Advanced Numerical Analysis
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Lecture - 04
Introduction to Normed Vector Spaces

Okay. So in last lecture we discussed about span of a set of vector and then we graduated to basis. So a linearly independent set of vectors is called basis. A linearly independent set is called basis if it generates the entire space.

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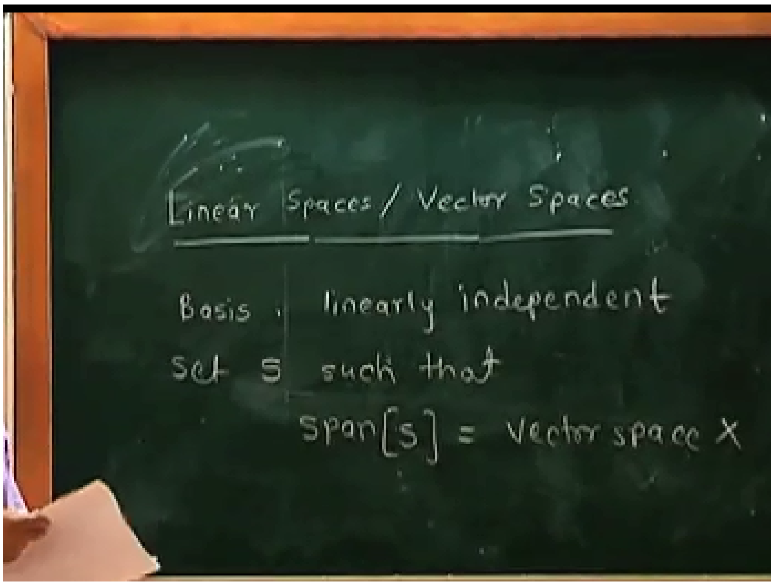

A linearly independent set S that belongs to vector space X which generates the entire space that means the span of that set is equal to entire space then it is called as a basis for that vector space. If the number of elements in the basis is finite, then we have a finite dimensional vector space and if this linearly independent set is an infinite set then we have an infinite dimensional vector space.

So number of linearly independent vectors that generate the entire space that defines basis and if that number is finite we have a finite dimensional vector space (()) (01:57) number is infinite. Infinite dimensional vector space example would be set of continuous functions over 0 to 1 or set of continuous functions over 0 to 2 pi, set of continuous functions over –pi to pi. All these vector spaces will encounter many such examples.

But these 2, 3 examples I gave you just now because we have seen something like this in some examples. We have look at a boundary value problem in which we have to look at set of continuous function over 0 to 1 or set of continuous functions over 0 to 2 pi. This is something which you have looked at when you studied Fourier series in your undergraduate or –pi to pi.

You have studied Fourier series and will be anyway revisiting Fourier series soon next week.

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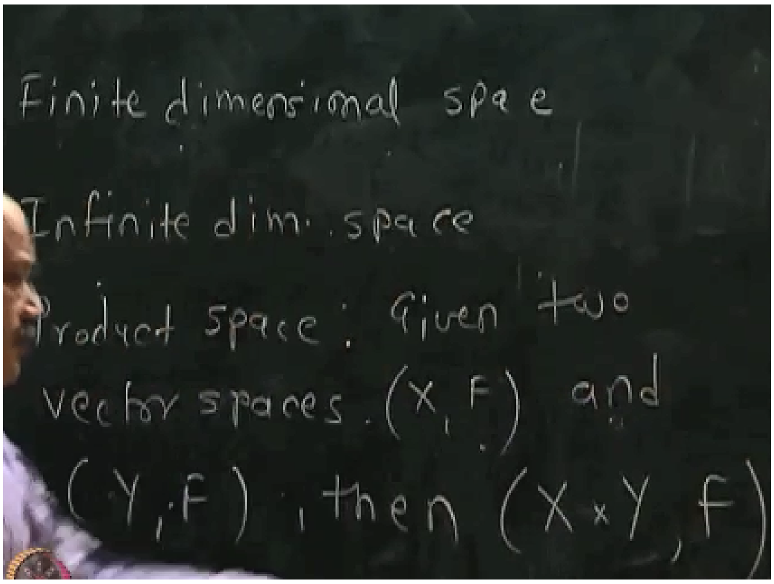

So we have these 2 concepts. Infinite dimensional space and then we will be using finite as well as infinite dimensional spaces throughout our study of numerical methods. So this is not something which I am just introducing out of completion or something. This is something which we are going to use and you will see when we start talking about transformation how this spaces come into play.

We need to introduce one more concept which is called as a product space. So a product space is composed of 2 vector spaces. I can create a new vector space by combining 2 vectors spaces or more vector spaces that is called as a product space. While the simplest example of product space is RN where R is a vector space line one dimensional vector space line and I can create R2/R cross R.

I can create R3/R cross R cross R. and so on. So likewise I can create a vector space by merging 2 or more vector spaces those are called as product spaces. So if I take 2 vector spaces X and Y both defined on field F. Then X cross Y what I mean here is this is X cross Y.

