Advanced Numerical Analysis
Prof. Sachin Patwardhan
Department of Chemical Engineering
Indian Institute of Technology - Bombay

Lecture - 03
Basis, Dimension and Sub-space of a Vector Space

In a combination set of vectors and so on so I started by defining the set of vectors and I had a special notation for it.

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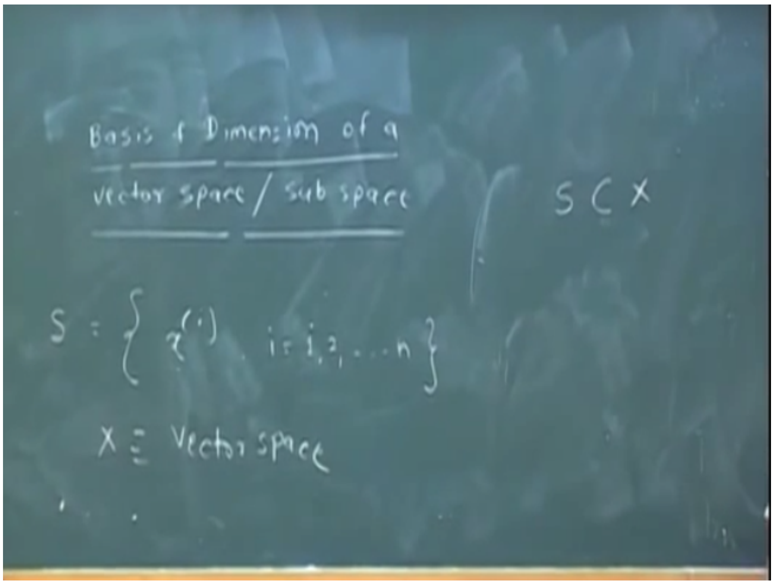

So today we are going to look at specific things like so basis and dimension of a vector space or a subspace and other than that just giving definitions, I want to give some examples for it. So that why these ideas are being pursued should become a little bit clear? Why there we get these funny spaces (()) (00:58) look at set of all continuous functions and then call it a space and why we look at set of polynomials and call it a subspace.

So yesterday we had developed this notation. This notation was a set S let us say was defined as xi where i=1, 2. This would be a finite set, in some situations this could be an infinite set. I am currently defining a finite set of n vectors, which means there are n vectors, X belongs to some vectors so X is my vector space and S is the subset of X.

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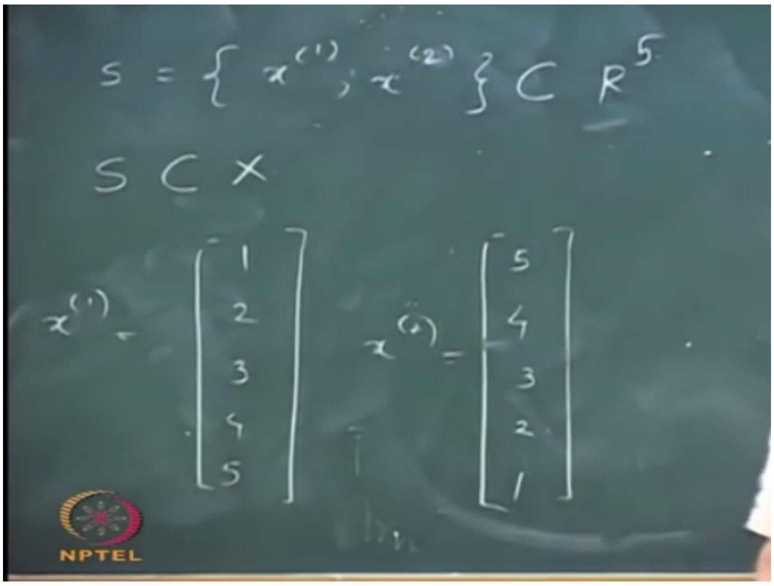

So one example that I gave was from R5 so I said that you know 2 vectors say x1 this was 1, 2, 3, 4, 5 and x2 which is 5, 4, 3, 2, 1 and putting some (()) (02:26) vectors from R5. So I can define the set S which is x1 and x2, these are 2 elements in the set where S is subset of R5 okay.

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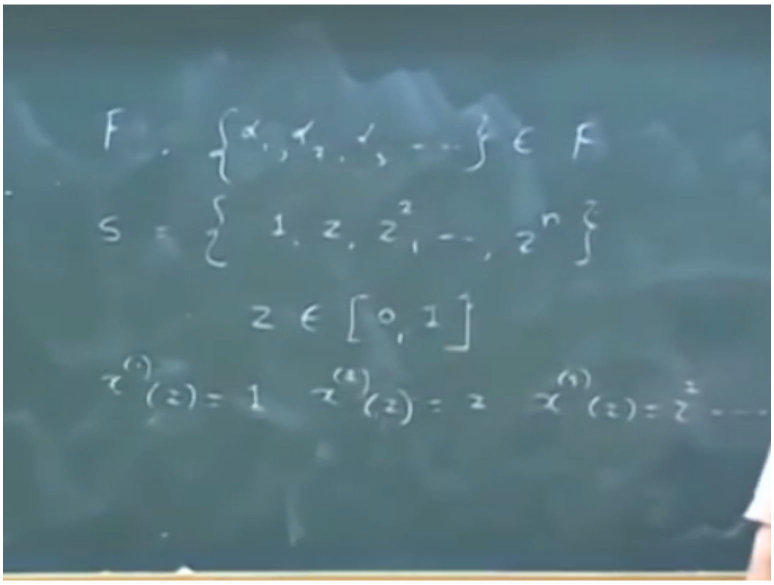

In some cases, I will define a set S to be set of polynomials. I define a set S which is set of polynomials, so my first vector here, my first vector x1z is 1, x2z. So in this particular set, I have n+1 vectors, which are denominated like this. This is the first function, this is the second function x2z second vector okay, this is the third vector, this is the n+1th vector and so on. So the notation here to generate could be used in a context of n-dimensional spaces.

It will be used in the context of function space and it will be used in a context of polynomials depends upon what context they are okay so that is first thing. Then yesterday the segment that we talked about linear combinations so F is my finite and if I take some scalars say alpha 1, alpha 2, alpha 3, which are the scalar set this belongs to F.

If this scalar set belongs to F then we could find a set of vector, which is linear combination, which is linear combination of the original vectors.

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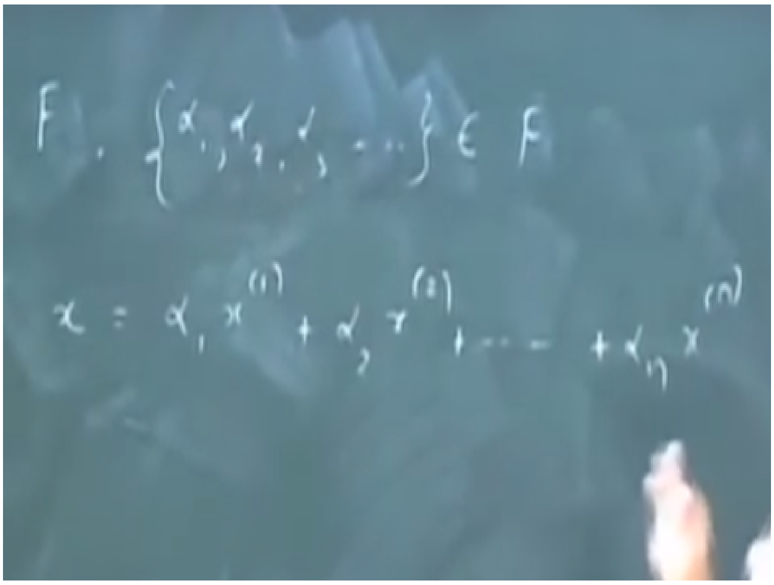

So I can construct the new vector x, which is alpha 1 x1+alpha 2 x2 and so on + alpha n xn. So this linear combination in the later example where we took vectors as polynomials would be a nth order polynomial okay then for n-1 polynomial the vector to the n. In the case of n dimension, it will be a vector in alpha n okay.

