Advanced Numerical Analysis
Prof. Sachin Patwardhan
Department of Chemical Engineering
Indian Institute of Technology – Bombay

Lecture – 31
Iterative Methods for Solving Linear Algebraic Equations: Convergence Analysis (Contd.)

So, in the last few lectures we have been looking at convergence of iterative schemes for solving linear algebraic equations. Starting from the basic equation for way the error evolves.

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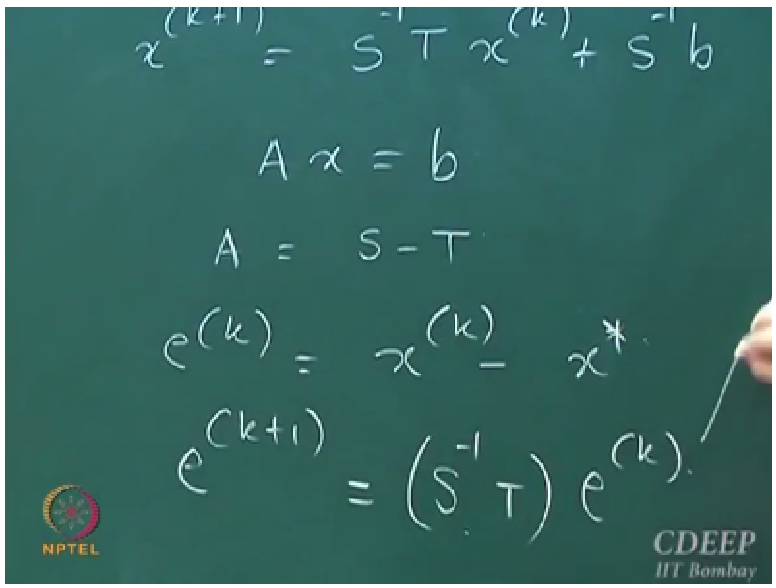

So, we have this iteration scheme to solve Ax=b and A was written as S-T and we said that the error which is defined as iterative xk-the true solution x star. This evolves according to e k+1, this is a linear difference equation. It evolves according to this linear difference equation.

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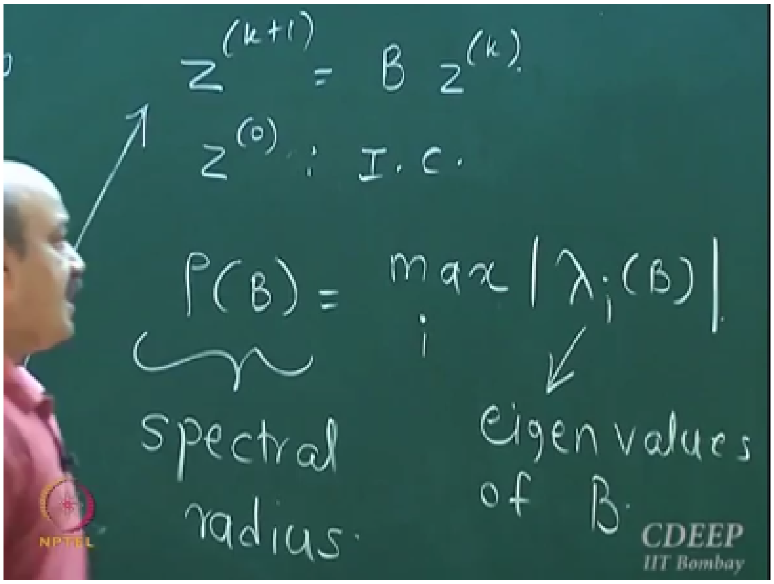

Now, from this we abstracted a linear difference equation problem. We said essentially we have to look at equations of this type and Z 0 is initial condition and then we wanted to come up with the way of analyzing asymptotic behaviour of the solution as A tends to infinity. So, we came up with analysis based on Eigenvalues, we came up with a condition that if rho B is nothing but m a x/i lambda i, that means if lambda i are Eigenvalues of matrix B, we find out its absolute value where Eigenvalues can be complex.

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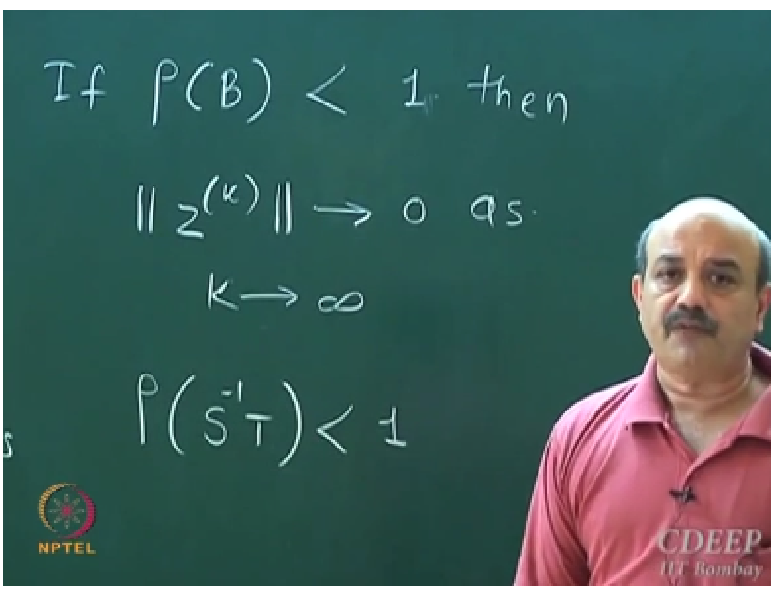

So, we find the absolute value and rho B this is called as spectral radius and we showed that the necessary and sufficient condition if rho B that means spectral radius of matrix be strictly < 1, we said that if the spectral radius of matrix B is strictly < 1, then the sequence Zk, norm of that, we tend to 0 as k tends to infinity and from this, we again connected to our original problem we said, which means that spectral radius of S inverse T is strictly < 1.

Then, this is necessary and sufficient condition for convergence of error, okay. The error between the true solution and the guess solution will diminish to 0 if this condition is satisfied, okay. I am just doing a recap of what we have done till now. So, from this point we again had some difficulty because we have to compute Eigenvalues. So, we said that Eigenvalues computations are difficult and then we used one more result to come up with a sufficient condition.

