Advanced Numerical Analysis
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Lecture – 05
Examples of Norms, Cauchy Sequence and Convergence, Introduction to Banach Spaces

So, let us begin with this is a quick review of axioms for norm.

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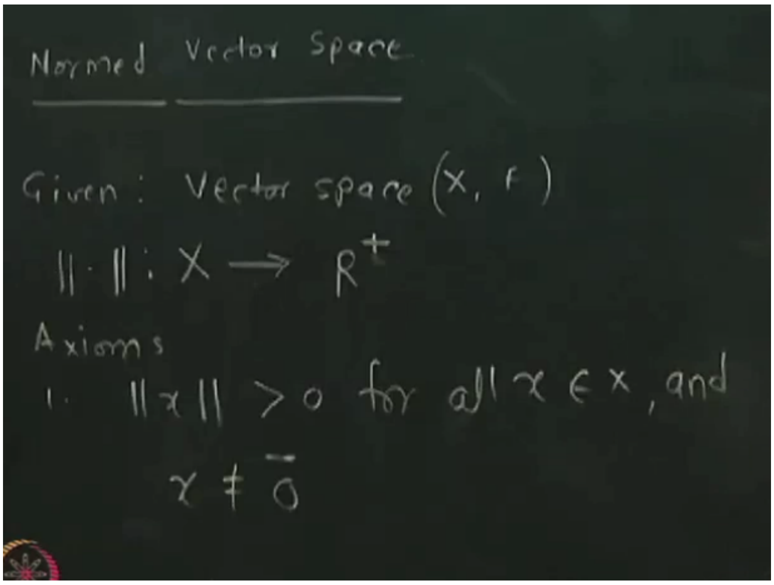

So given the vector space x and of course the field f, we wrote 3 axioms for defining the norm on this. So, this function called norm was a function that was from x to R+, R+ is real positive real numbers including 0. So R+ a set of positive real numbers including 0, so this is a function from an element in x which is the compact way of writing for element in x to R+ and we said the 3 axioms, one is that norm x is > 0 for all x that belong to x and x != 0 vector.

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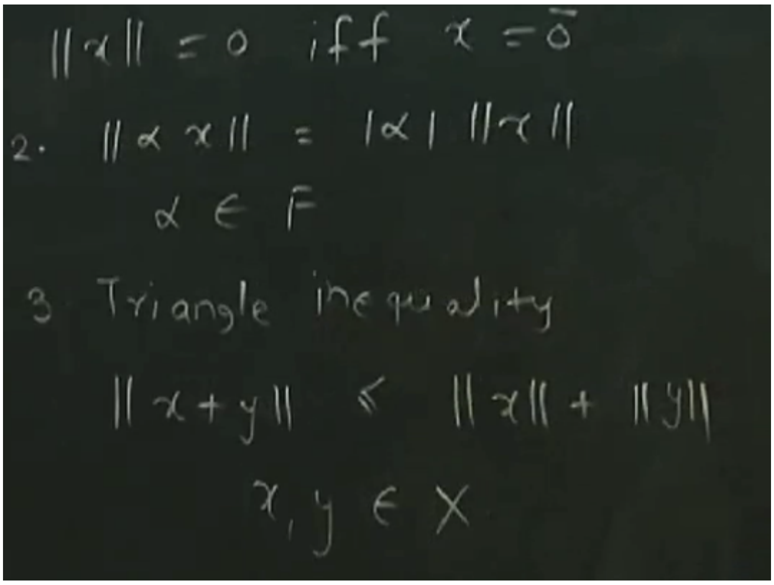

And norm x = 0 if and only if x = 0 vector, so this was the first axiom. The second axiom was norm of alpha * x is norm x mod alpha or absolute value of alpha * norm x where alpha is a scalar belonging to field f and a third axiom is triangle in equality. So this states that distance of x + y take any 2 elements from vector space x. Distance of the vector x + y is always <= or the length of vector, norm of vector of x + y is always <= norm x + norm y.

This is generalization of triangle equality that you know for one dimension or in 3 dimensions for triangles generalized to any other space. So we said any function that satisfies these criteria it should be a real positive function. It should give you a real number. It should be a nonzero real number or when x is not 0 it should be 0 real when x != 0 vector and so on and then we saw couple of examples that are functions that can be classified as a norm or that cannot be classified as a norm.

So both are important because you understand something better when you see where these one of these axiom fails. So there are multiple ways of defining norms not a unique way. A pair of a vector space together with or a linear space together with a definition of norm gives you a norm vector space. So that is take home message. Well, why did we do this?