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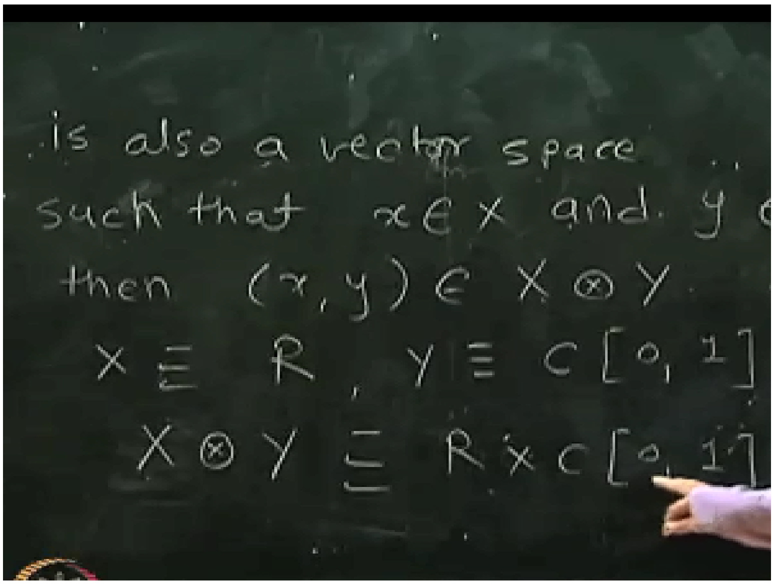

This is also a vector space. So if I take an element X belonging to vector space X particular element Y belonging to vector space Y then this combination XY it belongs to the product space. As I said simplest example of this is R square R cube RN or C, C square, C cube CN N dimensional vector spaces are the simplest example of this product spaces. See the way you can go about thinking about it is that you start with R. R cross R is a vector space then R square cross R is a vector space.

Okay I get R cube. So R cube cross R is a vector space so I get R to the power 4 and so on. R to the power not power 4 R 4 and so on. So my vector cases could be compositions of multiple vector spaces, but this I can extend to some other to create more complex spaces. See for example my X can be set of real numbers and my Y can be set of continuous functions over 0 to 1.

My Y can be set of continuous functions over 0 to 1. I can create a vector space which is X cross Y which is I can create a vector space which is R cross X cross Y. It is possible to have a product space which is defined like this. We will hit into these kinds of spaces when we start talking about boundary value problems and so on.

Examples of this will come in ample later when we study about transformations of mathematical problems into computable forms. So that is where we are going to need this. So I am not going to give examples right now because many examples will come a little later, but this is an important concept product spaces. Now having defined linear vector spaces sometimes they are called as linear spaces sometimes people refer to it as linear vector spaces.

Words are used interchangeably. Now here we need to define more structures and as I said the first thing that comes to your mind is length of a vector. So I need to define what is length of a vector when I go to set of continuous functions over some interval. What is length of a vector now. So we have to systematically define something equivalent to magnitude or length.

What we define here is a concept called norm. Now you can have a vector space defined in mathematics in functional analysis. You can have a vector space defined without definition of a norm or a length just set of objects which satisfy certain criteria. So what I want to say here is that norm is an additional structure that we are putting on the space.

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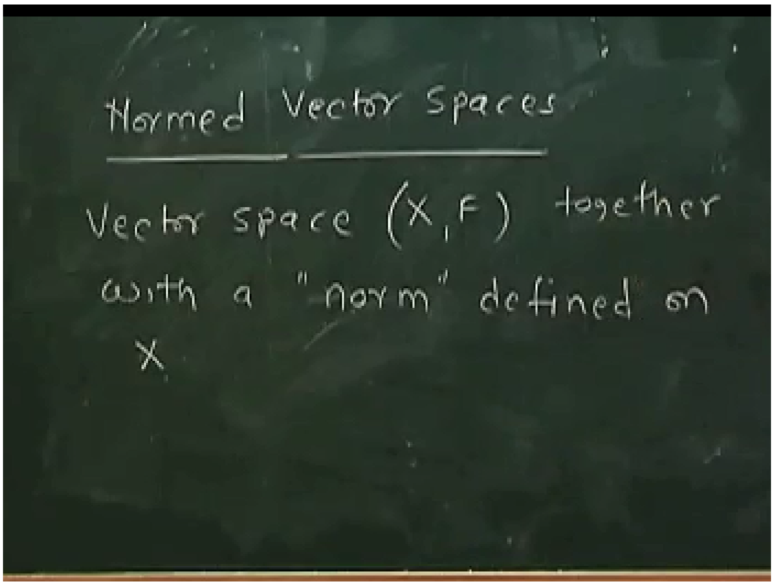

So we have normed vector spaces or normed linearly spaces as they are often referred to normed vector spaces is a vector space together with a definition of a norm. A vector space X, F. F here is the field associated with the vector space. So norm vector space is nothing, but a vector space together with a norm defined on it and the definition of norm and the definition of vector space it is not that given a vector space there is a unique way of defining norm. We will see what is a norm and then you will understand why I am saying it is a pair.

So for example in 3 dimension itself we will define different norms and 3 dimensional space with 1 norm will give you 1 norm space 3 dimensional space with another norm will give you another norm space. So it is not that the 3-dimension space if you are working 3 dimensions means you have to use 1 particular norm nothing like that. So a vector space together with a norm definition will give you a norm linear space.