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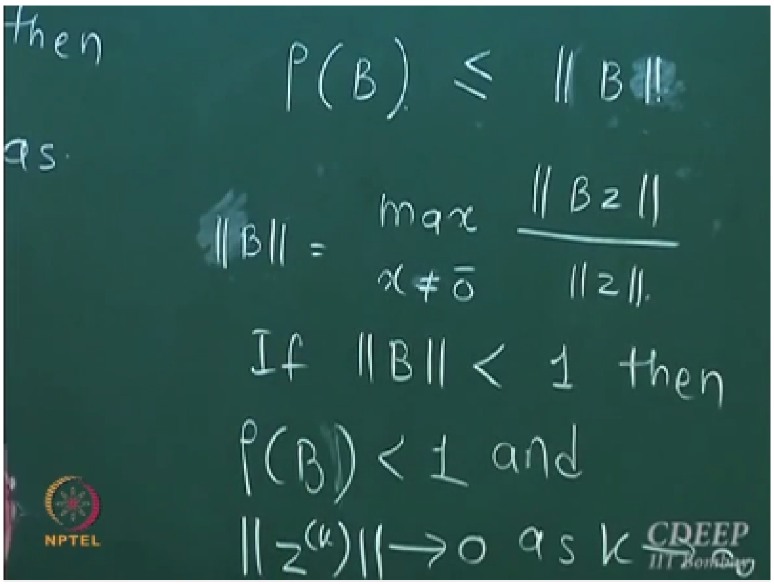

So, we said that spectral radius of matrix B is always <= any induced norm of matrix B, spectral radius of matrix B is < induced norm of matrix B. What is induced norm. This is induced norm, induced by the norm defined on the rain space and the domain space. So, this norm of matrix is nothing but amplification power or something like gain of a matrix if you can think about it as a gain or amplification power.

Then, we came up with a sufficient condition that if induced norm is < 1, obviously spectral radius is < 1 and convergence is guaranteed. If induced norm is > 1, we cannot say anything, okay. If induced norm is < 1, we are sure so we had another condition that if induced norm is strictly < 1, then spectral radius of B is strictly < 1 and then this implies that asymptotically norm of iterate Zk or difference equation Zk will go to 0 as k goes to infinity.

So, this we can say without actually having to solve it. Now, in particular we talked about infinite norm or one norm which are more convenient to do calculations. Now, based on this I wanted to derive some results which is even more simpler. I do not probably have to even compute the norm. I can compute what is called as diagonal dominance. So, in my last lecture, I talked about diagonal dominance.

So, I wanted to further cash on this result that if the induced norm is < 1, then of course the spectral radius is < 1. Induced norm is very-very easy to compute, particularly (()) (06:37) infinite norm as compared to computing the spectral radius, so checking whether a particular iteration will converge or not is very-very easy, okay. Now, let us move back to the thing that we have done in my last lecture.

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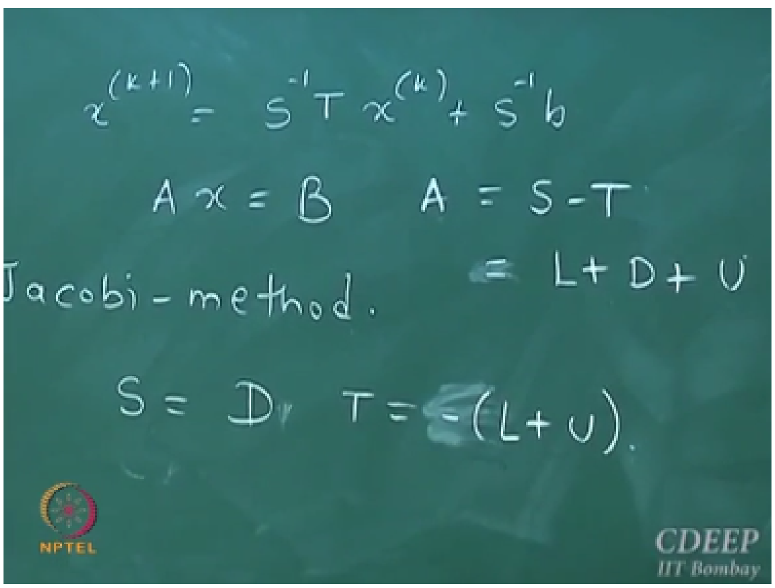

So, this was overview of the entire stability arguments that we have been giving but I wanted to derive something more specific from the previous results. So, we come back here. We are trying to solve for Ax=B and A has been split as S-T. So, for Jacobi method, in particular I analyzed Jacobi method, okay. For Jacobi method, S=D. Well, we are also writing A=L+D+U. This is strictly lower triangular part of A, this is diagonal part of A and this is strictly upper triangular part of A.

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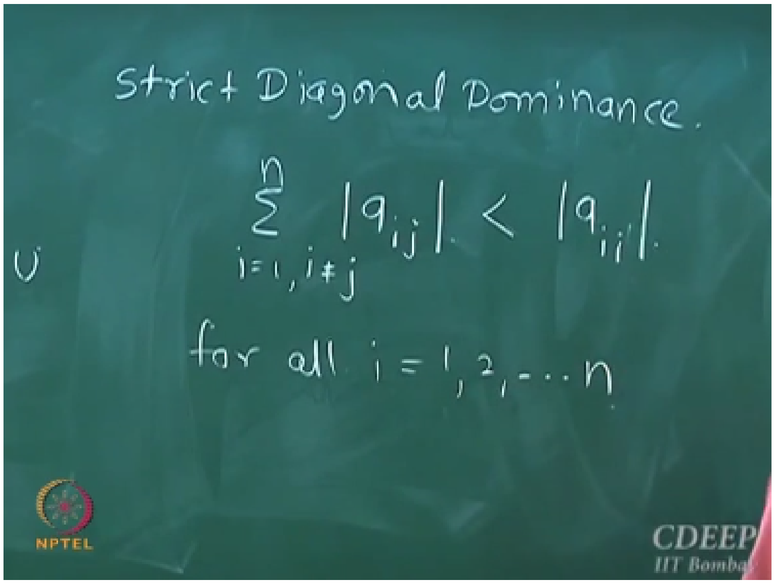

So, S=D and T=-L-U and then I defined the concept of strict diagonal dominance. If you take sum of absolute values of elements of matrix A in a row, except the diagonal element and if that sum is strictly less than the diagonal element, then the matrix is called as diagonally dominant matrix, okay. Just to give you a simple example.