I yesterday said that we are doing all this because you know want to talk about (()) (04:20) point about limits and sequences so why do I need to talk about limits. When we work in numerical methods we are forced to look at sequences of vector. I just give you a very brief example. We will actually do this much more in detail later. Let see I want to solve this equation.

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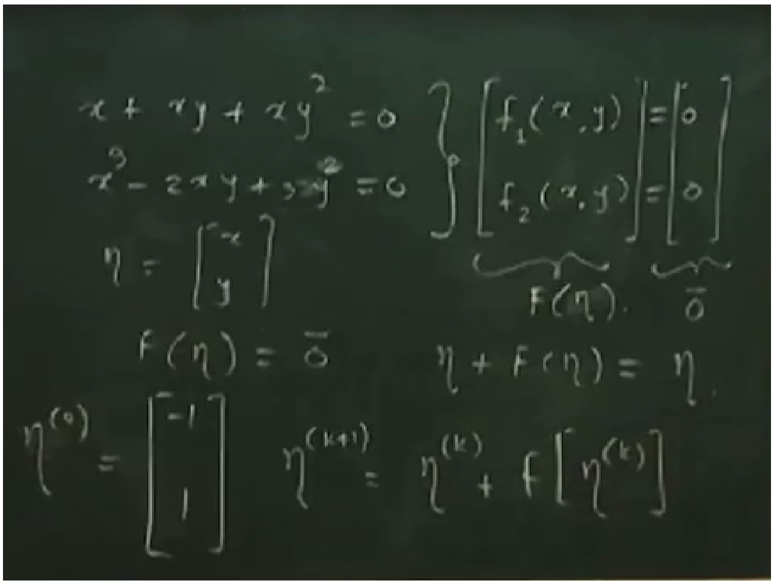

These 2 are coupled equations and I want to solve them simultaneously. I want to solve them simultaneously. These kinds of problems I am writing it in an abstract form very often we encounter these kind of problems. Well I am going to write this in abstract way as f function 1 xy = 0 and function 2 xy = 0. There are 2 functions. F1 xy = 0, F2 xy = 0 and this kind of equations arise steady state of a CSTR concentration and temperature are linked.

So first equation could be energy balance, second could be material balance and then you get 2 equations into one concentration temperature let us solve them for. I am going to define a vector. This is my function vector. I am going to call this as f of x of well let me call some new variable = 0. So my eta is a vector which comprises of x and y and then I want to solve for f eta = 0 vector.

This is my 0 vector. I am just writing the same thing in a different format. Now what method you know for solving this? How do you solve this? **"Professor - student conversation starts"** (()) (06:48) bijection method pardon me bijection method. Bijection method is for difficult to scale to 2 variables. One variable well defined bijection method is there. **"Professor - student conversation ends"** You can have bisection method for 2 variables, but well, let us take a very simple iterative scheme.

Let us construct a very simple iterative scheme. I will write eta + f eta. I will add this vector eta on both sides and then I construct an iteration, whether it will convergent or it is a different story, but I will construct an iterative process. So I will start with some gas vector eta 0 that is let say well, I do not know what the solution is so I am going to guess some solution so let say I start with say - 1 and 1.

This is x, this is y and then what I want to do is to say that eta k + 1 = eta k + f of eta k is it okay. I have just formulated an iteration scheme in which I start with vector 0. I take the 0 vector substitute here I will get vector 1. I take vector 1 substitute here; I will get vector 2. **"Professor - student conversation starts"** How do I know whether this sequence of vectors is converging to something? (()) (08:31) Pardon me. The difference between eta k + 1. What is difference?

See it is a 2 dimensional vector, now I will just further convenience it into 2 dimensional vector I could have done this in n dimensions. I could have written this in an n-dimensions n equations in n unknowns very, very common. Chemical engineering starting trying to solve steady state energy material balance for a plant. You can get 1000 equations and 1000 unknowns okay. The difference vector, Pardon me the difference vector, but what of difference vector? Norm of the difference vector.

So, we have to talk about a vector converging to another vector, a vector converging to a solution. What should happen at the solution?

