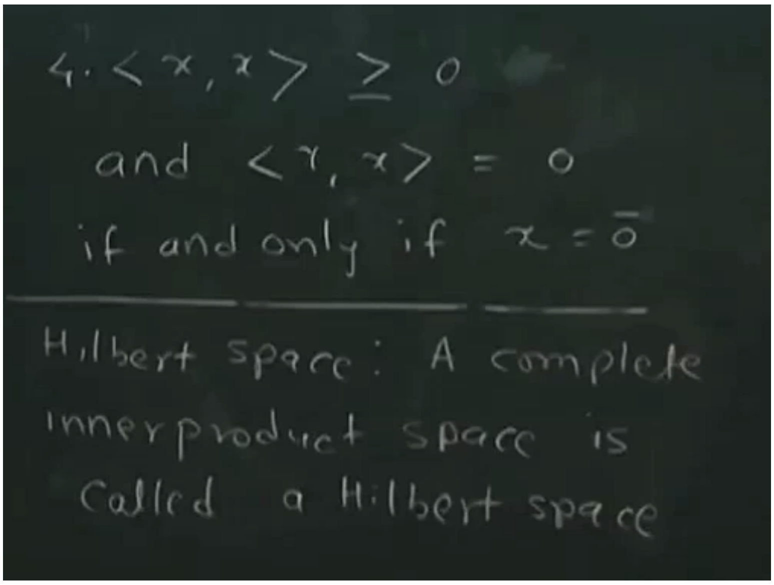

The last axiom is the 3th axiom is if I take inner product with let us look at here let us look at this property. if I take inner product of a vector x with itself what do I get I get 2 norm in 3 dimensions. What is 2 norm? 2 norm is if x1, x2, x3 are 3 components x1 square + x2 square + x3 square whole to the power half. So this particular property is quite important in light of generalizing this.

So this should always be > or = 0 and the inner product of x with itself should be 0 only if x is 0. This is also very, very important here in 3 dimensions only inner product of x will be = 0, x transpose x will be 0 only if it is origin the same property is being generalized here. in fact, this is what helps us in defining a norm which is tied up with inner product. A norm which is tied with inner product.

A norm which is tied up with inner product plays very, very important role in numerical analysis because this is a norm which comes with a definition of angle that is why 2 norm is something which is very, very often used in applied mathematics. So now let us start looking at can we define there in 3 dimensions we defined a norm using inner product can I do it here in a general vector space.

So I said any function that obeys these 3 axioms qualifies to be an inner product so it is not necessary that we have to you know we have only one particular way of defining inner product. We have a generic way of coming up with a definition of inner product that is suitable to our application. What I mean by this will become clear as we go along. So let me define some examples of inner products which are even on R3. I will show you that there are different ways of defining inner products on 3 dimensions.

But before that let me just state what is the Hilbert space. Sometime back in the last lecture you heard about Banach spaces. **“Professor - student conversation starts”** What are Banach spaces? complete norm linear spaces. So what happens in complete spaces. Every Cauchy sequence is convergent in the space. **“Professor - student conversation ends”** So Hilbert space.

A complete inner product space is called as a Hilbert space. This name is given after a great mathematician Hilbert who laid foundations of functional analysis. So now some examples of inner products.

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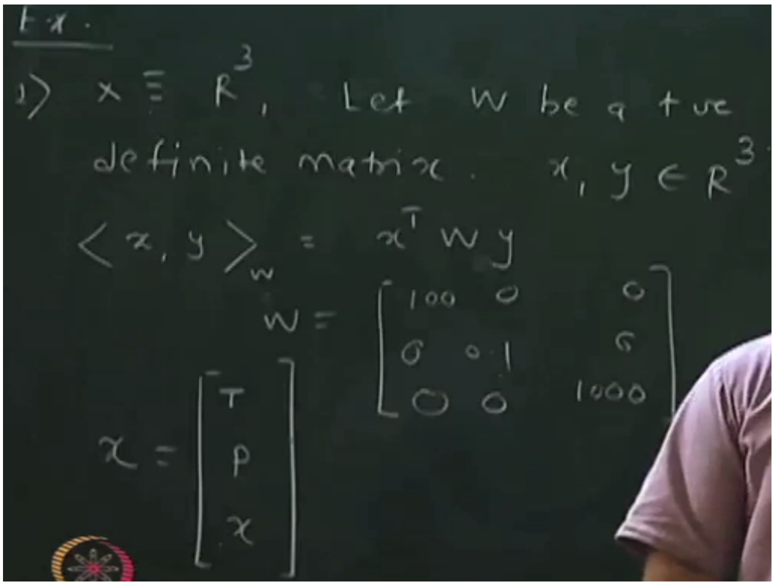

My first example is to show that there is no unique way of defining inner product in 3 dimensions also. Let me take my x as R3 now when you say R3, the field is R I am not going to write it every time this to keep the things simple I am going to define the inner product on this now which is different from what we have done earlier you know x transpose x. So let W be a positive definite matrix.