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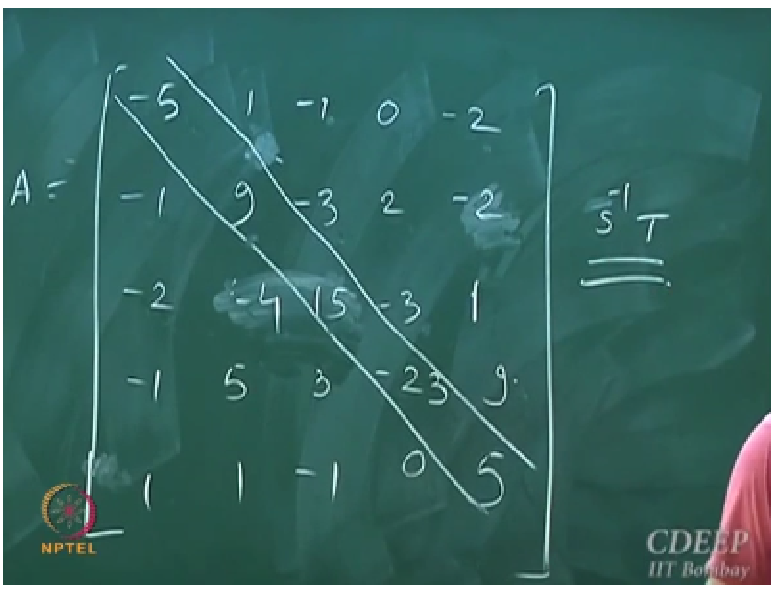

Well, you just write any matrix. Let us say, this is my matrix A. These are the diagonal elements here. Just have a look, if I take absolute sum of this, this, this, it is smaller than this. I take absolute sum of this, this, this, this, smaller than this, okay. I just look at this matrix. I look at its diagonal elements, okay. This particular matrix will obey this condition. This is a strictly diagonal dominant matrix. Just look at this, this is 2+4+3+1 is always < 15, okay.

Same thing here, okay, 5+3+1+9 is < 23. So, I am taking absolute values, okay. So, for this particular matrix can you calculate what is going to be Jacobi matrix which is S inverse T, can you calculate that. Just do it. What is S inverse T? Well mind you again that Jacobi matrix when you actually do computations, you never compute S inverse, you do row by row calculations, okay. This is for analysis, this is for getting insights but what you will realise is that you just look at the diagonal elements, you look at the sum of all diagonal elements.

You can say whether the iteration are going to converge or not which is very-very powerful reason. You do not have to actually solve it and this is true of 5x5, for 10x10, for 1000x1000. If this condition holds, iterations will converge, okay. So, you have guarantee convergence if this condition is satisfied.

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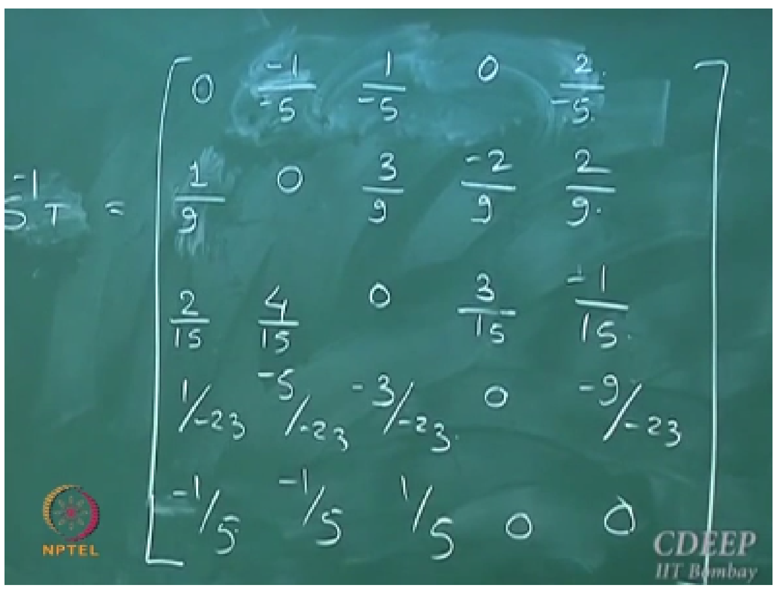

So, what will be S inverse T? What will be Jacobi matrix? let us call this Jacobi matrix. For Jacobi matrix, S inverse T, what will it be. This will be 0, it will be -1, 1, 0,2/-5, -5, -5. Then, this will be 1, 0, 3, -2, 2/9, 9, 9, 9, okay, -L-U. Then, what is this 2, 4, 0, 3, -1. This divided by 15, 15, 15 and 15, okay. Then, the next one is 1, -5, -3, 0 and -9/-23, -23, -23 and the last row is -1/5,-1/5, 1/5, 0, 0, okay. What will be the infinite norm of this matrix? You take absolute of rows, so absolute of this plus absolute of this plus absolute of this, okay. Absolute of these things, all these numbers.

Is it always going to be < 1. It is always going to be < 1 because this matrix is strictly diagonally dominant, okay. In the numerator this will appear, okay. Actually for a diagonally dominant matrix what you know is that this divided by norm Aii, this will be strictly < 1. You can see here. You add absolute of each one of these rows, okay. If each one of them is < 1, the maximum is also going to be < 1. What does it mean.

Spectral radius of this matrix is strictly < 1. So, if A is diagonally dominant, the Jacobi matrix which you get by S inverse T has spectral radius strictly < 1 which means Jacobi iterations will converge. Without having to solve it for arbitrary initial guess, very-very important, for an arbitrary initial guess, okay. So, any initial guess I give even if it is completely wrong, my iterations will converge to the true solution, okay if diagonal dominance condition is met.

So, you can just check diagonal dominance of a matrix very-very easily and then you know whether the solution is going to be obtained or not, that is straight forward. Now, there are many more results of how do you analyse the convergence behaviour and all of them I am not going to prove. I have stated those results here and I am just going to state them and show you how to apply them and the proofs for each one of them or at least most of them.

Some of them you can derive yourself, for most of them are included at the end of the chapter notes in the appendix, okay. I do not want to go over it in the class. You go the philosophy of how it is done and you have to look at the proofs in the appendix to understand more of this because we cannot spend time on this beyond a certain point. As long as you get the philosophy it is fine. Now, what are the more results.

There are some more results which exploit the structure of matrix A. One structure that we exploit is diagonal dominance, right. The other thing we will show is that if matrix A is symmetric positive definite, okay. If matrix A is symmetric positive definite, then Jacobi and Gauss-Seidel method converges. Also you can show that if matrix A is diagonally dominant, Gauss-Seidel method will converge, okay.

The proof is little more involved and you should look at the proof given in the appendix. I have given details of the proof in the appendix. So, if matrix A is diagonally dominant, Jacobi iterations will converge. It is also true that if matrix A is diagonally dominant, then Gauss-Seidel iteration also will converge, okay. So, you just have to check for diagonal dominance. You know that Gauss-Seidel iteration will converge and in fact (()) (17:18) is that most of the time Gauss-Seidel iterations converge faster than Jacobi iterations, okay.

So, if you know that a matrix is diagonally dominant, your preferred choice of using the method should be Gauss-Seidel method, not Jacobi method, okay. For other theorems; so, the first theorem that you should know about convergence of iteration scheme is that if matrix is diagonally dominant A matrix, okay; then Jacobi method as well as Gauss-Seidel methods will converge to the true solution, okay.

By the way, remember this, this is a sufficient condition, this is not a necessary condition. What does it mean that if this condition holds? Jacobi and Gauss-Seidel methods will converge. You cannot say if this condition does not hold, you cannot say anything about convergence. You have to go back and check something else. You have to go back and check spectral radius, okay. So, this is only a sufficient condition.