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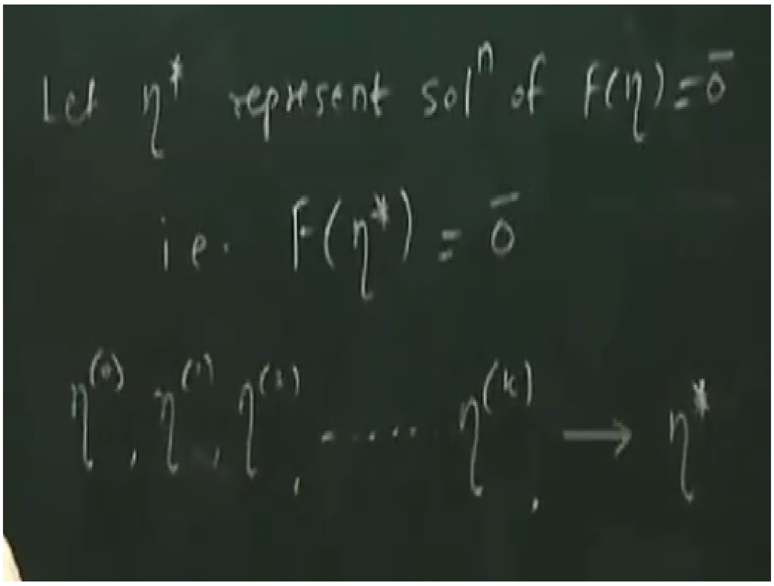

Let us say if x star is a solution eta star is a solution. f eta = 0. What she says is correct that one thing is that you know should be = 0 of course at the solution so at eta start so that is f eta star = 0 fine, but I am starting an iterative process. So what I am going to get is I am going to get this vector sequence x eta 0, eta 1, 2 and so on and I am going to get this vector. The question is the sequence is the sequence converging to eta star?

Does this go to eta star? That is the question I need to answer. See this is the solution. If I had pluck eta k here it is not going to be equal to zero, it is not going to be equal to 0 so it is going to some other small number probably. Is it small? So how do you answer this question? In general n dimensional spaces or function spaces that is where we need to now talk about I may have scenario where I have a sequence of functions.

I have sequence of function and I will give an example I am going to show you a small demo also sequence of functions. **"Professor - student conversation ends"** So the question is, is this sequence convergent? So this kind of problems are always encountered in numerical analysis because almost every method that you have for solving you know most of the problems through computing is iterative you start with the guess and you come up with a new guess and so on. So there is this need to look at convergence of sequences.

So we are going to define 2 notions, one is Cauchy sequence.

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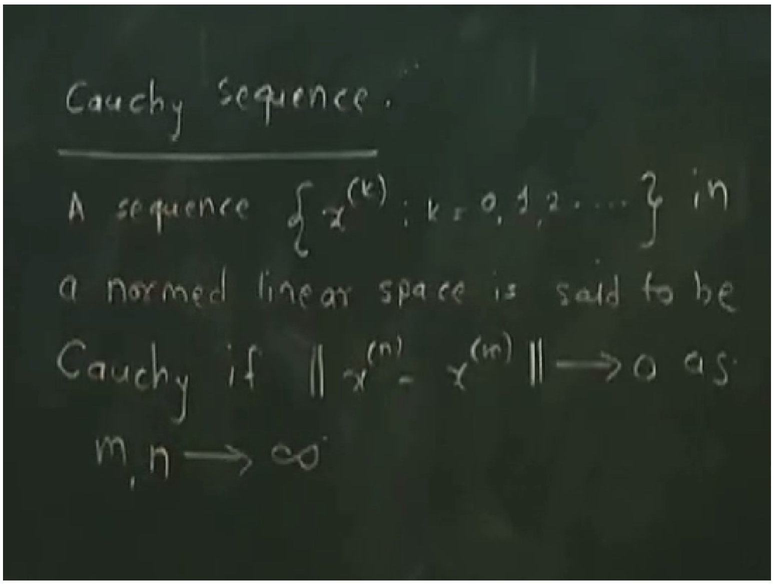

So I am taking a set of infinite set of sequences or infinite set of vectors which are generated by some process. You know it could be some iterative scheme by which you are working or whatever it is. Now I want to know how do I formally define convergence? A sequence of vectors is said to Cauchy if difference between xn - that is nth element in the sequence and m filament in the sequence if this tends to 0 norm of this tends to 0 as m and n become infinitive. So more and more elements are generating this, the vectors come closer and closer.

Well in one dimensional vector space. So in one dimensional vector space that is a set of real numbers. Well when a sequence is Cauchy it convergence to a limit inside a set, but depends upon the space funny things can happen if the space is not complete what is this business of completeness we will come to that soon before that let me define convergence sequence. So there are 2 different notions one is Cauchy sequence other is convergence sequence.

These (()) (13:46) just for the sake of nice mathematics. These are very, very relevant to computing. What is the convergence sequence?