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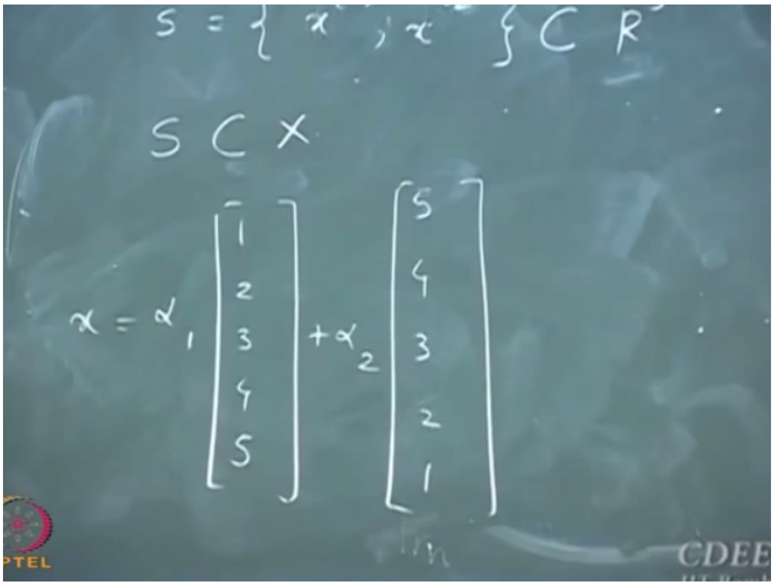

I can talk about a new vector x which is alpha 1, 1, 2, 3, 4, 5 and alpha 2, 5, 4, 3, 2, 1. When alpha 1 and alpha 2 are any 2 arbitrary scalars drawn from R okay. This is linear combination and then we said that we have this notion of span of a set.

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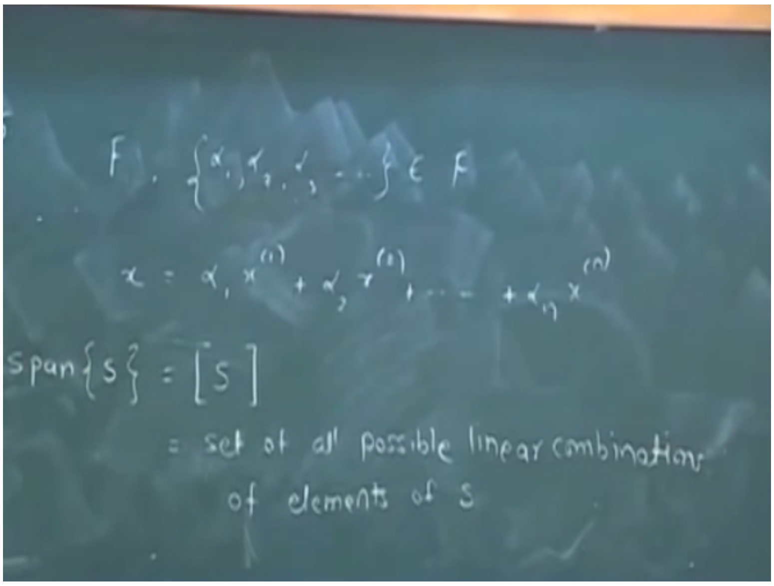

The span of a set is set of all possible linear combination. Span of S which is many times denoted as square bracket S. So we take all possible combinations of elements of S and then what I get is span of. If I take all possible linear combinations in this set, then what I get is span of this particular set. It is also example in 3 dimensions, I took 2 vectors and said span of these 2 vectors will be a plane cutting through the origin.

If I take 3 vectors in the same plane, span will be again plane cutting through the origin that is not going to be different. Well this sounds very abstract. Let us get on to some concrete examples which probably you have seen in your undergraduate and then let us connect why do I need span? Why do I need to have (()) (06:41) of spaces?

Why do I need them? And then we will move on to this concept of linear independence and linear dependence and then talk about basis. Before I do that let us get some concepts clear about this span. So why do we need span? Why it is used? Why it is important? So something that we actually keep using all the time these concepts are used when you solve linear algebraic equations, differential equations.

You will keep doing it unknowingly without understanding what eventually happened because some of these concepts are not introduced in undergraduate. Let me start with some example, which is simple solving linear algebraic equations.

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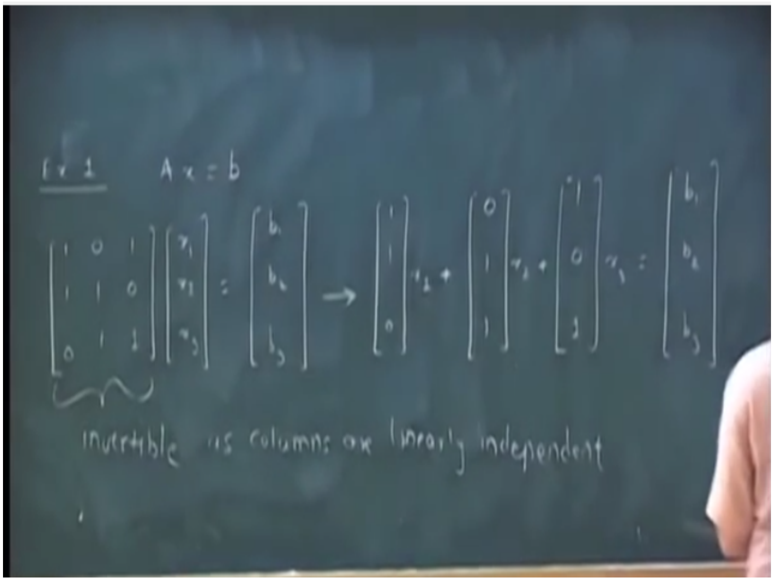

So my example 1 is I want to solve Ax=b where A is an invertible matrix. So I am taking one single matrix here 1 1 0 0 1 1 1 0 1*x1 x2 x3 gives me b1 b2 b3 okay. Actually, this equation I can rewrite slightly differently you will start getting insights to why I am talking about this span. I can rewrite this as 1 1 0*x1+0 1 1*x2+1 0 1*x3=b1 b2 b3. Am I correct? I am just rewriting this equation.

Here b is a vector which is linear combination of first vector, second column vector and third column vector okay. What is span of column vector 1, column vector 2, column vector 3? What is the span? The span of these 3 vectors, I want to show is same as 3-dimensional vector space. What is span? All possible linear approximations. Are these vectors linearly independent?

By inspection for this particular case, you know about linear algebra, you can see that these 3 vectors are linearly independent. What will be span? What will be all possible linear combinations of 3 linearly independent vectors in 3 dimensions (()) (09:56) okay.