So now we define an inner product using this positive definite matrix W. So my inner product for any 2 vectors x, y that belong to R3 so I have this x and y belong to R3 and x inner product y I am going to give a little subscript here w is going to be defined as x transpose w

y. A simplest example of w would be a diagonal matrix. See for example simplest example of w would be you know matrix which is 100.1 and 1000.

Now you might ask me why do you want to define some funny matrix w? and then call it as inner product where is it useful that is why I am working with a reactor and my x is a vector that consist of say temperature, pressure, and concentration fractional. So this is in 10s and 20s. It is a degree centigrade temperature. Pressure let say is you know defining Pascals mega Pascals so it is in 10 to the power 5 something here you know x is in fractions.

If I use my old good old way of defining inner product or length, I have trouble because this mod fraction is always going to be a small number. Square of it is going to be smaller number. So many times I need to work with scaled variables. I need to work with scaled variables. At such times it is useful to have an inner product definition which normalizes the unique differences between different variables.

I am not just defining this w matrix just like that there is purpose behind this under many, many situations value you will get into this kind of normalization business where you have to use a matrix w. Now let us see whether this particular inner product satisfies properties that are specified. What is the first property? Just go back and look at yours.

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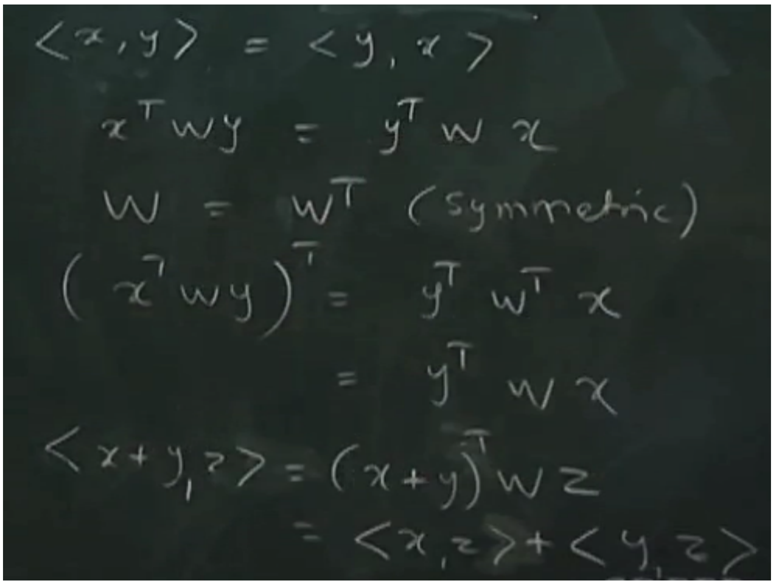

So first property is. My first property is that x y should be y x bar, but we are not working with complex numbers. We are not working with complex numbers. Since we are not working with complex numbers in this particular case it is if I interchange it should not matter. So this I do not have to prove to you that x transpose wy is same as y transpose wx. Why this is true?

This is always true I need one more property. I have missed out something here. **“Professor - student conversation starts”** (()) (20:19) No, I just said it is a positive definite matrix, I need something more (()) (20:25) yes, so I need this matrix W with positive definite and also this W has to be symmetric. w= w transpose. This should be symmetric otherwise this does not hold. Otherwise this does not hold. **“Professor - student conversation ends”**

So this matrix being positive if it is not sufficient, it should be a symmetric positive definite matrix. Then what will happen if I take x transpose wy transpose that is y transpose w transpose x which is y transpose wx because w transpose = w. So symmetry is very, very important symmetry is very, very important. So I need a positive definite matrix and a symmetric matrix.

What next? lambda * x what will happen? What is the next property? I think x + y the second property distribution is very obvious we do not have to prove this x + y z = x + y transpose wz which is nothing but xz + yz. I think this is just it just follows very simple. What is the third thing? if I multiply one of the vectors by a scalar inner product should get multiplied by now we do not have to do mod here, there is no bar here we just have to take the scalar out because we are working with real number so the third property is very, very obvious.