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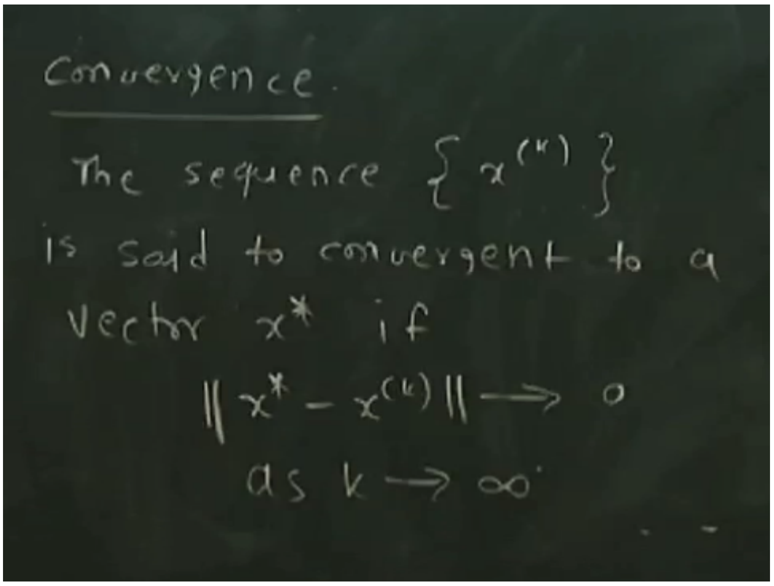

So I am considering this sequence again in fact this is a short hand notation for sequence I am not going to write every time k going from 0 to infinity or k going from 0 to n whatever it is curly braces x superscript k is a sequence in a norm linear space or a norm vector space. Now this is said to be convergent to a vector x star if this is said to be convergent to an element x star if difference between x star and x k goes to zero difference between x star and x k goes to 0 as k goes to infinity.

So what I want to show you is that it is not obvious that a Cauchy sequence will always be convergent it depends upon the space that you are considering. A convergence sequence is always a Cauchy sequence, but vice versa is not necessarily true. A Cauchy sequence may not be convergent. A convergent sequence is always a Cauchy sequence.

Now examples will make it clear why I am talking of this funny things and we will also realize that this is something that you deal with every day when you use computers. So I am going to take an example of a vector space in which a Cauchy sequence is not convergent. I am going to take an example of a vector space in which a Cauchy sequence is not convergent.

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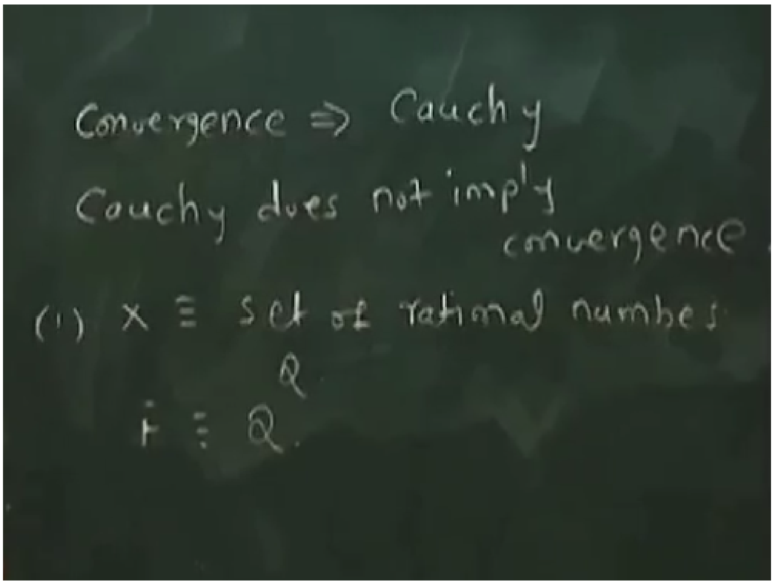

So basically I want to give an example of this idea that convergence to a particular element is something different when it depends upon the space. My first example here is my space x is. My first example here is a set of rational numbers Q and I am taking field f also to be Q. I am taking a field also to be Q. So, this combination will form a vector space, and I can find very easily a sequence in this vector space which is Cauchy, but not convergent. A simple example is. Now consider sequence.