If this happens, you are guaranteed convergence will occur. If this does not happen, we do not know, we cannot say anything, okay. So, this is a sufficient condition, not necessary condition.

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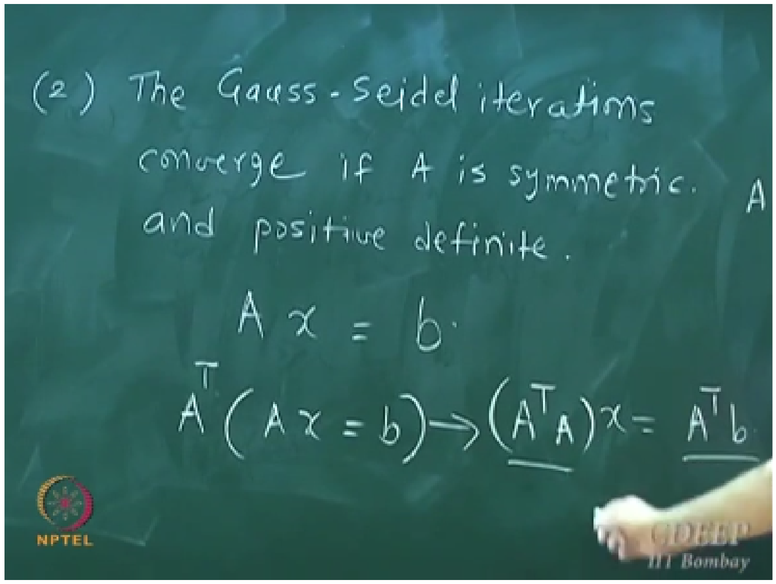

So, the second result I would say, very-very important result. So, the symmetric and positive definite seems to be something very-very nice. It seems to help us everywhere we go, okay. Now, you might start saying well I have been given this matrix A and you know only in very-very special cases, this A will be symmetric and positive definite, isn't it.

A is a square matrix. I am not talking about the square where we had a tall matrix. I am just talking about A is a square matrix and the problem which is given to me is such that A is not symmetric and positive definite, okay. But I know that Gauss-Seidel method will converge, okay. Sufficient condition for convergence is that if the matrix in my problem is symmetric and positive definite, is there something I do to solve this problem, to convert this problem into symmetric and positive definite matrix.

I just pre-multiply this equation with A transpose. So, this gives me A transpose A, okay. I do not have to solve for Ax=B. I can instead solve for A transpose A=A transpose B, okay. I am guaranteed convergence. So, I am using my theory to change the problem in such a way that I am guaranteed to get converge solution, okay. I am going to solve this problem using Gauss-Seidel iterations making use of this theorem, okay.

How do I make use of this theorem to modify my calculations, I pre-multiply both sides by A transpose. This becomes a symmetric positive definite matrix, okay. Now, if I apply Gauss-Seidel method to this matrix and this transform problem, I am guaranteed to get a solution. This solution is obviously a solution. If it is a solution of this, it is also a solution of this. You have no problem with that. So, I could solve this transform problem instead of solving this problem. I get a symmetric positive definite matrix here, okay. I am using theory to modify my calculations.

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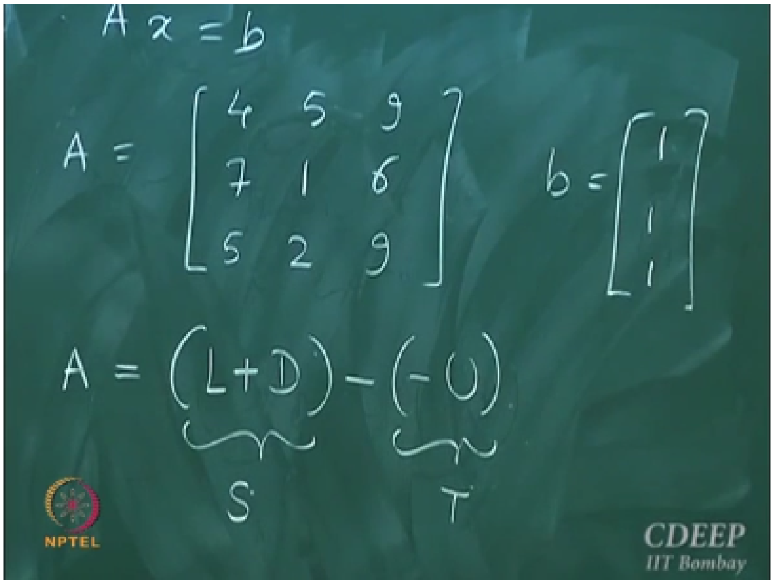

I will just give you an example here. So, I want to solve for Ax=b and my A matrix=4, 5, 9, 7, 1, 6, 5, 2, 9 and my n vector is 1, 1, 1, okay. Let say I want to solve this by Gauss-Seidel iterations, okay. Well what I will do is I know this is not a solution procedure, this is analysis okay.

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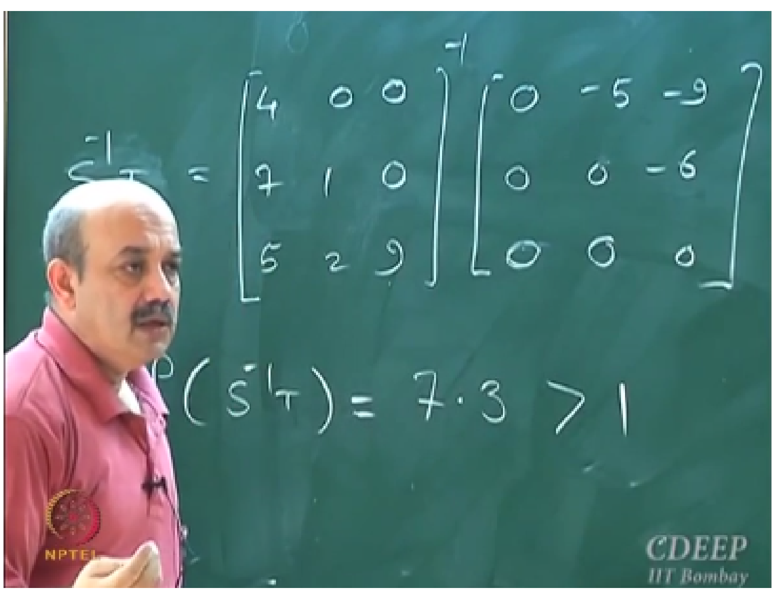

From analysis, what I know is that if I write this matrix as A matrix, if I call this S and if I call this as T, then doing Gauss-Seidel iterations is equivalent, then my S inverse T will be 4, 0, 0, 7, 1, 0. This is my S inverse T if I am able to use the row matrix A. In this case, the spectral radius of S inverse T turns out to be 7.3 which is strictly < 1, okay. If I use Gauss-Seidel iterations, iterations are not going to converge.