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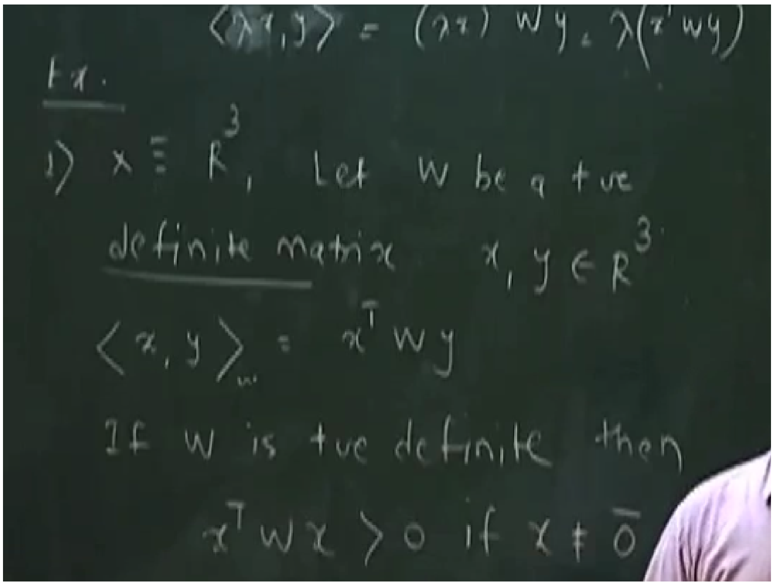

That is lambda *xy is lambda x transpose wy which is lambda x transpose wy. I do not want a complex conjugate because we are working with real numbers. What about the 3th property does it hold? x transpose wx if take inner product of a vector with itself what is the meaning of positive definiteness? All the Eigen values are > 0. There is no 0 Eigen values all the Eigen values are > 0.

The definition of positive definiteness itself means this is the definition of positive definiteness. A matrix is positive definite the fundamental definition of positive definiteness is that if x transpose wx is always > 0 if x is != 0. If x = 0 it will be = 0 so only vector that will give you x transpose wx = 0 is 0 vector that is what we wanted. All 3 examples are satisfied.

So this is another way of defining inner product on 3 dimensions. These kind of inner product be very, very routinely used in numerical methods because we need to do scaling of variables x will consist of pressures temperature concentrations all kinds of variables which have different units and then if you want to find out length of such a vector you cannot just say x1 square + x2 square + x3 square.

You need to multiply by a suitable waiting matrix that is why you need this. **"Professor - student conversation starts"** (()) (24:46) that is why I said w has to be positive definite and symmetric. Symmetric is important. (()) (24:59) x transpose wy will be symmetrical. So just positive definiteness is not enough we need symmetry also. So symmetric positive definite matrix is important and then. Sir what is lambda bar?

Lambda bar is complex conjugate, but we are working right now with real numbers so complex conjugate will be real number itself. **"Professor - student conversation ends"**

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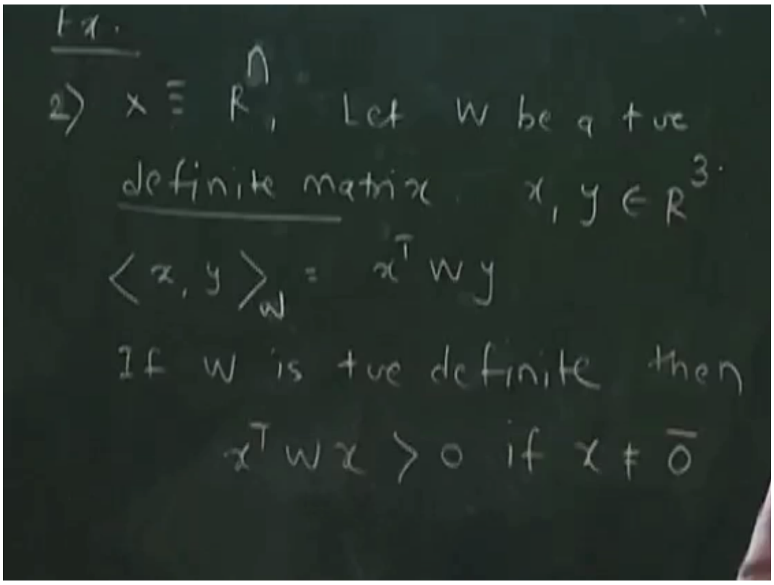

So this I can very easily change to my second example where you talk with Rn I could have talked with Rn and the same thing would hold. I have a symmetric positive definite matrix and I can define a norm which is I can define an inner product which is using any symmetric positive definite matrix which is n * n which will give me all the properties that are need for defining inner product. We still have not established the connection between the last axiom and the norm.