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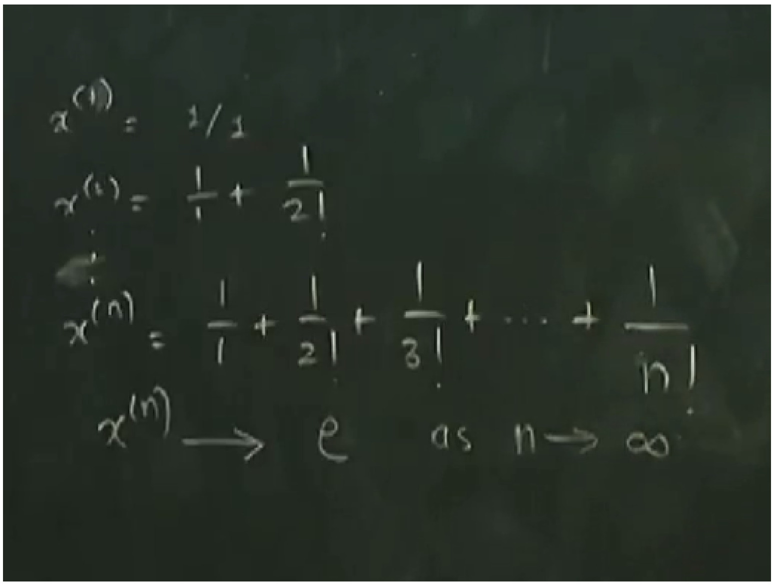

Whether I start index with 0 or 1 it does not matter I am starting with 1 x2 is 1 x 1 + 1/2 factorial and so on. So my nth element in this sequence is 1/1 + 1/2 factorial + 1/3 factorial. I think this is a well known series where does it converse to e, but e is it a Cauchy sequence? It is known to be a Cauchy sequence. It is a convergence sequence is real line. On real line where does it converse to?

So, this sequence xn this converges to element e as n tends to infinity. We know that this particular element tends to e, but e is not a rational number. So this element where it converges to is outside this space. So you have funny situation. You have a Cauchy sequence. If you apply the definition of Cauchy sequence if you take any 2 elements as n and m goes to infinity you take difference it goes to 0 that is very easy to show look at any book on real analysis.

You will see this proof it is just one or 2 pages of proof that this is a Cauchy sequence, but in this particular space it does not converge. It does not converge and in this space I can find many such sequences. I can find a sequence that is almost converging to pi, but pi is irrational number. Pi is not there inside this space. So likewise you know I have this sequence.

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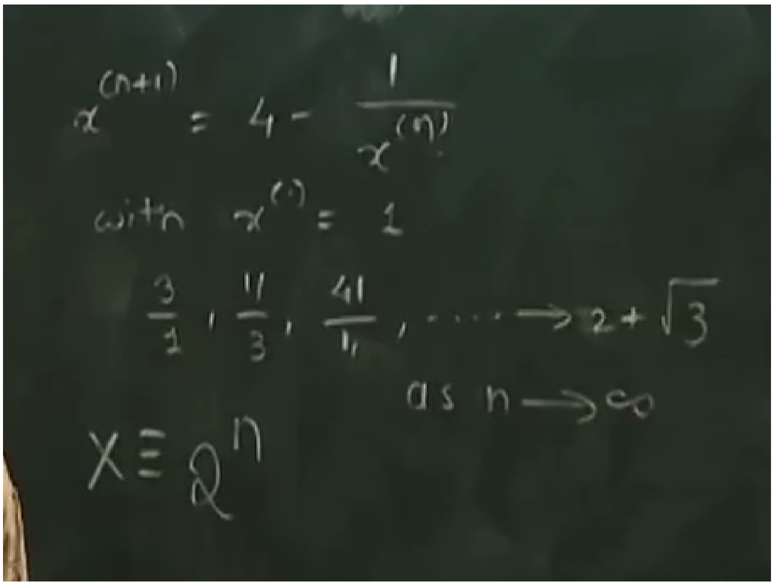

So this sequence that is 3/1, 11/3, 41/11 and so on it converges to not a rational number. I can find infinite such examples where you have a convergence sequence. You have a Cauchy sequence, but not converging to an element inside this particular space. **"Professor - student conversation starts"** (()) (19:52). Those are rational numbers. So this sequence is converging somewhere, but it is not converging inside this space. It will never converge inside the space.

(()) (20:00) so e does not belong to set of rational numbers that is what you are saying. We know that in a real line this will converge to e. See e is not the (()) (20:15) Why these are all rational number. Sir we individually (()) (20:19) x is being as 1/1 factorial + 1/2 factorial + 1/3 factorial. So you can always define one common denominator. It is a rational number. If it is just 1/1 factorial it is a rational number.

No, no all these are rational numbers. I think we can talk about it little later. This particular thing, these are all rational numbers. They are not irrational numbers. So you mean to say that 1/3 may not be expressible, but it is a summation of rational number. Rational number is whether you can write it as integer upon integer. I can always write integer upon integer. Whether you can express it as a continued fraction?