And then it will have certain properties. So what is this norm? So norm is a generalization of concept of length of a vector. So when I say length of a vector all of you think probably from your undergraduate experience in one direction. If I give you vector in 3 dimensions with say 3 coordinates X, Y, Z then X square+ Y square+ Z square whole raise to 1/2. This is what we know as norm of the vector or length of a vector.

We do not know any other way of looking at it. So a vector space together with a definition of norm defined on it forms a norm vector space.

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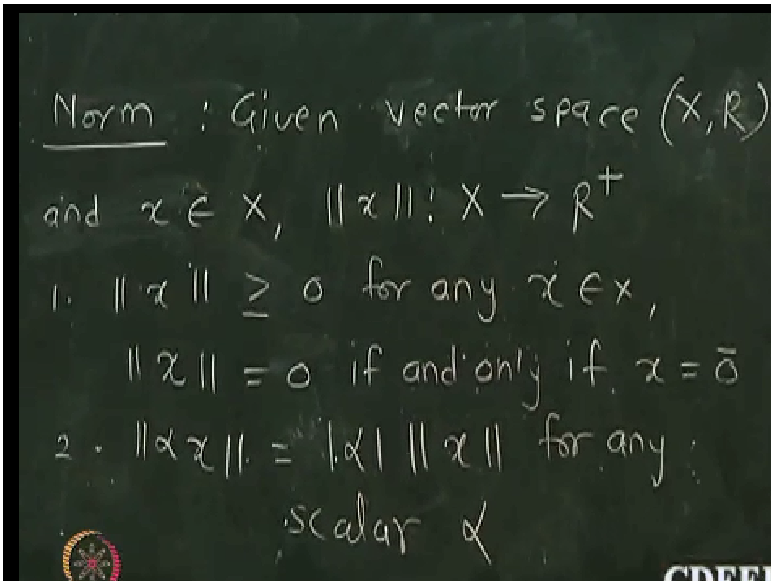

So what is a norm? Norm is a real valued function in fact it is a function that is defined from 0 to infinity. It is not defined on the negative side of the real line. It is a function defined from 0 to infinity. So given vector space or let us wok with set of real numbers. So given a vector space norm of a vector X that belongs to X. Norm of X is a function that is defined from X to R+ set of positive real numbers.

Well length if you look at why I am saying this because length of a vector in 3-dimensional is always positive it cannot be negative. So I need a generic function which exactly has similar properties as that of the concept of length in 3 dimensions. Do not forget that at any point that we are generalizing concept from 3 dimensions to look at notions in higher dimensions. So the first property that the norm function should satisfy is norm of X> or =0 for any X that belongs to X. Norm X=0.

So this norm function should be always > 0. It can be equal to 0 only for one vector that vector is the origin it if is 0 for any other vector which is a non-zero vector then that function cannot define a norm. So for the first criteria for a function to qualify as a norm is that it should be a positive function and then it should be 0 only for 0 vector should be non 0 for any non zero vector.

The second quality is if I multiply the vector X by a scalar alpha then what should happen? Norm X should be = mod alpha norm of alpha times X should be always = mod alpha times norm X for any scalar alpha it is very, very important second property. And the third property of the length function in 3 dimensions is triangle in equality. So this is way of saying that if I have given a point the shortest distance to the point is the straight line.

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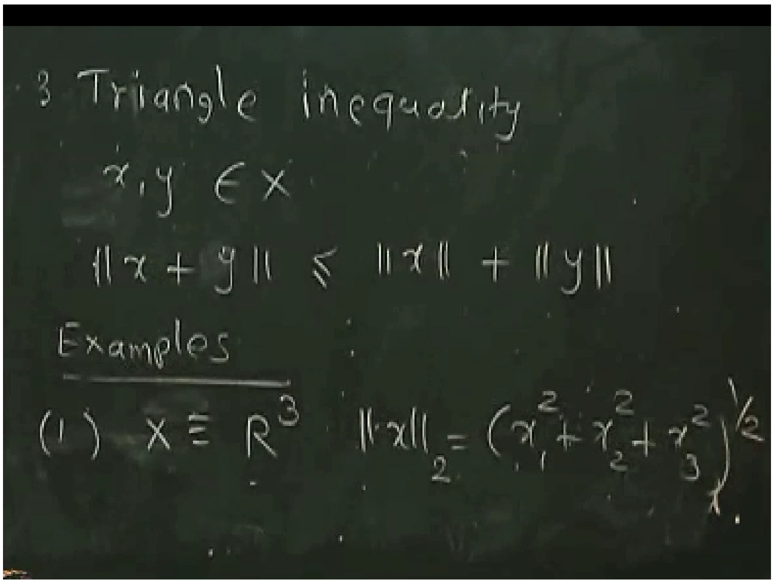

So the third property is now triangle inequality is a result which probably in your undergraduate to study in your high school. The sum of length of 2 sides of a triangle is always > third side that result is being generalized. So if I take any 2 points in the space the shortest distance in the straight line connecting the 2 points any other way I try to go to that point in the vector space by addition of 2 vectors that will be the larger part.