I have been just saying that well it is related to the inner product inner product gives you a norm which is here but actually we need to see that connection. So I will give one or 2 examples and move to proving, that actually inner product in a general space defines a norm just like in 3-dimensions x transpose x gives you a norm.

You will also get a norm defined through inner product. Before doing that let me give you one or 2 more examples of inner product spaces. So my second example would be Rn or I can easily move to Cn a complex valued (()) (27:04) and so on where the matrix there should be Hermitian not symmetric positive it should be Hermitian. Moving on from finite dimensional spaces let me give you third example.

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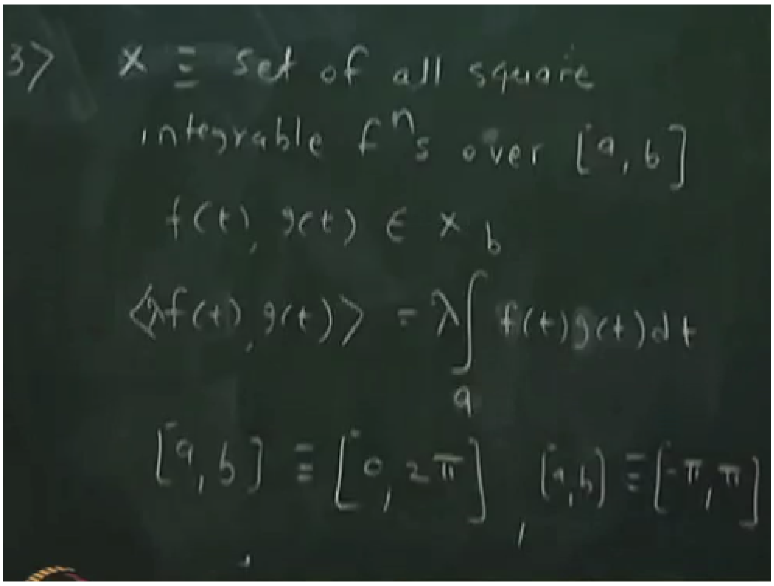

So set of square integrable functions over an interval ab set of square integrable functions over the interval ab you have come across this kind of a set when you worked with 3ier series expansion. Now you will soon realize what are the connections so if I am given any 2 functions say ft and gt that belong to x then I can define an inner product between ft and gt as integral a to b set of all square integral functions.

So integral over a to b typically when you study 3ier series in your undergraduate we look at a b that corresponds to 0 to 2 pi or we look at ab that correspond to - pi to pi. You remember something like this when you do 3ier series expansion you take sin theta or sin T * ft dt integral sign t Ftdt that is actually inner product and you can just check whether all 3 axioms are satisfied.

Let us look at first axiom what is the first axiom. If I interchange f and g will the integral be different? So first axiom is satisfied. If I multiply ft by some lambda what will happen to the integral it will be lambda times second property is satisfied what about distribution if I take f + g inner product with some ht it is very obvious. The third property.

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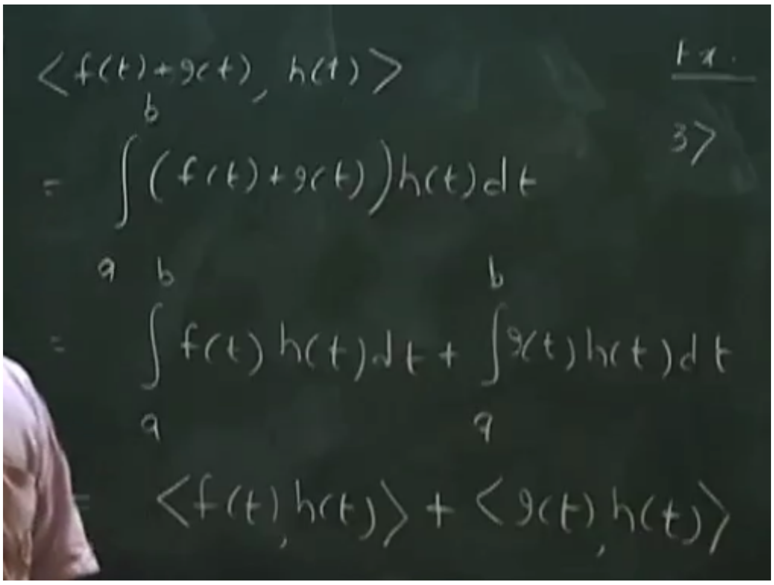