Because if I just choose the row matrix A that matrix is neither diagonally dominant, just check it is diagonally dominant, it is not. Is it symmetric matrix, it is not a symmetric matrix. Forget about positive definite, it is not symmetric matrix, but if I know this little bit of information, okay.

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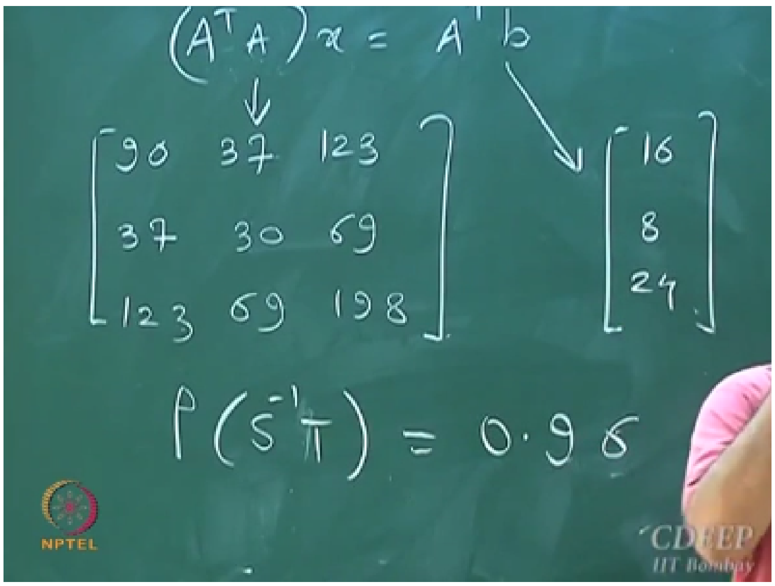

If I do this transformation that is A transpose Ax=A transpose b, okay. Then, this A transpose A matrix turns out to be 90, 37, okay and this is A transpose A. A transpose b becomes 16, 8, 24 and now if I apply Gauss-Seidel method to this transformed equation, then spectral radius of S inverse T turns out to be 0.96, okay. So, for the transform problem, guaranteed convergence of Gauss-Seidel method. This is a symmetric matrix, just see this, symmetric matrix. It is a positive definite matrix by definition.

A transpose A is always positive definite, even if A is not positive definite. We have seen this several times, okay. This is positive definite matrix, symmetric matrix convergence is guaranteed, just pre-multiplying both sides by A transpose, I can ensure that I will get convergence by iterative method, okay. So, in the case where obvious things like diagonal dominance are not there, if you want to ensure that you get convergence, just pre-multiply by A transpose both sides and then use Gauss-Seidel, you have guaranteed convergence, very-very powerful result.

"Professor - student conversation starts" Yeah. For any given (()) (25:50) no matter how would (()) (25:55). Always (()) (25:56). So, that spectral radius should be < 1 is necessary and sufficient condition. If necessary if the convergence occurs, spectral radius should be < 1. If spectral radius is < 1, convergence will occur, okay. **"Professor - student conversation ends."**

But that is not the case with the norm. If induced norm is < 1, convergence will occur but if induced norm is > 1, convergence may or may not occur, you may not know. That is not the case with spectral radius. Spectral radius is the absolute measure which is necessary and sufficient condition, okay. So, it is possible to transform. There are more results of this type. Again, I am not going to go into the proof.

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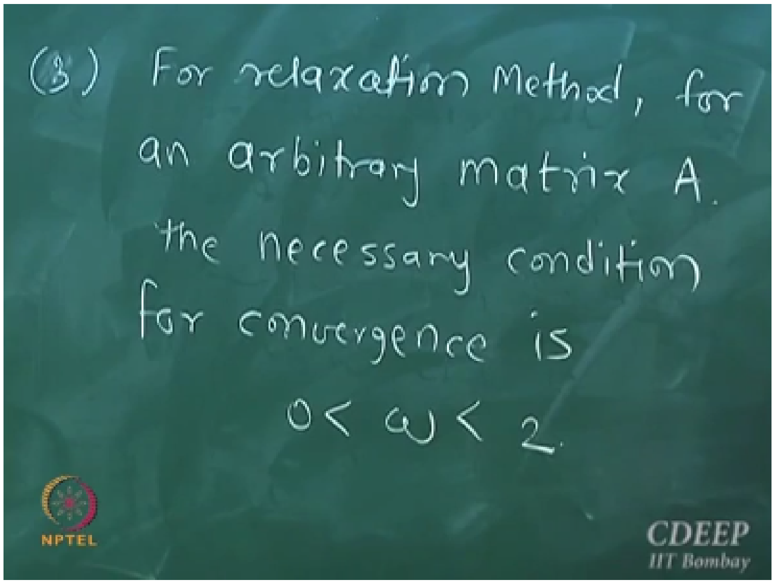

For relaxation method, we have this result. For relaxation method, what you can show again is the proof again given in the appendix. You should go and have a look at it. If omega is chosen between 0 and 2. Well, actually for relaxation method we want to choose it between 1 and 2 because we showed that omega = 1 is equivalent to Gauss iteration. So, we want to choose between 1 and 2 but in general, if omega is between 0 and 2, okay. This is a necessary condition for convergence, okay.

So, you know how to choose omega, you have a guideline here, okay. So, again remember this is only a necessary condition. This is not sufficient. If you choose < 2 that does not mean convergence has to occur. But convergence occurs only when you choose omega is < 2. This is result 3 and the necessary condition becomes necessary and sufficient conditions if extension to this theorem is another result. This is for an arbitrary matrix, okay.

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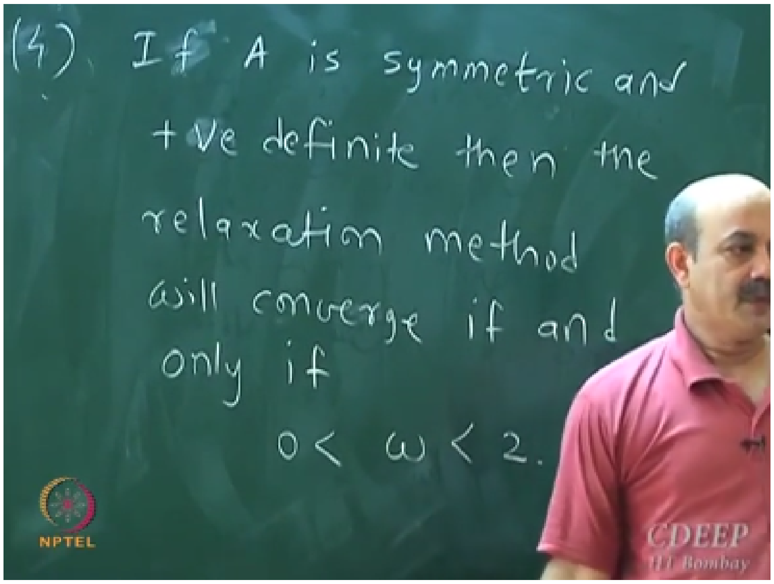

Now, if A is symmetric and positive definite. So, if matrix A is symmetric positive definite, okay, this necessary condition becomes necessary and sufficient condition, okay. Now, you know how to transform the problem which is usually not symmetric positive definite to symmetric positive definite matrix, okay. So, what I want to do, the take home message is that all these theorems are very-very useful in shaping your calculations.