So this is a very, very fundamental property of the concept of length in the 3 dimensions which we are generalizing to any other vector space. So now anything any function that satisfies these properties would qualify to be a norm function. For example, in 3 dimensions well one way to define norm is very, very common. So my first example is R3 X corresponds to R3 and norm x there are 3 components x1, x2, x3 then x1 square+ x2 square+x3 square whole raise is to ½.

This is the definition of the norm. In fact, this will be called as 2 norm. So x this space together with this definition of norm will give you 1 norm linear space. But this is not the only way to define norm all these 3 properties are very obviously satisfied before this. I do not know to even check.

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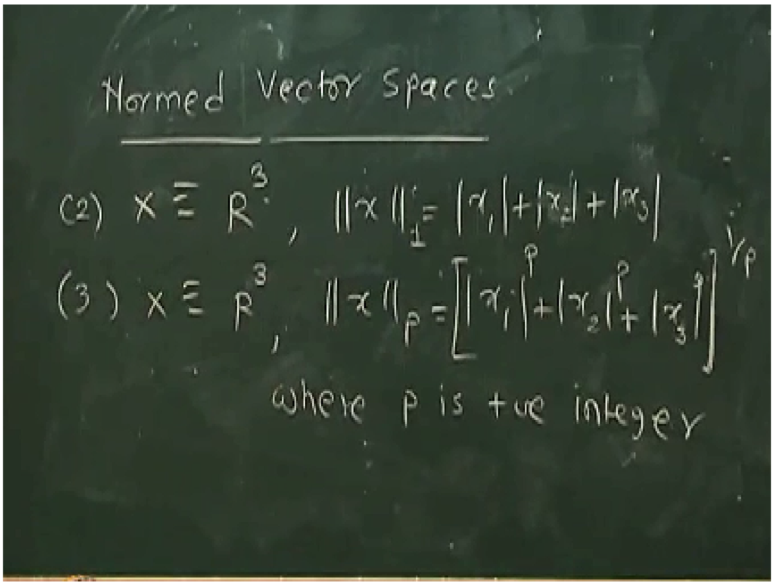

Well my second example is x corresponds to R3 and then my norm definition is absolute of x1+absolute of x2+abs of x3. You can check whether this qualifies to be a norm. Will it be 0 only when x1 is 0, x2 is 0, x3 is 0 and if anyone is not 0 then the value will be>than 0. So first property satisfied what about the second property if I multiply a vector by a scalar alpha the norm will get multiplied by alpha.

The third property follows also in a straight forward manner from the inequality that if I take any 2 scalars is always less than mod alpha+ mod beta. This is a simple inequality for scalars if I look at each component as a scalar I can apply this inequality to each one of them. It will follow that the third triangle inequalities also satisfied by this norm. So this function which is from the space x to R+ it will always give me a positive value.

Equally qualifies to be a norm of the vector in 3 dimensions I do not have to always thinking this term. What I wanted to realize is that this vector space and this vector space are 2 separate vector spaces 2 separate norm vector spaces because a norm space comes with a definition of norm on it. So do not say that if I am working with R3 which means I have to have 2 norm.

I am working in 3 dimensional vector space and I have 1 norm this is called 1 norm. So in general I can show that any function if x is R3 and I can define what is called as a P norm x1 to power P+ x2 to power P+ absolute of x3 to power P whole raise to 1/P where P is the positive integer this also forms a vector space with P norm defined on it. This is I am calling xp.

I can extend these definitions of R3 to RN dimensional vector space. For 2 norm on N dimensional vector space will be x1 square + x2 square + x3 square up to Xn square whole raise to 1/2. In general, actually I can define this for any P which is an integer. So 1 norm 2 norm are some special examples of a P norm vector space and I can very easily extend this definition too.

I have written here for R3 I can extend this to RN, I can extend this to CN and so on, but is important to note that these 3 are 3 different norm vector spaces. The definition of norm is different it is not same. What about set of continuous functions, how do we define norm on it. Well, before I move to that let me give me one more example which is in line with the P norm.

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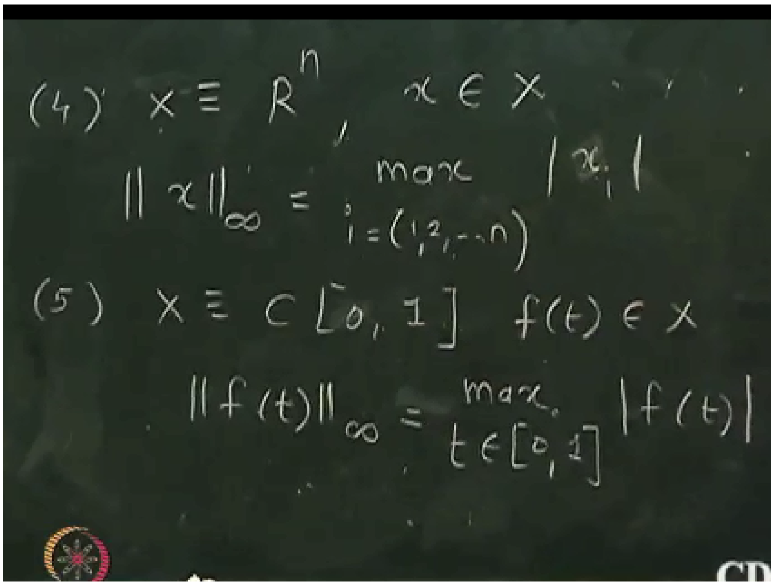

My fourth example is going to be X. Now, instead of writing R3 I will move to Rn and then I am going to define what is called as infinite norm. So for any element X that belongs to X infinite norm is defined as max over I=1, 2, to N mod X1. Infinite norm this is an N dimensional vector space. If I give you a vector that are n components, I find out absolute of each components and take max of that.