We are not looking at that problem right now. The true representation is integer upon integer. I can have a common denominator. For this it becomes a rational number. You are confusing between its representations in this computer I am coming to that. So do not confuse between the 2. So do not confuse one third with 0.33. Do not confuse that with 0.33. **"Professor - student conversation ends"**

If this is true about q it is also true about qn I can define a product space which is qn, n dimensional space. My x can be qn I can take a space which is where do you get qn? When I am doing computing in a computer, I can deal only with finite dimensional vectors. I can only leave with finite dimensional vectors and in computer you cannot represent many of these you know irrational numbers because computer has a finite precession.

If I take 64 beat precession the resulting number which you approximate as e actually will be a rational number something divided by I have to truncate right. I cannot have a representation do you understand what I am saying. In a computer whatever is the precession 64 bit, you know 128 bit, you go to very high precession computer. Any number is actually represented as you know using binary 1 0 1 0 1 0 sequence and there is finite number of bits used to represent the number.

So that number will always be representable as a rational number. Something divided by something. I truncate it. So the point which I want to make is that incomplete spaces are not so (()) (23:13) you know when you work with computer you are working with incomplete spaces and we have to bother we have Cauchy sequence which does not converge. Cauchy sequence this does not converge in a computer I will have a Cauchy sequence which does not converge to a number.

No it is true value. See for all practical purposes we say that well this is almost close to e, but it is not e. We take an approximation of pi may be you know correct up to 1000 decimals, but it is not pi okay. So, we are working with this incomplete spaces and then let me give you one more example and I want to show a demonstration here of an incomplete space.

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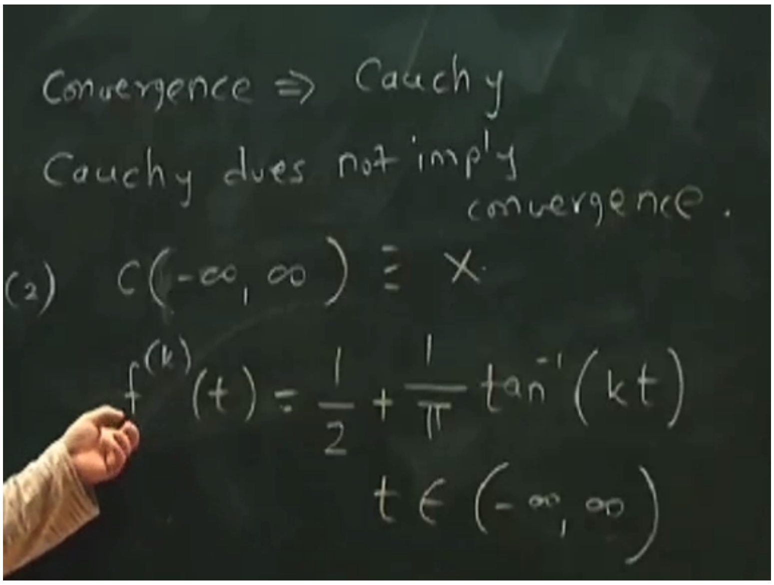

So my second example is set of continuous functions over - infinity, set of continuous functions over - infinity to infinity. This is my second example and I am going to construct a sequence in this particular vector space and what I want to demonstrate is that this sequence will converge to a discontinuous function. I have a sequence of continuous functions converging to a discontinuous function. So you are trying to solve some partial differential equation.

Or some problem you construct the solution as a sequence of continuous functions or continuously differentiable functions the sequence might converge to a non-differentiable, non-continuous function. So you can have funny situations. So my sequence here is this 1/2 plus my sequence here is a sequence of functions. These are continuous functions defined over interval - infinity to + infinity.

This is a function sequence define so t goes from - infinity to + infinity my k changes. K would be 1, 2, 3, 4, 5. I will get different functions for each value of k. So I will get k goes from 1, 2, and so on. K goes from.

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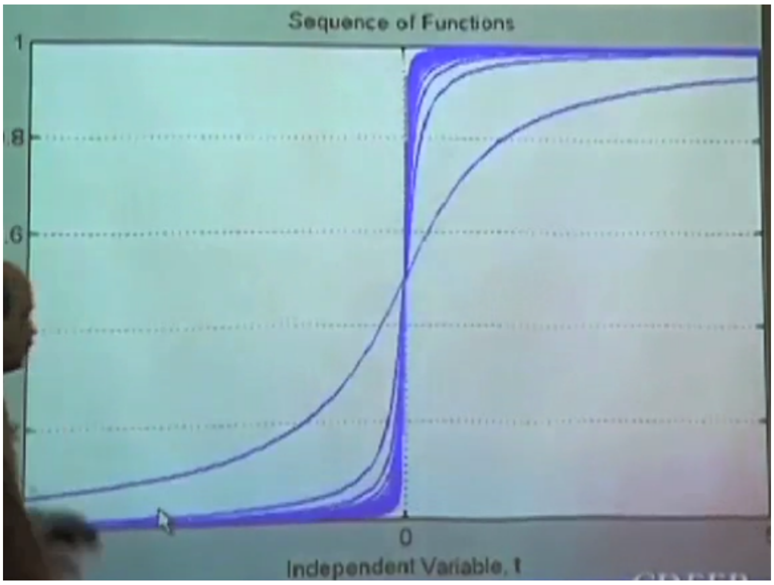