You should know how to make sure that convergences occur. Convergence is very-very important. Whenever you are not sure in an arbitrary large scale problem, you are not sure of A matrix, how it is going to be. If you want to use iterative schemes for solving Ax=b it is better to use a relaxation method in which you transform the problem because in general relaxation method will converge faster than Gauss-Seidel method.

I will just show you a very small example that Jacobi method is the slowest to converge typically. Gauss-Seidel method is faster and if you choose omega properly, then the relaxation method will even converge faster, okay. Now, how do you choose omega such that you get very-very fast convergence. It is very difficult to tell the (()) (30:11). You probably have to compute Eigenvalues but that is not desirable. You do not want to really compute Eigenvalues.

So, you have to develop some kind of experience beyond the point. You have use all these theorems and understand the theory and then develop experience to tweak with the calculations, that is very-very important, okay.

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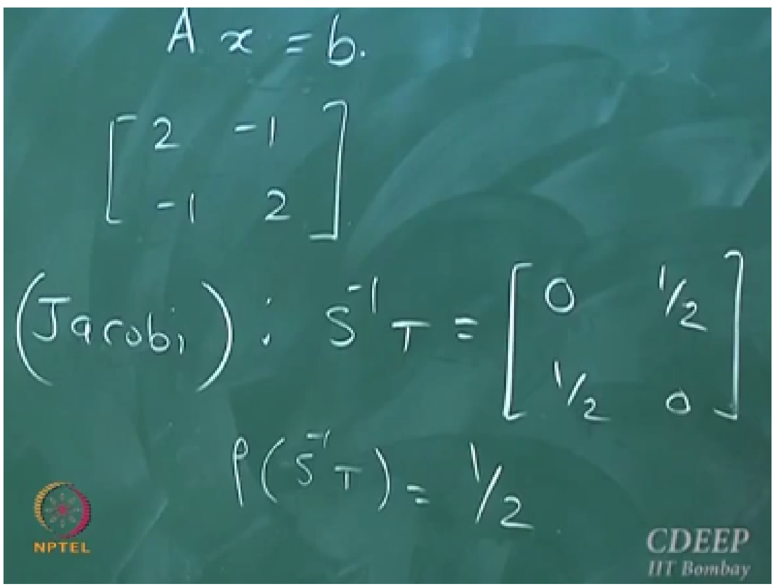

So, I will just show you one simple example. This is taken from (()) (30:39) book but it is very-very illustrative. Very simple problem. So, I want to solve and such a simple problem of course you do not need any of the iterative methods, 2x2 systems you can solve it by hand. So, my A matrix is 2, -1, -1, 2. Well, we will say that this is Jacobi and Gauss-Seidel will converge why, symmetric diagonal dominant, okay. Anyway that is not the point. The point is that for Jacobi method S inverse T will be 0, 1/2, 1/2, 0 and the spectral radius is = 1/2, okay.

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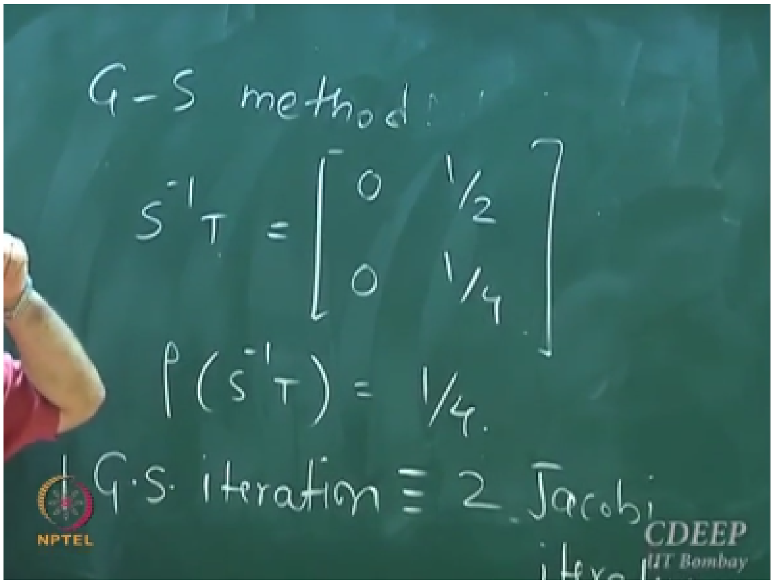

For Gauss-Seidel method S inverse T, this turns out to be 0, 0, 1/2, 1/4 and spectral radius is S inverse T, okay. The spectral radius is given by this. Actually spectral radius maximum magnitude Eigenvalue of S inverse T is an indicator also of the performance, okay. Now, there are 2 aspects, it should be < 1, okay. Now, how much it is < 1, how close it is to 0, that also matters in terms of the rate of convergence.

Whether the convergence is guaranteed or not is decided by whether it is strictly < 1, okay that is the stability criteria. The performance is given by how much it is < 1. So, this Jacobi method in which spectral radius is 1/2, okay converges slower than the Gauss-Seidel method, okay because the spectral radius here is 1/4. In fact, if you start with calculations you will see that one step of Gauss-Seidel will be almost equal to 2 steps of Jacobi, okay.

So, the Gauss-Seidel can move much faster. You cannot show it for every matrix. This is no proof that Gauss-Seidel always converges but in general Gauss-Seidel convergence faster than and the reason is typically spectral radius of S inverse T for Gauss-Seidel is <, okay, that is the reason. Now, what if I formulate the relaxation method. So you can almost show that because of this one GS iteration is equivalent to 2 Jacobi iterations, okay because spectral radius in this case is even smaller.