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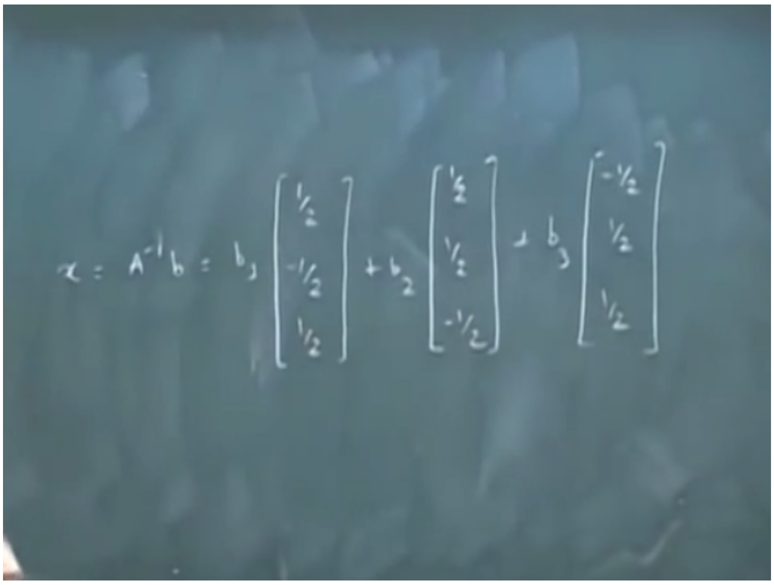

So I can write the solution, which is x, this is A inverse b. Well I am not right now worried about how (()) (10:08) inverse I suppose you know this and this A inverse b comes out to be b1 times 1/2 -1/2 1/2+b2 times 1/2 1/2 -1/2+b3 times -1/2 1/2 1/2. If I actually compute a solution of this system of equations okay we can do it very easily by Gauss elimination, Gauss-Legendre method and compute inverse okay.

I am just rewriting the solution in terms of A inverse b except that I have chosen to write like this, I have chosen to write like this okay. Give me any b vector, I can write the solution like this okay. So where does the solution belong to? The solution x belongs to span of this vector, this vector and this vector. Again from undergraduate experience, will know that these 3 vectors are linearly independent okay.

So the solution actually belongs to the span of these vectors okay. Now this is a well behaved example of matrix A is invertible. I am going to do another example if matrix A is non-invertible, I still want to find a solution and then we will get this concept of the solution belonging to a space which is not (()) (12:11). In this particular case, what is the solution space?

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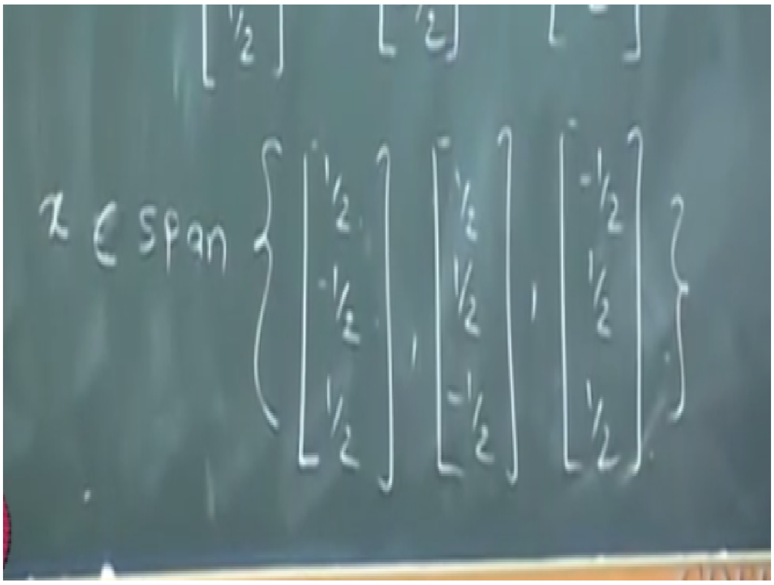

So x belongs to span of 1/2 -1/2 1/2*1*1/2 1/2 -1/2*1*-1/2 1/2 1/2. The solution actually belongs to span of this all possible linear combinations right. If you specify b I will get one particular x. How many ways I can specify b? Infinite possible ways b1 b2 b3 can be chosen, all vectors are put here, I will get one particular solution. So this is an example where you have to look at the span.

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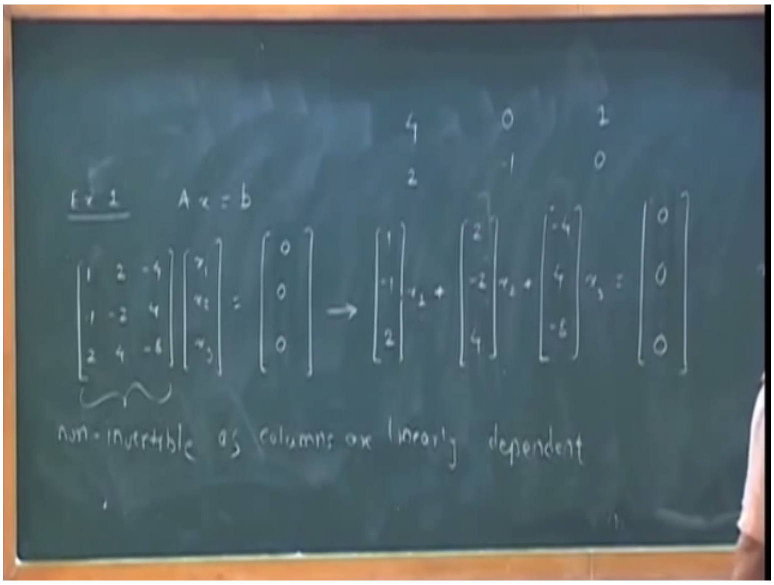

Now let us look another example then the things will become clearer okay. I am just going to change this matrix to slightly different matrix 1 2 -4 -1 -2 4 and 2 4 -8 and then I am going to look for a solution which is for 0 0 0. I am going to look for a solution which is 0 0 0. Well so basically I want to solve for 1 -1 2 3 -2 4 and -4 4 -8, this becomes 0 0 0. Is this matrix invertible first of all? **"Professor - student conversation starts."** No sir. Why?

The columns are linearly dependent. I just multiplied this column by a vector and just multiplied this column by another vector I will get this column okay. Is this problem solving? This is going to 0? Yes why? I told why not what, 3 unknowns, 3 equations yeah linearly dependence. So they are linearly dependent and I should be able to get infinite solutions **"Professor - student conversation ends."** so this matrix is non-invertible.

So as the columns are linearly dependent, this matrix is non-invertible. Can I find the solution? What are the possible solutions to this problem? So what about 2, I am not interested in 0 0 0. I am not interested in trivial solution. What I want to know is are there nonlinear solutions. **"Professor - student conversation starts."** Yeah, so tell me the solution **"Professor - student conversation ends."**

So what about x2=2 x1=-1 and x3=0, will this form a solution? Yeah. Right multiply this by 2, multiply this by -1 okay. I will get 0 the last column is multiplied by 0. Other candidate solution of course is multiplied by 4, multiply this by 0 and multiply this by 1 right. I get 2 possible candidate solutions. Here my first solution well remember these are linearly dependent columns, which means there are same equations repeated again multiplied by scalars okay.

I am not taking the equation view point I am taking the column view point, I am not taking the row view point. What is different between row viewpoint and column viewpoint will come to that little later okay.