If I take ft + gt inner product with xt this will be integral a to b ft + gt which is same as integral a to b everyone with me on this? so the third axiom is satisfied what about the 3th axiom? If I take inner product of a function f with itself what will happen? Will always be a positive number why?

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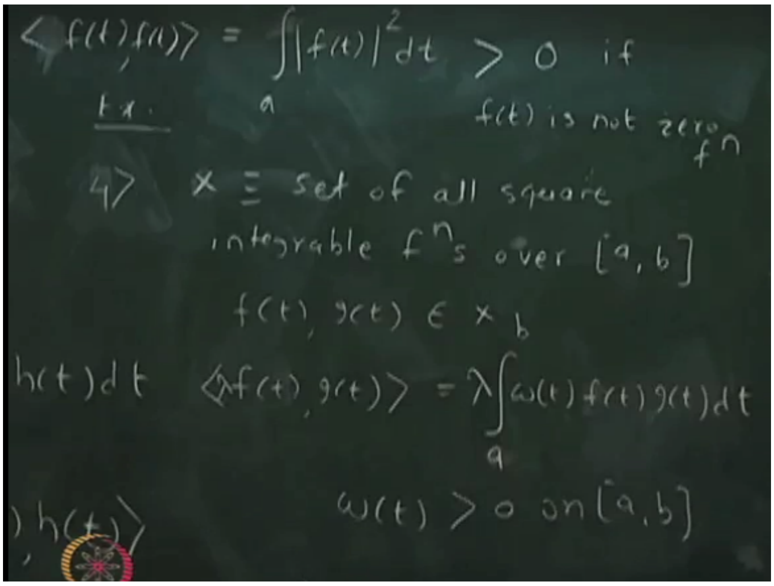

So my 3th axiom is integral of ft with ft this is nothing but integral a to b ft square dt which is always > 0 if ft is not a 0 function. Am I correct? if ft has even one non-zero value in interval a to be ft square will be positive, ft square dt will be positive so ft as long as this will be 0 when f is 0 everywhere on a b. If f has non-zero values this integral will always be non-zero.

So all the 3 properties that you need for an inner product space or inner product to be defined are satisfied. I could further modify this inner product see just like from x transpose x from x, I said x transpose wx where w is a symmetric positive definite matrix is also inner product I could expand this definition by putting a positive waiting function here so I can have another definition my 3th example would be.

I will take waiting function wt ft gt dt wt is strictly > 0 on wt is strictly > 0 is a positive function wt is a positive function it has only positive values in the interval ab, this is my interval ab on which inner product is defined on which the space is defined just like you could use the positive definite symmetric matrix there if I modify my definition of inner product by multiplying a positive waiting function that also satisfies inner product.

And these kinds of waiting functions we are going to hit up on soon in when we come up with different ways of defining inner product on set of continuous functions which are square integrable. We will also come up with these kind of inner products. We will need them when you solve partial differential equations so boundary value problems when you solve in the mathematical methods scores.

So there are different ways of defining inner products yet we have to establish 2 major connections one is with the angle and other is with the norm. So let me start preparing for this. I need to prove an inequality which is essence which exactly captures this part. in order to show that an inner product defines a norm I need to pull an inequality called as Cauchy Schwartz inequality.

And this inequality will help us to come up with connection between inner product and the so called 2 norm. This is 2 norm and we want a connection to be established in a general. So what all things that you need for a function to be a norm when do you call a function to be a norm what are the 3 axioms?