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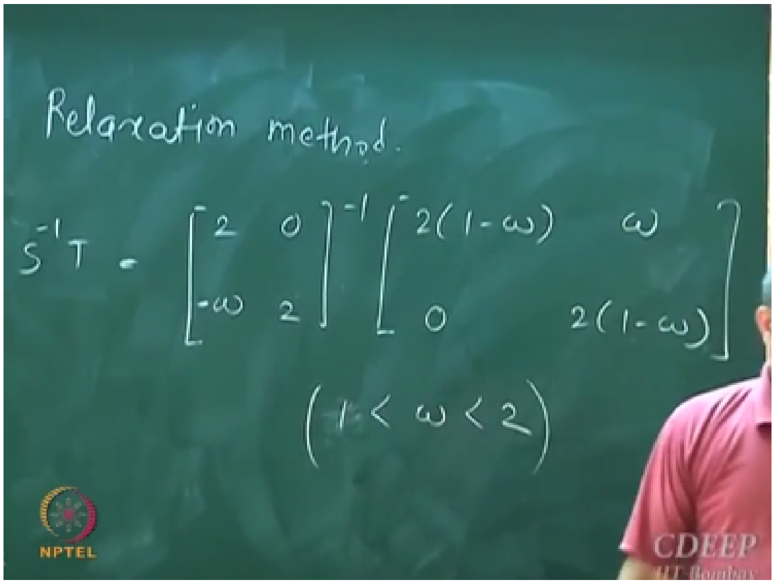

Now for relaxation method, S inverse T turns out to be inverse of this matrix 0, omega, -omega here, 2 inverse 2*1-omega, omega, 0, 2*1-omega and of course we should choose omega between 1 and 2. We want it to be > 1 because if it is = 1, it is nothing but Gauss-Seidel method, if we want to be > 1. Now, for this simple case 2x2 matrix you can actually find out what is the best value of omega that will enhance the convergence. What is the optimum value, okay.

For different choices of omega, you will get different spectral radius, okay. You can actually find out which value of omega, this is just again to tell you emphasis it, this is only to get insight. In real problem, I am not going compute optimum omega by doing some. I have to tune give a guess for omega.

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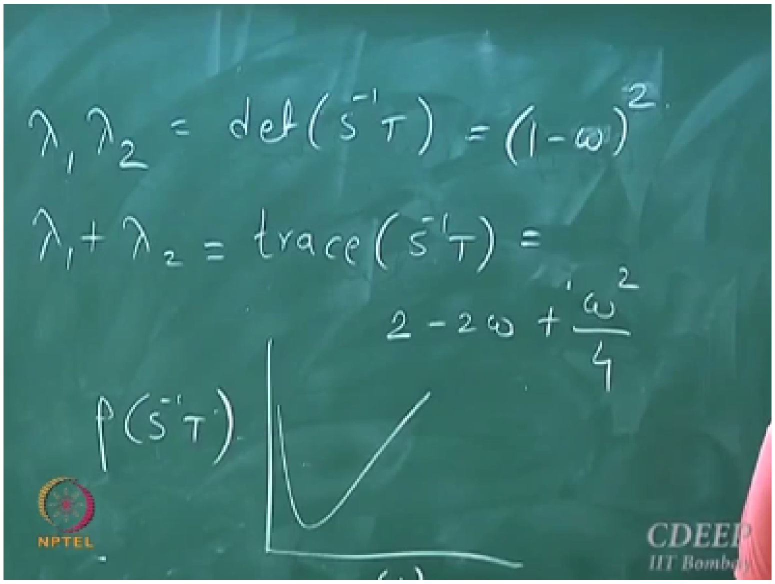

So, if you use some properties (()) (35:39), then you know that lambda 1 and lambda 2, if these 2 are Eigenvalues of S inverse T then that is equal to determent of S inverse T which turns out to be in this case, if you take determinant of that, it will be 1-omega whole square, okay. You know this property, multiplication of Eigenvalues for matrix is same as determinant and then what is the other property trace. So, lambda 1+lambda 2=trace of.

So, this will turn out to be 2-2omega+omega square/4. Now, if you plot this, that is if you plot S inverse T versus omega. If you plot spectral radius using these 2 relationships, you can find out lambda 1 and lambda 2 and spectral radius; and if you plot this, you will find that getting the optimum is not very difficult.

If you plot this, you will find that the optimum value for which the spectral radius is minimum, you know you will get a point where you will get a minimum value of the spectral radius.

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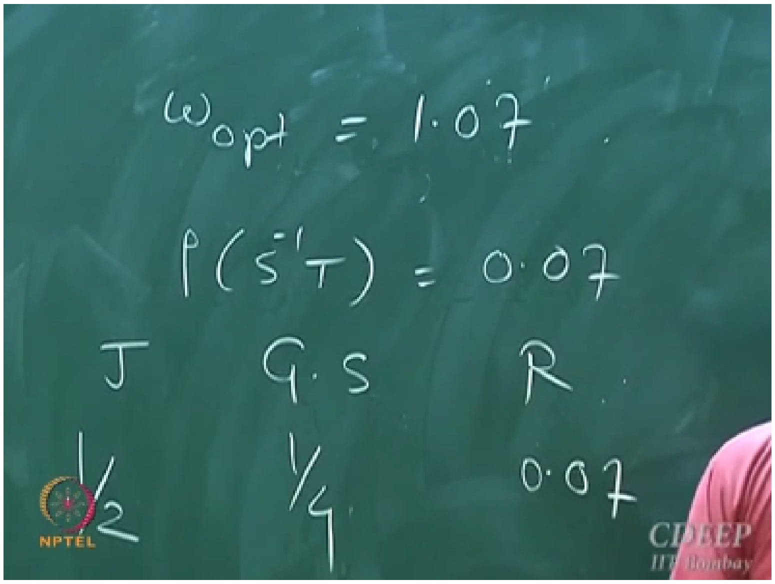

So, that value turns out to be omega optimum=1.07, okay and the spectral radius of S inverse T for this omega is 0.07, okay. I am skipping some steps, you can see here in the notes, that is not important. What I want to point out here is that if you are able to choose omega properly, then this is 0.07.

So, we had 3 situations; you know Jacobi method, then Gauss-Seidel method, and relaxation method, the S inverse T spectral radius in this case was 1/2, this was 1/4 and this is you know 0.7 which is almost 1/4 of this. So, we said one iteration of Gauss-Seidel was 2 Jacobi iterations. What can you say about one relaxation iteration. It is like one relaxation iteration is like 4 Gauss-Seidel iterations, almost like eight Jacobi iterations.

So, relaxation method can converge even faster. Typically, values close to 1, 1.1, 1.2 are used, this is thumb rule and not substantiated. I think (()) (38:57) gave some clue that you can use it close to 1.2 but it is hard to say generally what value of omega will make convergence very-very fast, okay. So, the tricks that you should use is first of all make sure that either the matrix is diagonally dominant. If is not, okay, to ensure convergence he should pre-multiply both the side by A transpose that will make it symmetric positive definite, I have guaranteed convergence, okay.