I find out absolute of each component and then take a max of that so there are n values absolute values and then find out the maximum you can show that this also forms norm it satisfies all the 3 properties of function to be a norm. It will always be greater than 0 if x is not = 0. It will be = 0 only when x=0 or the origin and then if you multiply a vector by a scalar the norm will get multiplied by mod of the scalar.

Triangle inequality will be satisfied so some of these will be exercise problem that is why I am not doing it on the board. My fifth example is X corresponds to set of continuous functions over 0 to 1. And then I am going to define a norm for an element say function ft which belongs to x is norm of ft. Well, let me extend norm of ft using the earlier notions. So I could define an infinite norm which is max/t belongs to 0 to 1.

Infinite norm of this function is maximum value of the function over the interval absolute of the maximum value very, very important. Why absolute is required? Because the norm has to be a positive number. When will this be 0 when will the maximum be 0 absolute of the maximum when you have 0 functions. Okay, if I multiply a function f by a scalar alpha what will happen to the norm.

Since it is mod absolute value of the scalar will be multiplying the mod.

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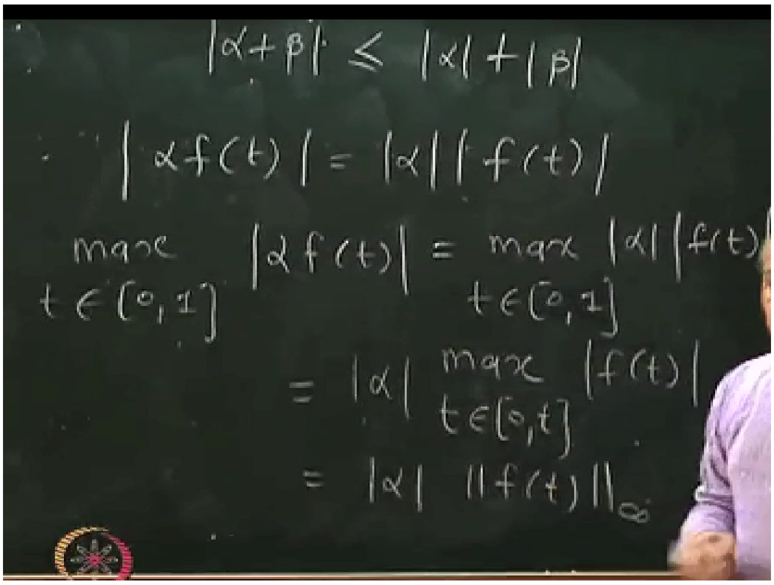

So for example if I take a scalar alpha so what is absolute of alpha times f of t this will be mod alpha* f of t. So if I put max operator that t belongs to 0 to 1. So which is same as all right. So I start with this and I show that it is mod alpha times f of t. I can prove the third triangle inequality. The norm is absolute of the maximum value over interval 0 to 1. **“Professor - student conversation starts”** It would all functions maximum values. **“Professor - student conversation ends”**

So given a function if I give you a vector you find norm of that vector. So if I give you for example sin t where t goes from 0 to 1. You are expected to find what is the norm of that find the maximum value of sin t over interval 0 to 1. Find absolute of that, that will give you norm of that. So not overall function it is over t the max is not over the functions. Max is over t okay. So likewise if I take any 2.

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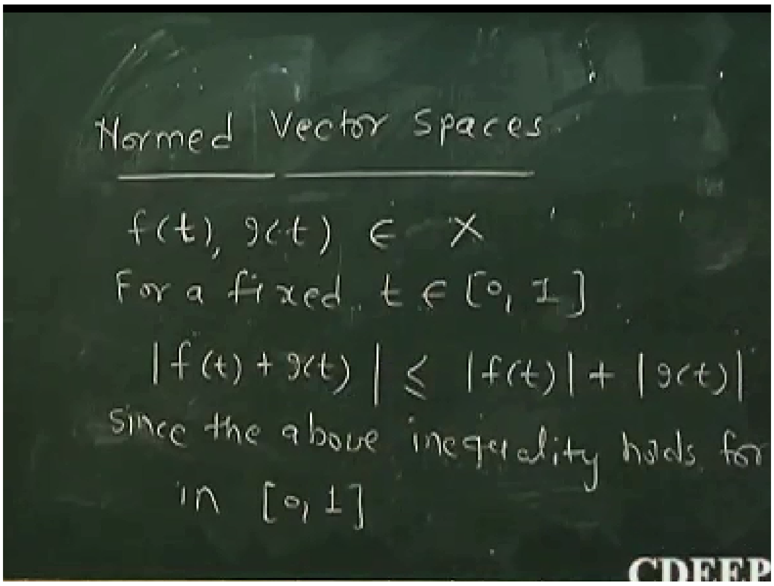

What about the third thing what about the triangle inequality. If I give you 2 functions say ft and gt which belong to that x set of continuous functions over 0 to 1 then what about triangle inequality. What we know that at a particular point if I fix t for a fixed t then I get function values. So these are real numbers we are talking about real valued functions by the way. So for real numbers what I know is ft+ for a fixed value of t I can write this.