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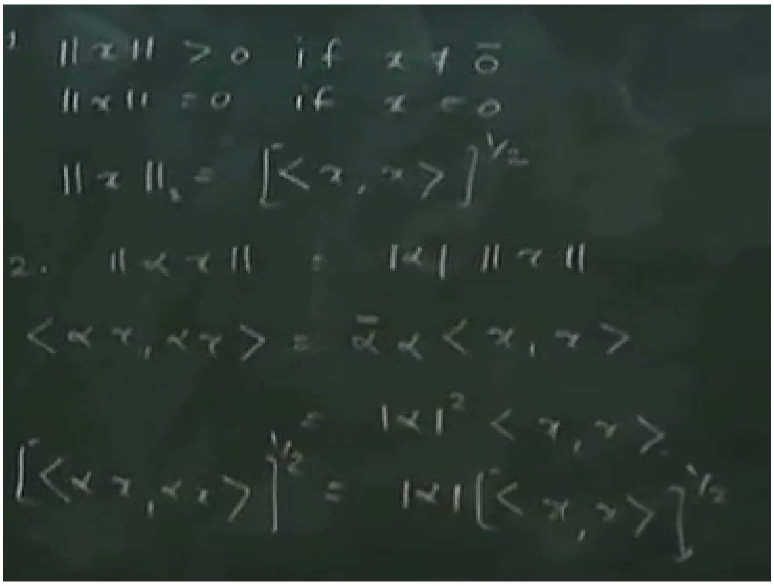

One is norm x > 0 if x not = 0 and this is =0 if x = 0. That is the first axiom. What is my candidate norm definition is I want to use in an inner product space I want to define a norm actually we will call it 2 norm, but right now let us keep calling it 2 norm here is I want to say x x raise to half that is what I want to do is x transpose x. This is my candidate function.

Now does this follow the first axiom for norm does it follow from the definition of the very definition it will follow nothing to worry. What is the second thing about norm scalar multiplication so if I take alpha * x then that is = mod alpha norm x what about this does it follow let me see alpha x alpha x what is this equal to alpha bar alpha x x right. First element alpha bar, second element alpha so which is mod alpha square x x. so alpha x alpha x raise to half = mod alpha x x raise to half we have proved whatever we wanted so far so good.

Now comes the third problem. What is the third thing? Triangle inequality. Triangle inequality is where we need this to be generalized. You cannot go to triangle inequality unless you generalize this result in inner product spaces and here we need a little bit of work. I am going to prove this on the board why this defines an inner product why this inner product defines a norm and how you can generalize this result in an inner product space is called as Cauchy- Schwartz inequality.

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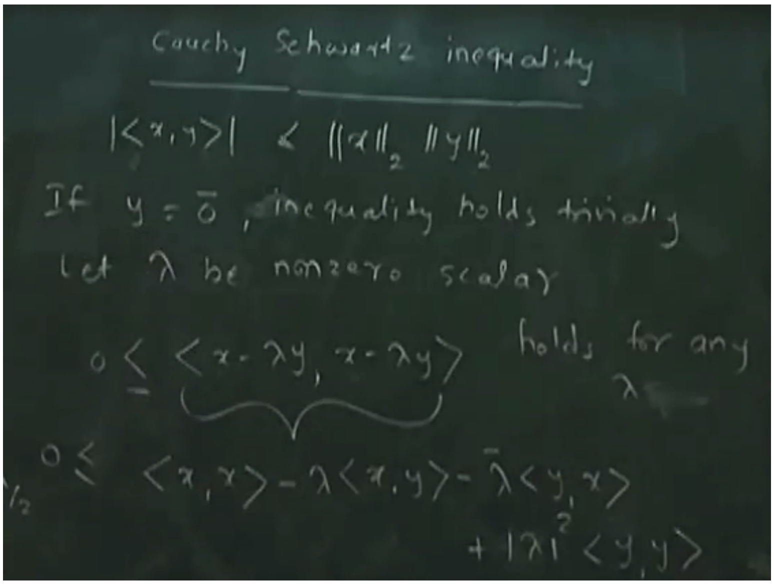

So that what is that I want to do I want to generalize this particular result from 3 dimensions except here it is written x transpose y I want to prove I want to arrive at mod of xy is <= norm x if I take inner product of any 2 vectors xy is absolute value is less than. This is what I want to show in any inner product space that is generalization of the result cos theta is < 1 and with that I will move to triangle inequality because I have to establish triangle inequality to come up with.

So how do I do that? Now let us first look at the situation were y= 0 vector. If y = 0 vector does this hold always because in a product with 0 will give you zero. 0 <= 0. So if y = 0, so we do not want to look at a trivial case 0 vector case. Now to prove this inequality now I am going to play a trick. So let lambda be a scalar non-zero scalar such that I am going to take a vector x - lambda y and take inner product of x - lambda y with itself everyone with me on this.