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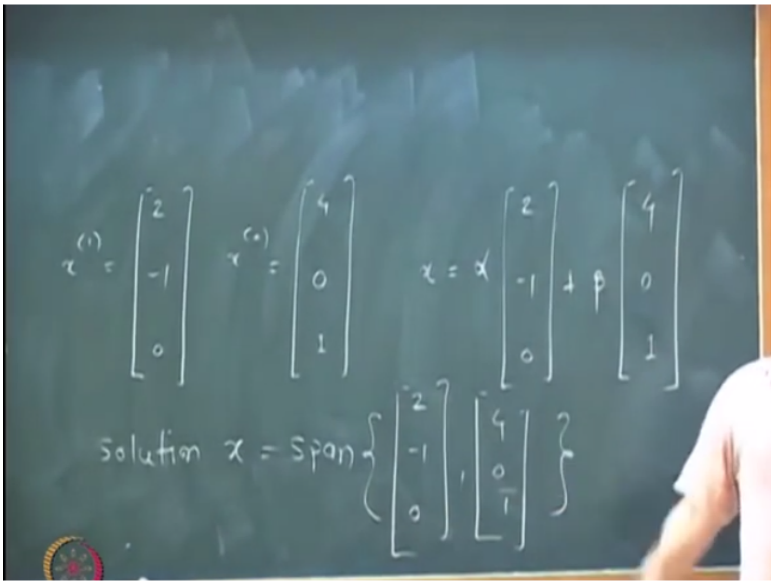

So now I have 2 possible solutions. I have 1 solution which is x1 so I would write this as 2 -1 0 and have x2 which is 4 0 1 okay. I found 2 possible equations that are infinite solutions. How do I get 2 different solutions? **"Professor - student conversation starts."** Any linear combination. **"Professor - student conversation ends."** So if I construct the vector which is alpha times 2 -1 0 and beta times 4 0 1 just check this also will be a solution.

Because A times this vector is 0 +beta times this vector 0 so you know the solutions at 0 so A vector which is what do I mean by any vector which is span of these. So what I want to say here is that a solution lies in the span of these true vectors. Are these linearly independent? Yes are linearly independent and the solution lies in the span on these 2 vectors. What is the dimension of space created by the span of these?

"Professor - student conversation starts." Two. Because there are 2 linearly independent vectors. All possible linear combinations of these 2 vectors will give rise to a 2-dimensional subspace of 3 dimensions okay **"Professor - student conversation ends."** will give rise to a 2-dimensional subspace of all 3 okay. So span is something which you have been using without knowing what is happening now.

Let us graduate from these examples to differential equations okay. So that we get a feel of how we can generalize these concepts to some other spaces that what we have been looking for okay.

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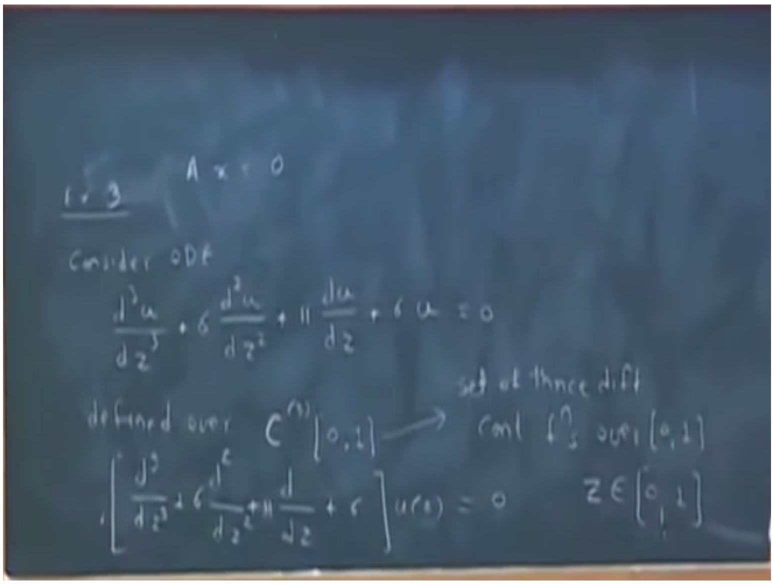

So my third example, I have already covered 2 examples, so third example is not Ax=b well conceptually it is not different to Ax=b but you know take some time to realize that what I am talking about is 1 and the same thing okay. Let us look at this example. So now I want to look at I want a transition to example which is different from linear algebraic equations. I am looking at differential equations while this equation is not too different from Ax=b.

A here would be this differential operator operating on vector U and giving me 0 vector okay. So this is exactly my Ax=0 except I have to write as d3/dz cube+6 d2/dz square+11 d/dz+6. This operator operating on Uz is=0. Actually 0 is the 0 vector in this space. What is this funny space? I have not talked about it earlier. This is set of twice differentiable continuous functions.

All intervals 0 to 1 so z belongs to this is the set of 0 to 1 and this particular set is thrice differential continuous function C3 0 to 1 is thrice differentiable continuous function while you know how to solve this problem. I have given the initial conditions I want to get a general solution. I want to get a general solution for this problem. How do you do that? You write a calculus in equation mode and then you write the solution.

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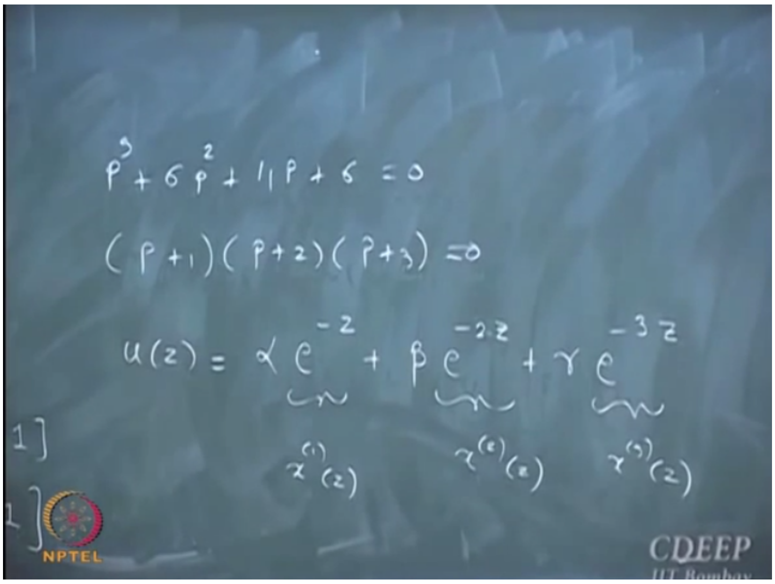

So let us write all what you know the characteristic equation is will be p3+6p square+11 p+6=0 and then I can write p+1 p+2 p+3=0 so I have 3 routes, -1, -2, -3 and all of you know how to write a solution but what I want to point out here is something different. I am going to talk about this span okay. So the general solution to this particular problem is given by Uz=alpha e to the power –z+beta e to the power -2z+gamma e to the power -3z okay.

Look at these 3 vectors, where are the vectors coming here? The vectors are 1, 2, 3 so this is my x1z, this is my x2z and this is my x3z okay. So a general solution to this problem, it belongs to span of all possible linear combinations. What is this alpha, beta and gamma? The moment we specify initial conditions, alpha, beta, gamma will get this.

Well if you have not done this (()) (23:45) we will be doing these solutions little later. This is the linear combination of these 3 vectors.