Function sequence I just want to animate and show you what is happening so this is for k = 1. This is for k = 6 I am going to increment by 5 and see what is happening. This is k = 11, 16, and so on I just go on right I am going closer and closer towards this step kind of a function. I am going closer and closer to the step function. So if you do this I have gone only up to 100 (()) (26:38). If I do this by incrementing k much, much longer much to a larger value, this will converge to a step function.

So moral of the story is that I am starting with a set of continuous functions. I am generating a sequence in this set, but this sequence does not converge to element in the set. The sequence does not converge to an element in the set. So there is a problem. So if what is nice about real line that every real line every sequence which is Cauchy will converge to an element inside them. Ever Cauchy sequence on the real line will converge to a number on the real line. So in some sense real line is a complete set.

There is nothing outside it whereas set of all rational numbers is incomplete. There is something outside and the sequences here seem to converge to something which is outside the space. seem to converge or something which is outside the space. So what is nice about real line its complete space. What is nice about because real line is the complete space same thing is sure about R2 2 dimensional vector space.

Any sequence in 2 dimensional vector space will converge to the point in 2 dimensions. Any sequence in you n dimensional real rn will converge to element in rn, but in qn there are holes you know so where the sequence be Cauchy, but it will not converge. So this spaces you know in which all sequences converge within the space are called as complete vector spaces and these are special vector spaces.

So there is something different about the spaces in which so we move back to the black board. So we want this nice property to hold even in the vector spaces. So we call this vector spaces which have the special property as complete vector spaces or they are named after a famous mathematicians Banach who actually founded this one of the founders of functional analysis.

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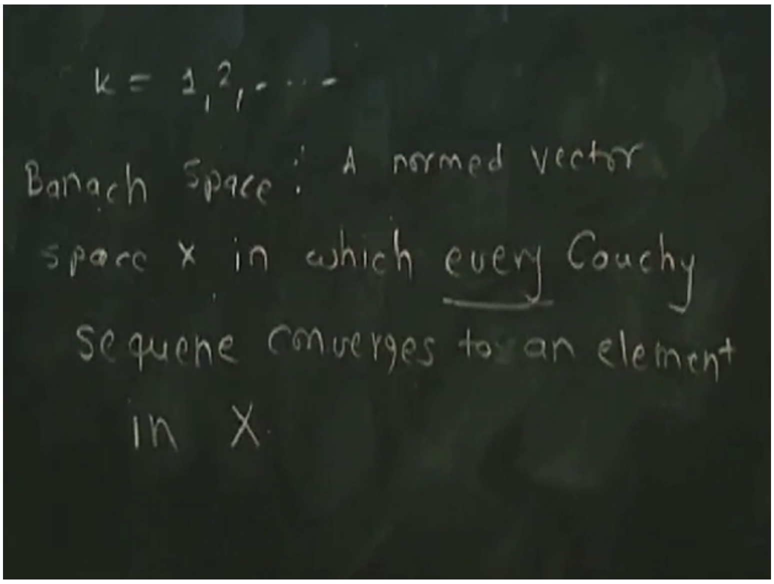

So what is Banach space so every Cauchy sequence to converge to an element is space this word here every is important, every Cauchy sequence. If I can find one sequence which does not converge the space is not a Banach space. Every Cauchy sequence should converge so the real line or rn or equivalently if you take complex numbers cn they have some very nice property. They are all complete spaces.

Function spaces need not be complete spaces. Set of continuous function we saw is not a complete space. Well in functional analysis you talk about completion of an incomplete space you add all the elements and then create a new space which is complete and so on, but we do not want to go into those details right now. I just wanted to sensitize you about the fact that even in a computer we are working with incomplete vector spaces and then you can get into funny situations in advance computing because of this incomplete behaviour well.

So far so good we talk about we started generalizing notions from 3 dimensions do not forget that. We talked about a vector and then we said there are essential properties of a set which the 2 essential properties vector addition and scalar multiplications. So these 2 things hold in a set then or if a set is closed under vector addition and scalar multiplication we call it a vector space any set. So we freed ourselves from the notion of vector space which is just 3 dimensional.