But I want convergence faster than Gauss-Seidel, Gauss-Seidel is better than Jacobi, so I will apply Gauss-Seidel and I can make convergence faster even going to relaxation method. So, probably I should all these tricks and use relaxation method to enhance my convergence, that is how I should proceed with arranging my calculations. So, this brings us to end of this analysis. What is important here is that there are many take home messages.

One of the things is that Eigenvalues is one of the prime tools for analyzing behaviour qualitatively, asymptotically. I do not have to solve, that is the beauty of this tool. I do not have to solve the problem. I can just look at Eigenvalues or in this case it turns out finally that I can just look at diagonal dominance, I can see whether to convert the problem to a symmetric positive definite matrix, I am guaranteed convergence of my iterative scheme, very-very powerful result.

In fact, Eigenvalues are used for convergence analysis in engineering literature, in many, many, many ways, okay. Well most of you I think have done the first course in chemical engineering or process control and in process control well you may not have connected it to the Eigenvalues in the first course but actually what you can show is that if you write a differential equation for local linear differential equation for evaluation of the system dynamics, then the so-called roots of the characteristic polynomial are nothing but Eigenvalues of certain matrix which governs the system dynamics, okay.

If the Eigenvalues are on the left of plane and then you know, what was nice thing about Eigenvalues there or roots of the characteristic polynomial, you do not have to solve. You just look at the roots whether they are lying on this half of the plane or this half of the plane. You can tell how the system is going to behave asymptotically without having to solve, okay. Same thing is here, without having to solve, I can tell whether my iterations will converge or not, okay.

Also using necessary sufficient conditions, I can go and modify my problem to make sure that convergence will occur, okay. This is more important than the algorithm per se. Algorithm you will learn to program it or nowadays I think these algorithms will be available on the net, you might download it. Algorithm might be very well written that does not mean that does not mean that convergence is going to occur, okay.

You should know why convergence occurs and then make sure that you transform the problem in such a way that convergence occurs, that is important, okay. There are 2 more things that I need to do, because we missed one lecture, some timetable is disturbed but I will try to make up for it. I will try to cover optimization based iterative methods for solving Ax=b, okay. So, till now I formulated iterations in one particular way by splitting the matrix, in fact row by row calculations, not really splitting the matrix.

The way the iterations were derived where doing row by row calculations.

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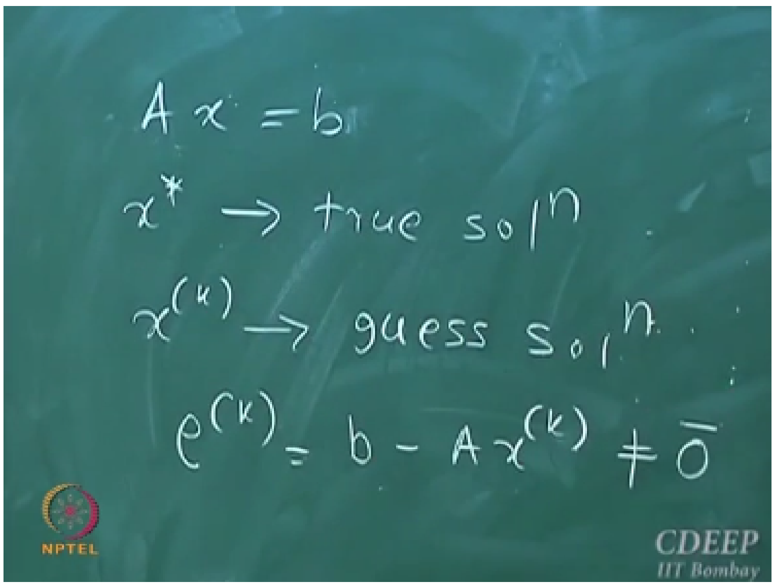

My next aim is going to be instead of doing that can I iteratively solve this problem, well. If I want to solve Ax=b, okay. Well, if I take a guess solution, the true solution is let us say x star, okay and if xk is my guess solution, then obviously ek, now my ek has a different definition, ek is b-Axk, okay. This is not going to be 0. When this is equal to x star, it will be = 0. If xk is = x star, this is = 0.

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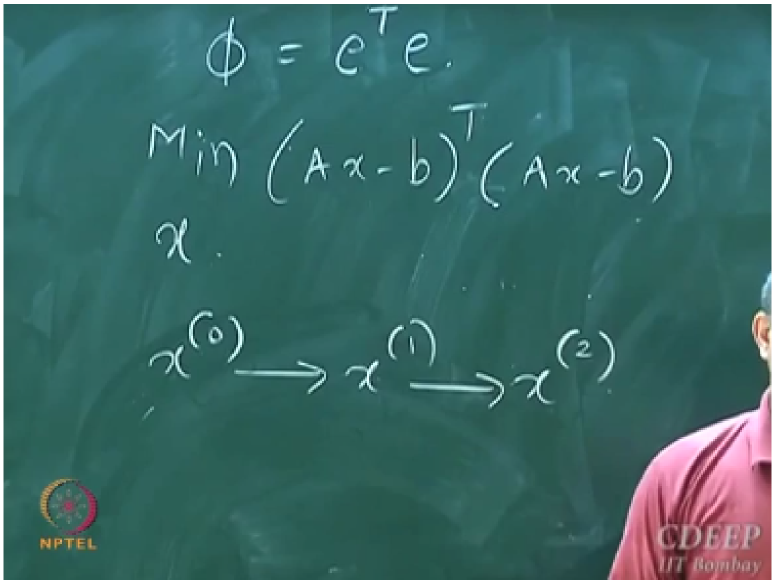

The way I want to solve this problem is minimized scalar objective function phi is defined as e transpose e, that is Ax-b transpose Ax-b with respect to x, okay. I do not want to solve this. Well, you will say that if you apply it on a necessary condition for optimality, you will get you know dou phi/dou x=0 and we will give you A transpose Ax. If you apply this condition, dou phi/dou x-=0, then you will get A transpose Ax=A transpose b.

I do not want to solve this, I do not want to go by this route. I want to go iteratively, okay. I want to guess X0 and by some method I want to go to x1, then I want to go to x2, and so on and then I want to see whether this iteration converges. We are going to use what is called as the gradient search, okay. One of the fundamental methods in optimization, gradient based search. So, we will look at gradient search and then there is one more method called conjugate gradient search which we will look at next, okay.

That is one thing which I want to do. After having done that, we have talked about iterative schemes for solving Ax=b and then we move on to a very-very fundamental issue, matrix conditioning, which problems are inherently ill-conditioned, which problems are well-conditioned, how do I classify and say that this is ill-conditioned problem, whatever I do I am going to end up into some trouble. This is a well-conditioned problem.

If I am getting wrong solution, I have made a mistake. So, well-conditioned problems you know, absurd solutions, you have made a mistake. Ill-conditioned problems, absurd solutions, you cannot do much. How do you classify ill-condition from well-condition is the next thing, that will bring this to end of this module.