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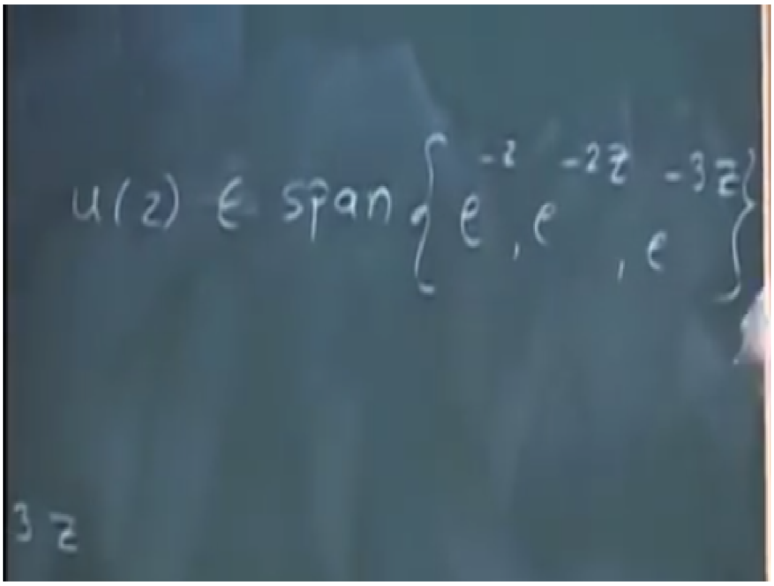

So I can say that Uz actually belongs to span of e to the power –z, e to the power -2z and e to the power -3z okay. So span is something which I need to know; something that I need to know; if you remember, there are 2 solutions to our problem of this side, if there is something on the right hand side; right, if there is something on the right hand side, it is a forcing function.

Then you have a particular solution and you have general solution which is; the general solution will be this, then we have a particular solution depending upon the forcing function. The general solution is actually governed by the initial condition and span of these characteristic vectors.

Any solution; any solution, which is where forcing function in 0 will always lie in the span of these 3 vectors; will always lie in the span of these 3 vectors, you cannot leave this; solution of this cannot leave this sub space. What is this? This is a; what is this is the subspace, right, is it a subspace? It is a sub space.

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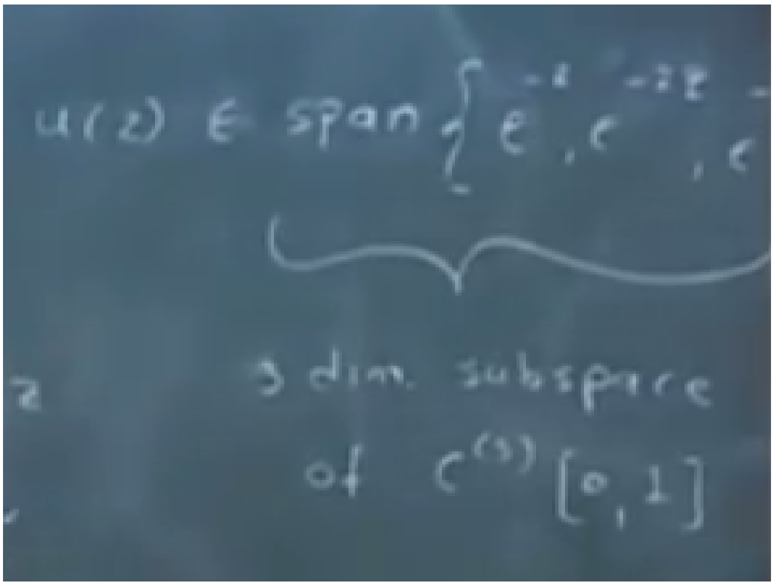

All possible linear combinations of these 3 is a 3 dimensional subspace. What is the underlying space, which we are talking about? The underlying space which we are talking about is thrice differentiable continuous functions, each one of them is thrice differentiable; do you agree? All of them are thrice differentiable, okay. First of all, I wanted to understand that (()) (25:57) A operating on a finite dimensional vector X will be 0 zero.

And this operator operating on this vector belonging to this space or not wonder will be different ideas, they are same things; geometrically same things, nothing different. If you start doing; **"Professor-student conversation starts"** Sir, how can we visualize these are vectors; I told you, you can visualize these spaces, you can visualise only in 3 dimensions but believe me that if you understand in 3 dimension, what is happening very well, okay.

We talked about a 2 dimensional subspace of R3 just now, right okay. In the same sense, see what is the 2 dimensional subspaces, R3, is it a very thin set? There are too many vectors in R3 right and then R2 is the plane passing through the origin, so very thin set, pure vectors relatively, smaller infinity than this bigger infinity, right. Now, in the same sense well, I can construct a vectors in this space, which are you know -2, -3, -4, -5, -7 all of them belong to (()) (27:08) continuous functions, right.

Actually we will come to this idea of basis and what will show is that this particular space is an infinite dimensional space, there are infinite vectors in the spaces; linearly dependent vectors that we can find in this particular space are infinite. Let the right hand side not be 0, this will not be a sub space? Which one? The solution; No, no, no, you will get a specific solution, who might I specify an initial condition.

If I specify initial condition u0 is = something, say alpha, No, no take, a and then u dash 0 = b and u double prime 0 is = some c, moment I give you these 3 initial conditions, these are some specific numbers, know you can take -1, 2, -3, I have taken some arbitrary numbers, okay. Moment I fix this, okay, I will get one specific value of alpha, beta, gamma; I will get 1 specific value alpha, beta, gamma, which is one element okay, now if I change the initial condition, what will happen?

I will get another value of alpha, beta, gamma okay but all of them have to belong to; sir, but if the right hand side of the main ODE had not been 0, then zero vector would not have been in the span, then it would not have been a subspace. No, no, when you have something here, f of z, how do you solve? There are 2 solutions; one is the characteristics solution; this part will anywhere appear plus something which is appear here.

So this part is invariant with there is something on the right hand side or nothing on the right hand side, I can say it can be zero vector, can be non-zero vector, this will appear all the time. So some part of the solution will always belong to this sub space, you cannot escape it, this is characteristic of this particular equation okay. So, these 3 vectors are kind of (()) (29:26) to this equation okay, this can change; right hand side can change.

This can be sin, cos, or whatever, forcing function can be done, okay but this is something like inherent nature of these differential equation, you cannot change this. Let me change, if you change these coefficients okay, you change if you change the coefficients, okay. **"Professor-student conversation ends". S**o, this is fine, let us move on to another example which is again differential equation and it will sort of (()) (30:06), what I am talking about as dimension.

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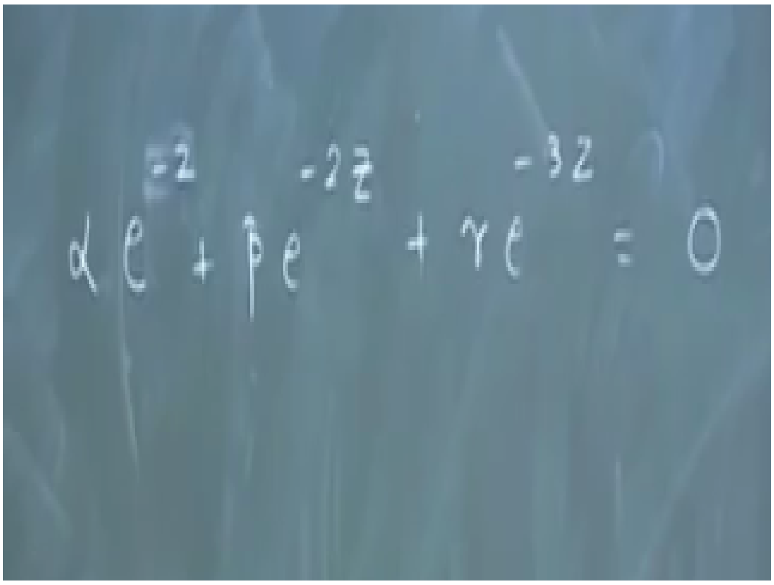

Now, I am going to introduce (()) (30:12), what is the dimension of the vector; vector space but this particular example we knew that idea of dimension. Well, there is one question; one question; these three vectors are they linearly independent? **"Professor-student conversation starts"** Yes, why? You have to prove it, so I should prove that alpha e to the power –z + beta e to the power 2z + gamma e to the power -3z is = 0.