Lambda is any arbitrary scalar. So does this hold for any non-zero lambda this inequality holds for in why inner product of a vector with itself is always > or = 0. So this always holds for any lambda not = 0. It holds for any lambda. So what is this quantity on the left hand side can you expand this. So this will be x inner product x I am taking first with first when x inner product lambda x y - lambda bar y x look carefully lambda times lambda and x distribution.

I am using the distribution property plus everyone with me on this. I have just expanded the right hand side. This also of course has to be > 0. This is x inner product x always > 0, mod lambda square y always > 0 now I have 2 quantities in between x and y and y in the product

x. so let us preserve this part here because this is what we are generalizing. so this holds for any lambda am I correct that in equality which we proved there holds for any lambda so I am going to pick one specific lambda now.

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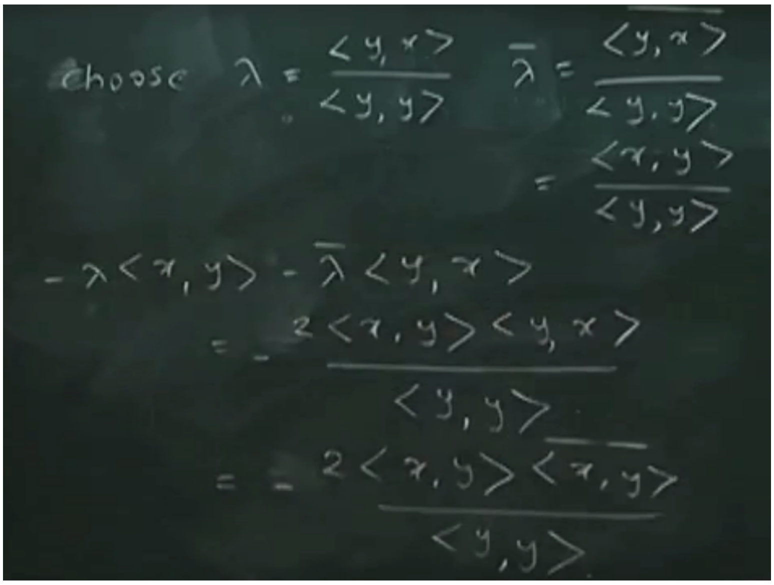

I am going to pick one specific lambda. Inner product is a scalar ratio of 2 scalars. This is y is not zero. So since y is not 0 this is a positive number and this lambda is a valid lambda so this should hold for this lambda also for this particular lambda. What is lambda bar? Is that right I just used the first property. Now I am going to substitute this lambda and this lambda bar in the inequality that we developed earlier so using these lambda and lambda bar I have 0 > x inner product x.

Before that let us do a little bit of work so this implies that - lambda x inner product y - lambda bar y inner product x. This is equal to if I just substitute this lambda and lambda bar then what I will get is that this is nothing but 2 times x y, y x. Just check what is lambda y x I am substituting in the first thing there xy and what is lambda bar xy, but there it comes yx. Just algebraic juggling. This is equal to well - is here of course. - sign will persist.

So this is equal to - 2 times x inner product y, x inner product y bar. y inner product y. I will move on to here now. Is everyone with me on that.

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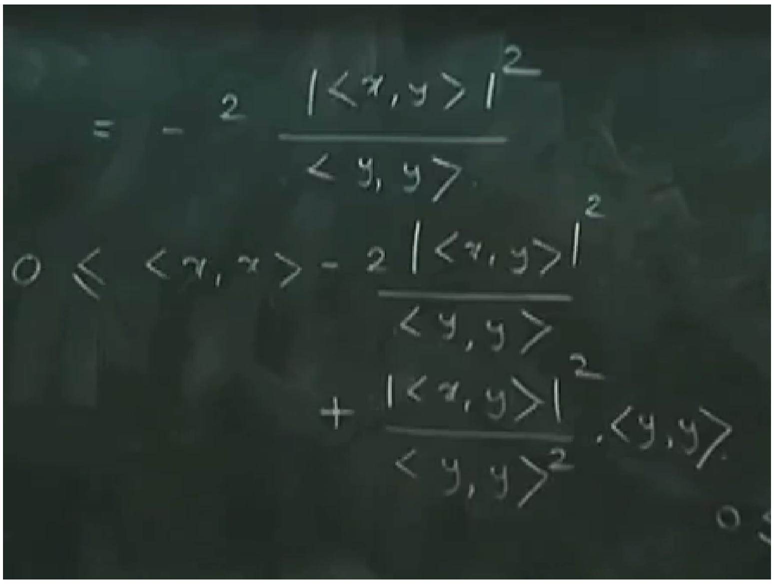

So this is = -2 mod. Now so this quantity here can be now replaced by our new value. so I get 0 > x - 2 and our lambda is if I substitute for lambda square where lambda square would be if I substitute for lambda square it will this and then finally the inequality that I get is.