Now I can use this inequality and through the triangle inequality. So now what I am going to do since this holds for every t, I can argue that max it holds for all t between 0 to 1.

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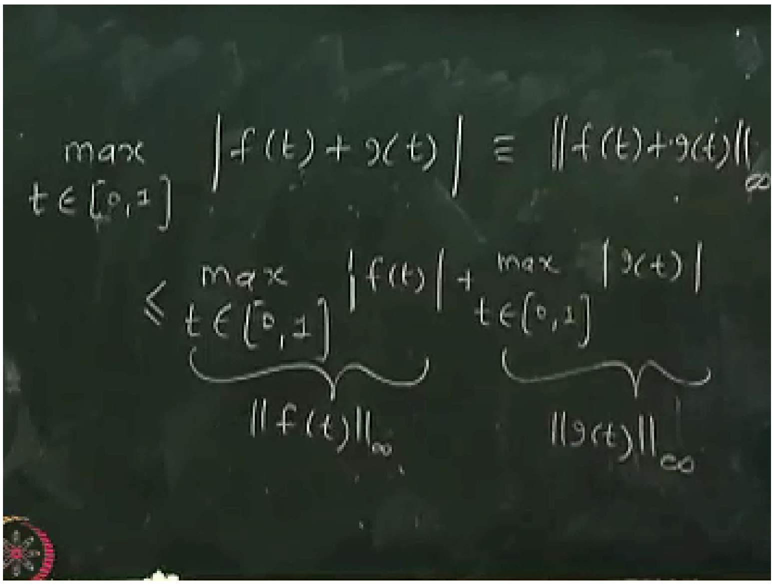

So I can write I can take a max operator. This is nothing, but this is equal to norm of ft+ gt right. This will be infinite norm for ft+ gt you agree with me, but using this inequality you can say that this is always less than not equal to max of it follows from this inequality that initially I am looking at these as 2 scalar numbers. If you give me 2 real numbers this inequality holds.

See if I take a function over an interval if I fixed myself to 1 t I will get one scalar value right. Let us this is sin t this is cos t. If I say that my t is 0.5 I will get some specific value of sin and cos. What we know from a very fundamental inequality is that if I add 2 real numbers then that sum is always less than or equal to sum of mod of 2 real numbers. Now actually I want to look at max over all t.

So I am graduating from this inequality to this inequality and what is this? This is norm ft infinity. What is this? So what I have proved is triangle inequality that if I take infinite norm of sum of 2 functions it is always less than or equal to this + this. This is norm of function ft this is infinite norm of function gt. Is that okay everyone with me on this?

Well I can define now norms which are similar to 1 norm or 2 norm or P norm on this basis. Infinite norm is not the only way to go about defining norm on this pace. So my next example this is sixth or seventh example.

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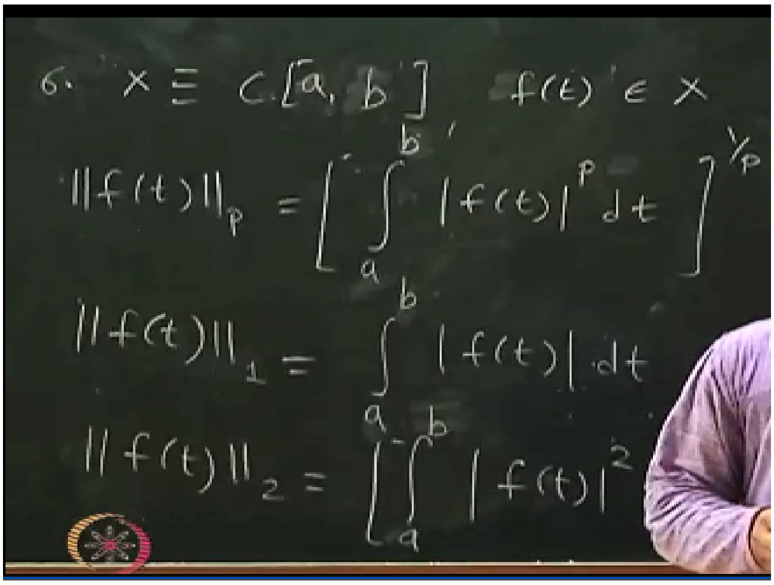

My sixth example is x corresponds to set of continuous functions over 0 to 1 and my norm definition. So for any ft that belongs to x my norm of ft is p norm is integral 0 to 1. I can define a norm which is integral 0 to 1. I need a norm to be a scalar value. If you look at n dimensions how was the norm defined? Sum of absolute values for 1 norm, sum of square absolute value then (()) (34:44) square root for the 2 norm.