Actually it means zero function; zero does not mean 0; 0 to 0 function, can be found only when alpha, beta, gamma are 0, so these are linearly independent okay. Can you prove it, well I leave you, this is one example, which is here, I do not know whether I get time to do it but we should look at this example, the example in my notes. How we prove that three vectors are linearly independent?

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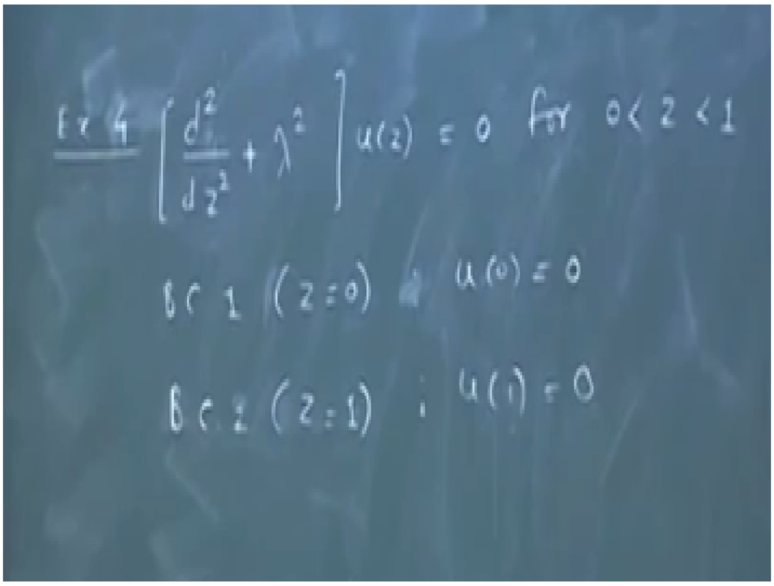

It is not that state of it; you have to do some; some amount of thinking, okay. **"Professor-student conversation ends"** Let us move on this fourth example, so my example 4 is; so, bounded value problem now; for I have 2 boundary conditions, this is the differential equation, which is satisfied in the domain, say going from 0 to 1 okay then, I have this boundary condition at z is = 0.

So, this u0 = 0 and there are I have second boundary condition z = 1, so this is u1 is = 0 and this kind of problems will appear when you start solving partial differential equations and so called (()) (32:31), which will be covered in the other course. So or; we might have seen this in your undergraduate, these kind of problems, we may see it in (()) (32:42) once in a while.

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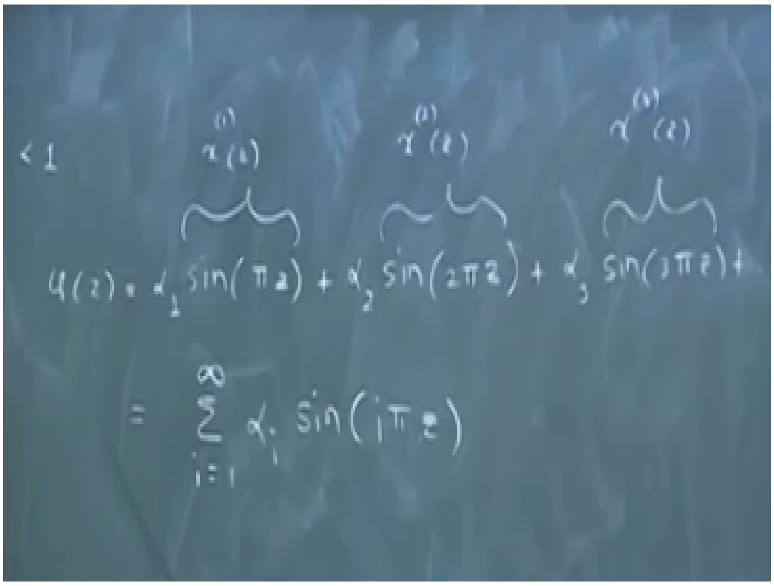

Uz = alpha 1 sin PI z + alpha 2 sin 2pi z, well, well, well, the general solution here cannot be expressed in terms of finite number vectors. For this particular problem, a general solution cannot be arrived at by (()) (33:22) finite number of vectors, what are vectors here? So, this is my x1z, this is my x2z, this is my x3z and so on. How many such vectors I have? Infinite, I have infinite vectors, okay.

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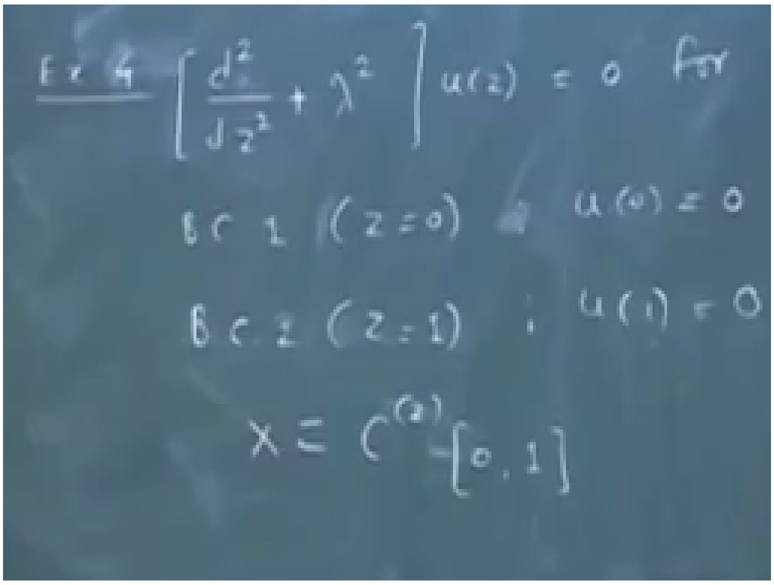

Where does this solution belong to? Span off, what is this; what is this? This is linear combination of vectors; linear combination of vectors, right, same idea which was we talked about earlier, like I said as, okay. What is the underlying space here? Somebody said it, c2 0, 1 okay. Now, underlying space here is C2, well you have to be careful while writing this notation, it has to be square values.

Because two ends are included, there are 2 boundary conditions, not just; so twice differentiable continuous functions all interval 0 to 1 that is underlying space okay. So, where does this solution belong to? Solutions belongs to the span of; these are vectors; these are vectors; these vectors belong to twice differentiable continuous functions okay, a linear combination of that gives me a solution to this problem okay, infinite linear combination not just one value.

These; well in a partial differential equation, these coefficients; alpha 1, alpha 2, alpha 3 to be fixed by some other particular solution, which this comes as a couple problem in the partial differential equation, so there will be one part of the partial differential equation is solution to this problem and the other part will be particular solution, so that is; you can look at book by Professor Pushapavanam, mathematical methods in chemical engineering (()) (35:52).