We can now talk about set of continuous functions, set of continuously differentiable functions, set of twice differentiable, 3 differentiable, and you can so now how many such spaces are there infinite spaces. Then we said well we now that is not enough to have just generalizing of vector space. We also need notion of length so we talk about norm right. We talked about norm.

Norm was in some saying generalization of notion of magnitude of a vector and we said there are so many ways of defining norms and a pair of a vector space and a norm defined on it will give you a normed vector space or norm linear space. So this up to here fine. Now we need something more I need angle one of the primary thing that you use in 3 dimensions one of the most fundamental result in our school geometry or in 3 dimensional geometry Pythagoras theorem and I need Pythagoras theorem in all these spaces what I am going to do.

I need Pythagoras theorem. So I need orthogonality. I need perpendicularity one of the most important concepts that you use in applied mathematics in modelling in physics, in chemistry and every where. Orthogonality is very, very quantum chemistry. Chemistry in the sense you might wonder where in chemistry. So orthogonality is very, very important and we need to generalize the notion of orthogonality and that is where we will start looking at in a product spaces.

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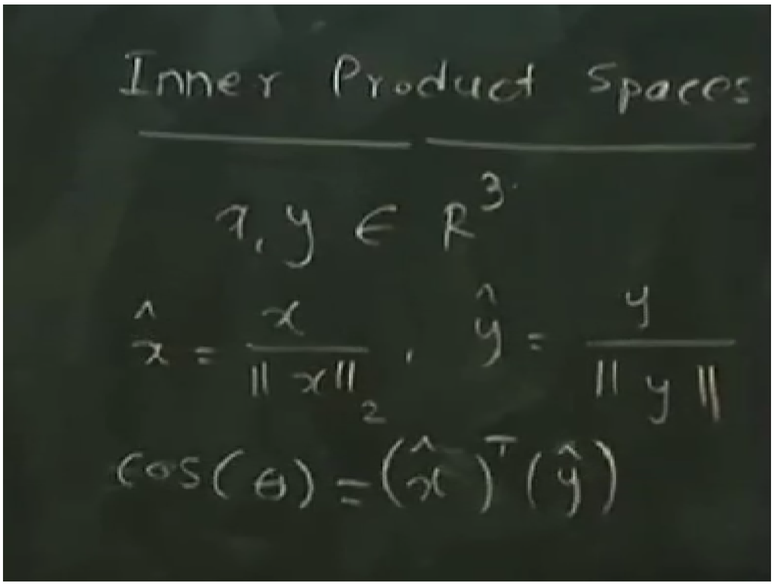

We will start looking at inner product spaces. So here the attempt is to generalize the concept of dot product. **"Professor - student conversation starts"** how do you define angle in 3 dimensions? Well if I am given any 2 vectors say x and y which belongs to R3. How do I find the angle between them? So what I do is I find out excap which is a unit vector in this direction normally I take a 2 norm here well why 2 norm we will come to that why not one norm.

So this is something special about this 2 norm and why cap = and then we say that dot product that is x cap cos theta angle between these 2 vectors is just x cap transpose y cap. This is the fundamental way by which we define angle between any 2 vectors in 3 dimensions. Now can I come up with something that we will generalize notion of angle in 3 dimensions.

When do you say 2 vectors are perpendicular in 3 dimensions. (()) (34:38) dot product. When dot product is 0 cos theta is 0, 2 vectors are perpendicular. So I am going to peg on to these ideas well that dot product between unit vectors is used to define angle. When dot product is 0 you call 2 vectors to be orthogonal and come up with a generalization in the product spaces of concepts of angle orthogonality and once the orthogonality you have Pythagoras theorem.

I can talk about Pythagoras theorem in any n dimensional infinite dimensional space of course it has to qualify certain properties what are those properties those are the properties of inner product space. So now we have to start questioning what is characteristics of an inner product? See we had 3 properties of magnitude. What were the 3 properties of magnitude?

Magnitude is always non-negative for a non-zero vector and 0 for a 0 vector. alpha times you get you know mod alpha gets multiplied to the norm and triangular equality likewise what are the essential properties of inner product in this which can be used to generalize in any other vector space? Those vector spaces are going to be called as inner product spaces because we are going to define a norm vector space in its additional structure is put called inner product.

"Professor - student conversation ends" All these spaces which are describing till now we did not talk about inner product. So now I am going to introduce something new which is the inner product space which will have definition of inner product. What you release there are umpteen number of ways to defining the product and so the way of defining generalizing orthogonality is not unique and so we will see from our next lecture.