For a p norm it was defined absolute value of each element is to P sum it and raise to 1/p and so on. So this definition should be logical extension because instead of sum I am getting integral because now t is very continuously. In n-dimensions vector space the index I was finite 1 to n. So we had a summation here we have an integral. Well p norms though they are defined we normally use only 1 norm, 2 norm and infinite norm. The three norm which are very, very commonly used are 1 norm, 2 norm.

So ft 1 norm would be integral 0 to 1 mod ft dt and 2 norm is integral 0 to 1 mod ft square dt whole raise to 1/2. So this is extension of the ideas from 3 dimensions or from n dimensions to a set of continuous functions and infinite dimension space. Now it is not necessary that this should be limit here should be between 0 to 1. I could actually define a set of continuous function over any interval say a to b any interval -5 to +10 whatever does not matter.

My integral here will change from a to b. My integral here will change from a to b and so on, but what you should realize is that this vector space together with a definition of 1 specific norm will be one norm linear space one normed space. So set of continuous functions with 1 norm is 1 norm linear space. Set of continuous functions with 2 norm is another norm linear space they are not same.

Definition of length is different. How we measure length in 1 system is not the way we measure length in the other system. Well there might be advantages of this norm over this norm or advantages of this norm over this infinite norm and indeed that is why we keep using so called 2 norm very, very often that will become clear as we graduate to in a product space little later.

So why is this guy so special, why we do not seem to use 1 norm of infinite norm so often as against this 2 norm will become clear. Nevertheless, do not think that 2 norm is the only way to define norm there are many other ways of defining norms. Now to understand this concept of norm I think we should have 1 or 2 more examples and then things will become clear. I have actually given here some arbitrary functions which could be used to define norm.

I will just talk about one of them right now here and then we will move on to some other concept which are important. Like convergence and I mean what is the use of defining norm the one of the use of defining norm is to talk about limit and convergence. Now what happens in iterative processes is that you have some idea about iterative process like Newton–Raphson. Everyone I think knows about Newton–Raphson. In Newton–Raphson what do you do you start with the guess value and then you get another guess.

From the new guess you concept third guess and fourth guess. So you get what is called as sequence of vectors. The question that you need to answer when you solve a numerical problem is this sequence converging to something. Is it convergent to my solution in fact that is what is in your mind, but at least before whether to know whether it is going to the solution or not you would like to know whether it is converging.

So convergence is a very, very critical thing in numerical analysis and when you work with n dimensional spaces you need to generalize the concept of convergence. We know about a sequence of real numbers converging to a point and we call it limit. All those things we have done in 12th standard right, but now what is the meaning of a function sequence converging to a limit. We will have to talk about these ideas so that is why we need this.

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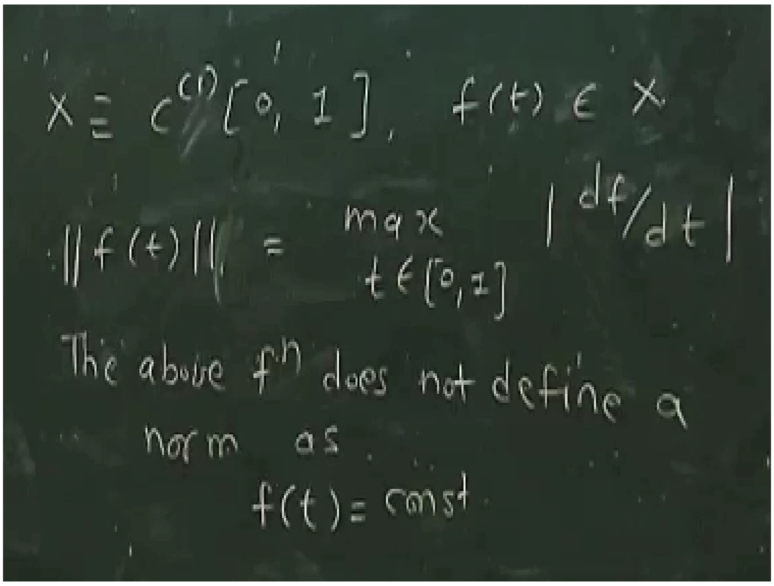

Now just to give you a feeling that the norm can be defined in different ways. I will just take the same x which is set of once differentiable continuous function. I will take set of once differentiable continuous function and let me define the function you have to tell me whether it will define a norm or not. So what about candidate function. So I have this function ft that is =, that belongs to this x.

So this is a once differentiable continuous function. Now I am defining a norm of ft as max over t belongs to 0 to 1 mod of d f/dt derivative. I am taking derivative of the function df/dt. Question is does this function define the norm why? So what does mean? A non-zero vector is giving me 0 value.