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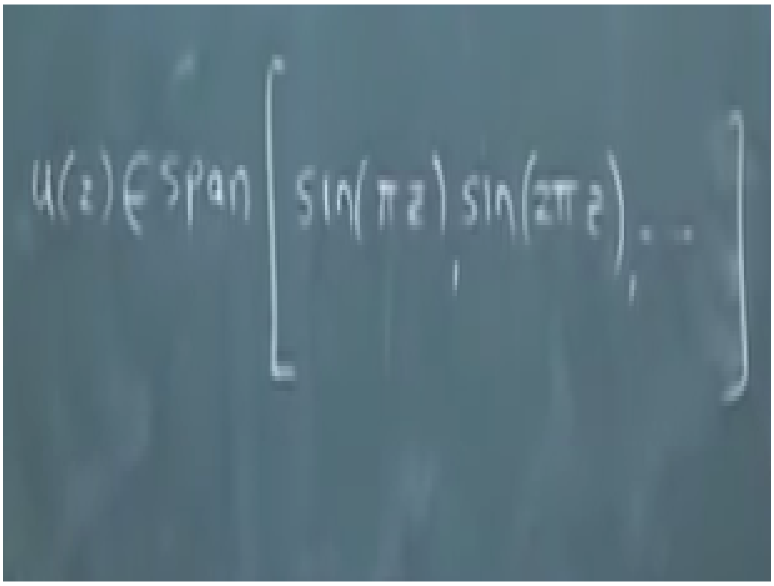

But the point that I want to make here is that uz actually belongs to the span of; it belongs to span of this set of vectors; belongs to span of this set of vectors, so span is a very very important concept and we keep require everywhere, not been introduce just for the sake of it okay. In this particular case, the space or the subspace of; subspace of twice differentiable continuous function, which is fine by these vectors okay is equally dimension, is infinite dimension, okay.

Because these are linearly independent vectors well, coming to the point where we have defined what are the linear dependence and then we have also worry about the basis business. Is this clear; (()) (36:52) clear? Somebody has any doubt, please; please feel free to stop me at any point. **"Professor-student conversation starts"** Is it possible? Well, if you take initial condition, it depends upon the initial condition, so if you take initial condition to be 0, 0, 0, only possible solution that you get this.

Depends upon what is the initial condition of u0, u prime 0, u double prime 0, all the 3 initial conditions are 0, 0, 0 (()) (37:25). Yeah, no, they are the functions; they are functions, there are not values of the functions. 0 to 1, according to the problem, this particular problem would be heat transfer problem; it will be something by conduction in a rod, 0 to 1 comes because non dimensionalize the space variable that one is; it could be 0 to n, if you want, does not matter.

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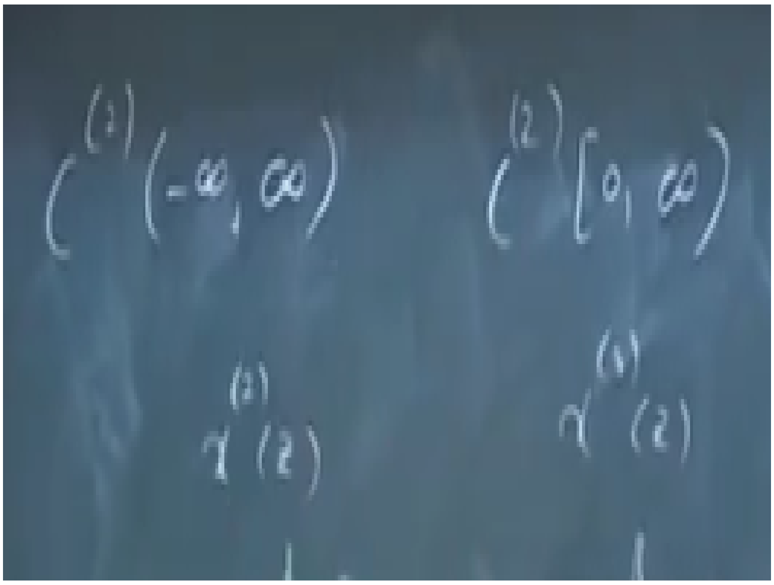

Well, if you know; no, no, these are different spaces. If you have; you may have a situation, see there are some problems in chemical engineering, where you have some infinite dimension of a rod and all, so you may have such scenario, where you know, which is C2 - infinity to infinity or C2, 0 to infinity but these are different spaces, this space is different than; No, no, a function defined over 0 to 1 is not same as function defined over 0 to pi.

Or –pi to pi, there are different functions maybe some part, which is not included here, okay, they are not same; exactly, that is another problem and it cannot be continues, it might be continuous in some region but not continuous in some other region, so all kinds of (()) (39:15) okay. **"Professor-student conversation ends".** Okay**,** let us move on to this basis and dimension business, so now what is basis, what is dimension of the space or subspace okay.

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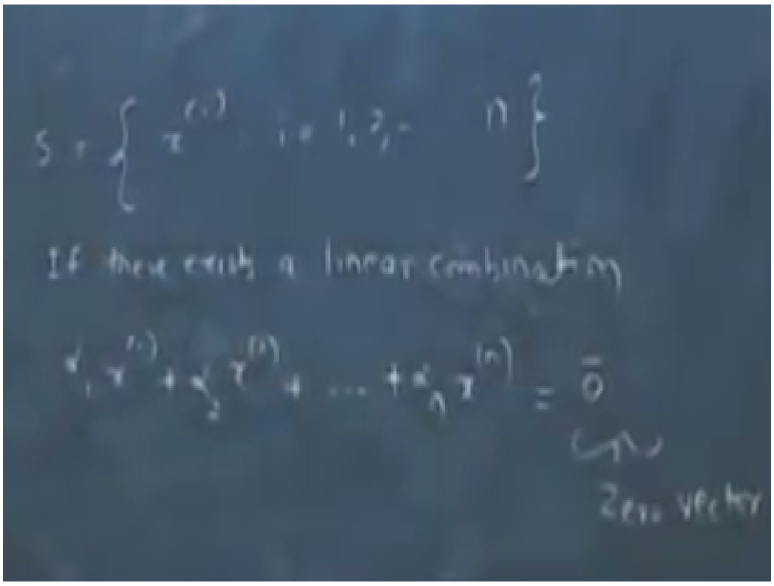

The formal definition of linear dependence that will find in the notes, so we again define abstract terms, we start with this set s, which is the; we start with set s okay, zero vector, well we are writing 0 on the right hand side, it means zero vector in that particular space, it is reverse now. If these vectors are n dimensional, what will be this 0? Well, let me qualify by put a bar over it; 0 bars and 0 vectors, 0 bar is 0 vectors.

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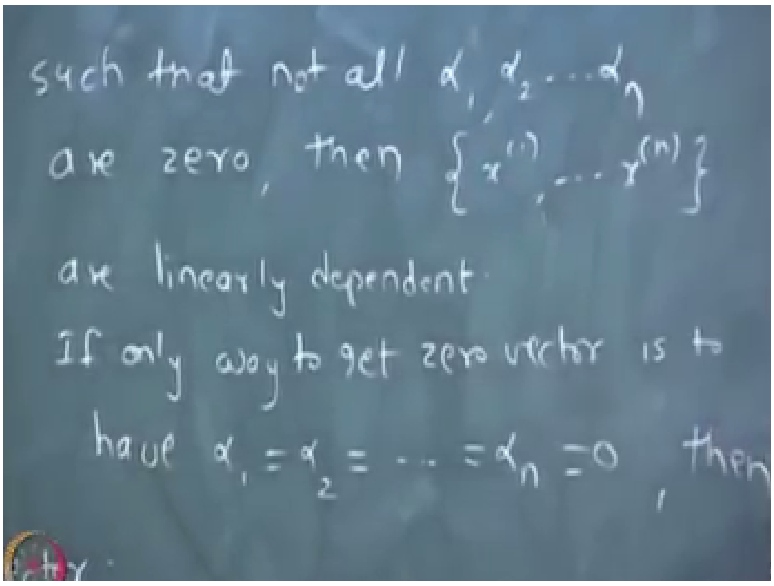

So, if this is = 0 bar that is zero vector okay, if this is function space; set of continuous function, so 0 to 1, what is the 0 vector? Zero function everywhere, okay, Fz = 0 on 0 to 1, that is the 0 function, okay that is what it means, it does not mean liberally 0 at one point, it depends upon what space, okay. So, if this is a linear combination such that; okay, so if there exists a linear combination such that okay, if there exists a nonzero linear combination, there exists a nonzero linear combination, which gives 0 vector.