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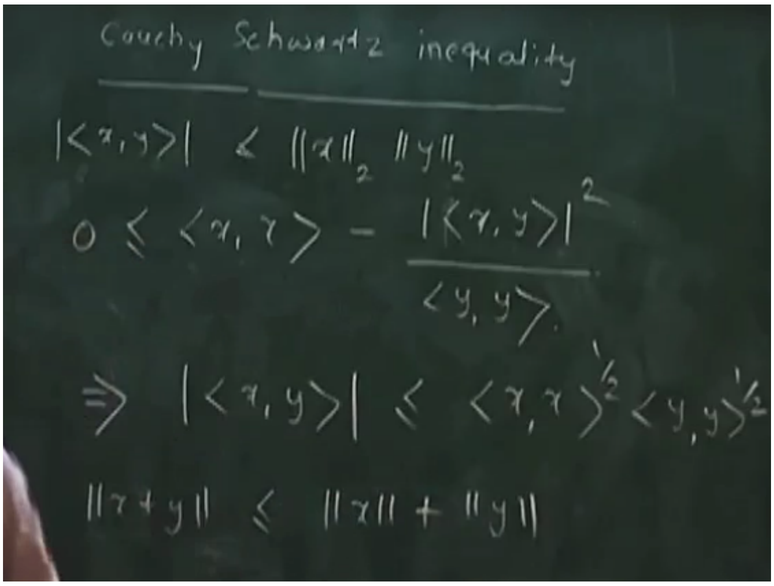

I finally get an equality which is 0 > x x -. So this minus, this is always positive and I am just doing algebraic juggling now there is nothing specific if you are not followed right now just goes through meticulously through the notes you will see the steps just substitutions. I have just eliminated by this is a scalar so this square this will cancel with the square and then you can do the juggling this is not so difficult. So what does this imply?

This implies that the above thing implies that mod of xy is <= x x raise to 1/2 yy raise to 1/2. I take this on the left hand side and take the square root. See this is square of this inner product of x and y. I take this quantity of left hand side because this is greater than this right otherwise this cannot be > 0 and then I am just doing multiplication yy I have brought it on this side I have just omitted one in between steps. Everyone clear about this no problems.

So what is this? This inequality is same as these results in 3 dimensions which we know no difference. x dot product y mod of that is always < this which is nothing but cos theta < 1. So I have proved an equality which is Cauchy-Schwartz inequality I have proved an equality called Cauchy-Schwartz inequality and this helps us to prove the triangle inequality. How will I prove triangle inequality now?

What is triangle inequality? So triangle inequality should be norm so we want to prove x + y to or x + y <= norm x + norm y. We want to prove this inequality finally and I want to use this. I want to use this result. This is Cauchy-Schwartz inequality. This is generalization of this result. Well once I declare x transpose x to be norm of x I can actually even move to this inequality because this is a scalar I can divide take it inside and so on.

We will move to that little later. In the next class, we will start from this inequality Cauchy-Schwartz inequality and move on to proofing triangle inequality. Once we prove triangle inequality we have done. Once you prove triangle inequality we have show that inner product defines the norm. 3 axioms of norm 2 of them we have already proved the third one was triangle inequality. To prove triangle inequality, we need Cauchy-Schwartz inequality.

But Cauchy-Schwartz inequality not only helps you to prove triangle inequality it also gives you a way of generalizing definition of angle. It will also give you a way of generalizing (()) (52:43). So we will be able to define orthogonal vectors in any inner product space. These vectors could be 2 functions like sin and cos or these vectors could be 2 polynomials. We will talk about orthogonal polynomials.

Why do we talk about orthogonal polynomials? Why do we talk about orthogonal functions? They are very, very useful when you do mathematics applied mathematics, but why where they called orthogonal? Why were they called orthonormal or whatever so those questions will get answered if you understand this basis that is why I am doing all this proofs? So in the next class, we will move on to triangle inequality and then more properties of inner product spaces.

We will see that the famous Pythagoras theorem which you studied in your 8 grade also holds in any of these inner product spaces what a relief you can work with orthogonal vectors.