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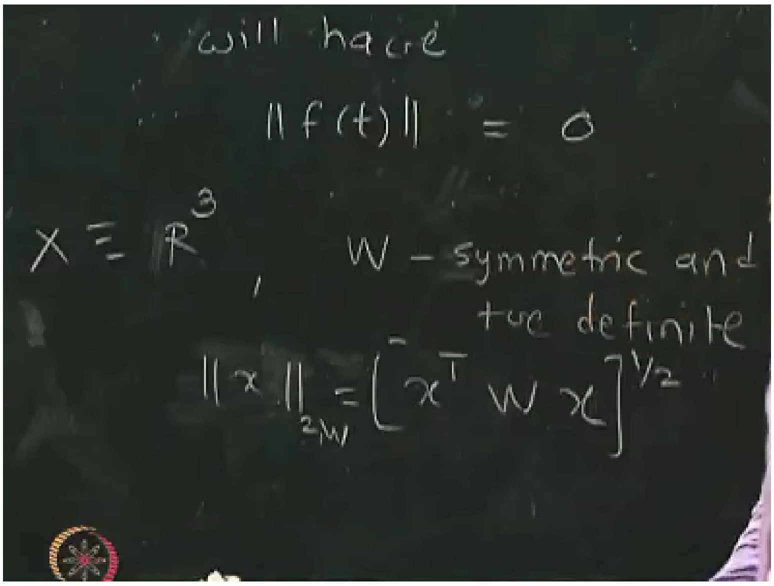

So this is a non 0 function f of t=constant and that will yield 0 norm if I use this definition. So this function cannot qualify as a norm. There are more such examples given listed here you should go through them this is on page 12. I have listed many other functions and I have solved systematically. If I give you a function and ask you to check whether this function satisfies to be a norm of not. You have to systematically look at all the 3 axioms.

First you have to look for whether non 0 element gives you a 0 vector whether the second thing is alpha times original vector that will give you mod alpha times. And a third property is triangle inequality. Even in 3 dimensions there are multiple ways of defining norms see for example you can define a norm in 3 dimensions if I have x is R3 if I give you a matrix w which is symmetric and positive definite.

Then I can define a norm of vector x in 3 dimensions as x transpose wx. You can show that x transpose wx will also raise to 1/2 x transpose wx. So we call this as 2 nom with matrix w where w is some symmetric positive definite matrix if this matrix is singular which means if it is semi definite and not semi definite. What is a semi definite matrix? Not all eigenvalues are positive so it means it has some eigenvalues =0.

So some eigenvalues are=0 we will visit this (()) (44:20) definite semi definite many times. So let us wait for that right now let us wait for that, but just think about this w cannot be singular if this W matrix is singular then it will not define a norm because a singular matrix can take a non-zero vector to 0. Yesterday we looked at that we have a singular matrix with 3 columns which are linearly dependent a singular matrix can take a non-zero vector to 0 vector.

So which means a non-zero vector can give you a 0 value for the norm which is not acceptable so it will not be a norm. So the next thing that I want to talk about is convergence of a sequence of vectors. So as I said we are going to deal with iterative processes.

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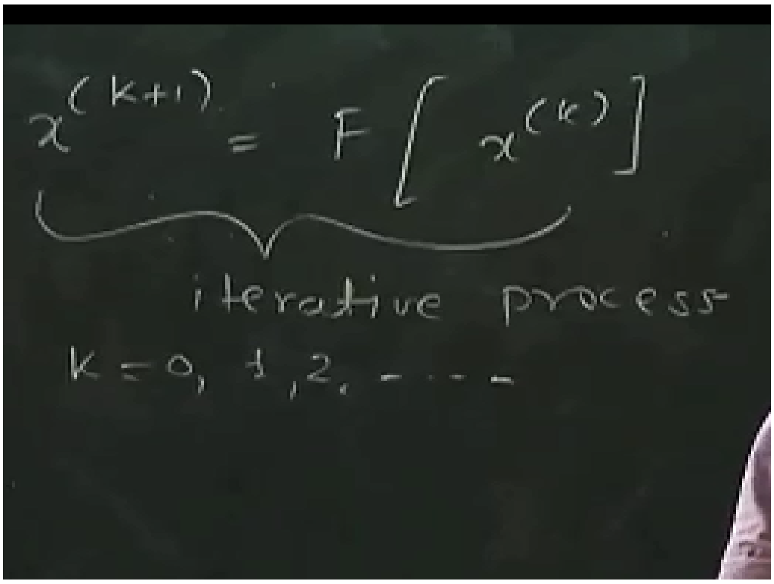

So let say you will have iterative processes which will give you x k+1= some function f of xk. We have iterative process in which we start with a guess solution and that guess is used to construct the next guess and then that guess is used to construct on next guess due to Newton–Raphson method all of you know about this. We are going to generalize a Newton–Raphson method from one dimension to n dimension or to a function space and so on.

So that time you will have sequence of vectors in n dimensions and I need to know whether they are close to each other or they are converging how do I know about that. I need to use definitions of norm. If I have a sequence of functions generated in a method how do I know whether this sequence of functions is converging to solution or it is not converging. I need to use definition of norms that is why we need this norm definition generalize because we need to talk about convergence of a sequence.

We will talk about limit of a sequence. So we will look at this concept in my next lecture. It is important to understand keep in mind that why it is being done. It is being done for dealing with iterative processes. So in the next lecture we will look at convergence and sequences. We will have something called Cauchy sequence which almost converges and a convergence sequence and we will also look at some funny properties, funny things that happened in infinite dimensions' spaces.

You have a sequence which converges, but the limit is not in the space. The limit is outside the space and so on. I will show you some examples where these funny things happened.