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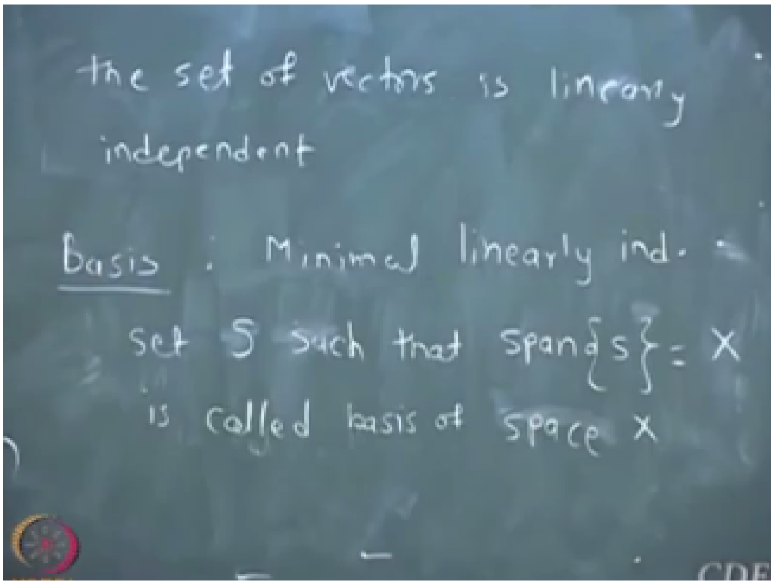

Which means; if I can find some nonzero elements; alpha 1, alpha 2, which will give the zero vector, when these elements are linearly dependent, else there if you cannot find; if you cannot find a nonzero combination, which means only when to get a zero vector is to put alpha 1 = 0, alpha 2 = 0, alpha 3 = 0, all of them are 0, okay, then they are linearly; then they are linearly independent, then the set of vectors is linearly independent, okay.

And what is spaces; yeah, basis of the vector is nothing but the set of linearly independent vectors, which generate the entire space; minimal set okay, which generates the entire space. So, the minimal linearly independent set okay minimal linearly independent set, which generates the entire space yeah; **"Professor-student conversation starts"** In 3 dimensions, I can take 4 vectors (()) (43:01) Sir, but they would not be linearly independent, one will be dependent.

Yeah, minimal set, okay, the minimal is not really required, the linearly independent set, which is; which generates the entire space; when you entire space actually, the minimal set. **"Professor-student conversation ends".** So, when do you say that a space as finite dimension, the number of elements in the set is finite, okay; number of elements in the set is finite, it is called a finite dimensional space.

If the number of elements in the basis side with number of linearly independent vectors is the basis set is infinite, that is called infinite dimension space; we saw one example of infinite dimension space just now. Linear combination of sin, sin, sin, pi x, sin 2 pi x, sin 3 pi x, sin 4 pi x, infinity, okay, all these vectors actually form a basis; all these vectors actually form a basis and we can show that it generates the entire subspace, okay.

The dimension of that particular subspace is (()) (44:25) why? Because number of linearly independent vector; you cannot find the combination, which you cannot find the nonzero combination, which will give the sum of all those vectors equal to 0, linear combination of (()) (44:40), 0. So, that means the dimension of the space generated by sin pi x, sin 2pi x, sin 3pi x, sin 4pi x is infinity, okay.

So, we started with; till I mention this space, all that we are doing is just trying to know generalize ideas from 3 dimensions just remember that. I am just trying to generalize the ideas in 3 dimensions. So, visualization geometric ideas, you should be clear in 3 dimensions, you understand 3 dimension very well, no difficulty understanding all these infinite dimension spaces, okay.

So, just be clear about this 3 dimensional geometry and then that is enough, all visualization can be done in 3 dimensions, the concepts are just be generalised, for this grand generalizations, we will apply mathematics in a period, you know, 1850 to 1950, these 100-year period and with these grand generalizations, you can actually view, you can have a very different view of engineering (()) (46:00).

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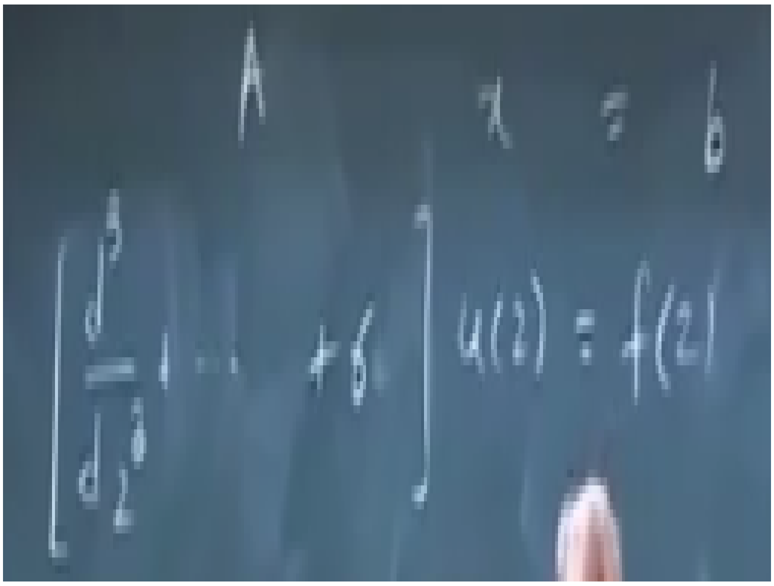

So, you will see that you are just working with the; if you start having this viewpoint then working with Ax = B finite dimensions or working with some boundary value problem, some differential equation is not different, operators change what we call is a vector change, what we call this B on the right hand side change. Actually, the problem that we talked about this problem of; d3/d z cube + something, something 6 * uz.

I put this = some fz, okay this is fundamentally not different from Ax = B, solving for Ax = B in 3 dimensional spaces is same as solving for this equation in thrice differentiable continuous functions (()) (47:02) fundamentally not different. This same equation, different space (()) (47:08) spaces is different, we are not doing something fundamentally, different when you solving this and solving okay. The specific solution which I wrote was f of z = 0 that is a specific solution.

And then to write; if you take some fz, whether you can solve this problem or not that they depend upon the span of; you know, the characteristic equation and so on. The same idea, which will apply here when you saw the x = b, the same idea will apply. So, we will continue in the next lecture; this is not enough; this is not enough, we are just defining the space; we define the span, linear dependence, and basis.

We need to go further let us say, now when in 3 dimensions, you know I have something more important I had length of vector. Now, how do I can generate the length of vector and why generalise the length of vector, that this how it because we will need to define something called conversions, limit and so on that is why we have to define structures like norm, we will talk about it in the next lecture.