Advanced Numerical Analysis
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Lecture – 28
Iterative Methods for Solving Linear Algebraic Equations: Convergence Analysis using Eigenvalues

We have been looking at conversions of iterative methods for solving linear algebraic equations.

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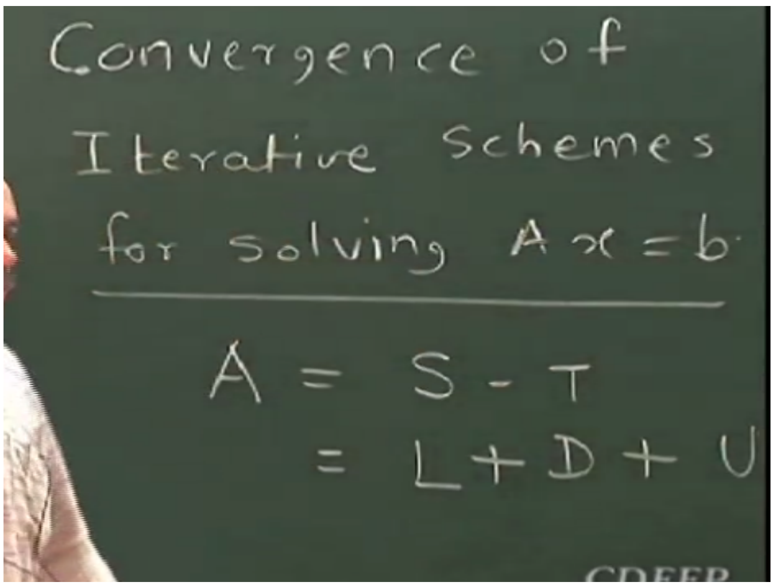

And we have made some progress. We have found out the difference equation that we need to analyze for solving the convergence problem. For solving the convergence problem we have looked at so I will just take a brief review of what we have done. So we have this matrix A which we write it as S - T. We split this matrix. Well we wrote this in two ways, one was L + D + U and then using this L, D, and U we formulated two matrices S and T.

And then we showed that the iteration scheme essentially any iteration scheme, Jacobi iteration scheme, Gauss-Seidel iteration scheme, relaxation iteration scheme.

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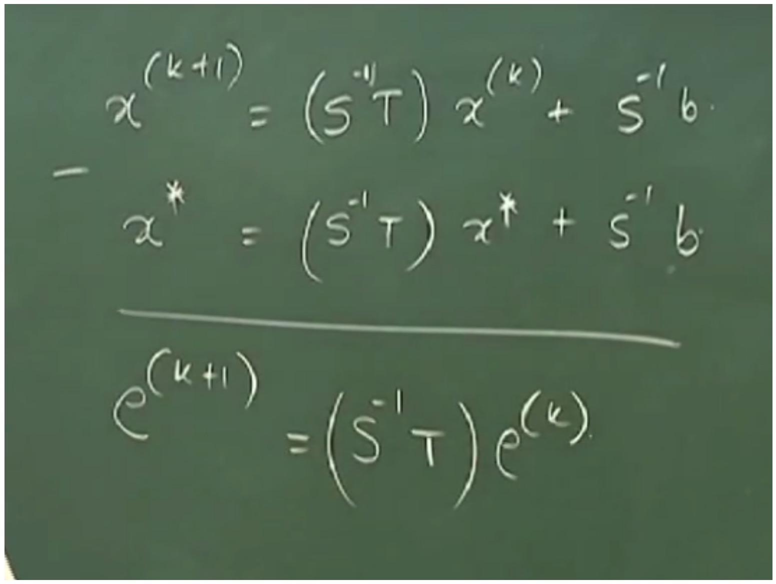

It can be written as x k + 1 = S inverse T x k + x S inverse b and finally we wanted to converge to the solution. Let us say x * = S inverse T x *. The solution is S * = S inverse t x *. This iteration scheme should converge to this fixed point. This is called fixed point of the equation because it when you substitute it on the right hand side you will get back the same vector. So x * is called a fixed point.

So we subtracted this and then we got this equation e error at point k + 1 is S inverse T error at k. We got this equation where error was defined as.

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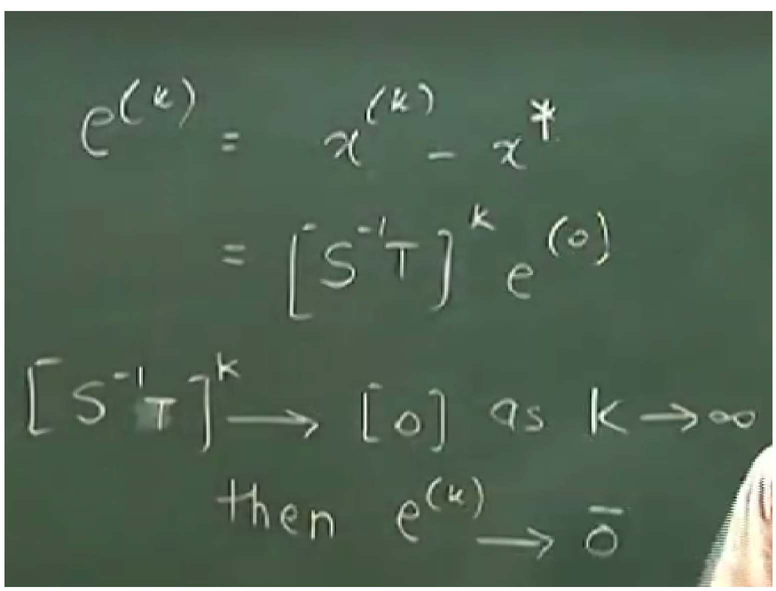

Error at iteration k was defined as x k - x *. This is the distance from the true solution. Well we do not know the true solution. We are going to reach it. Nevertheless, the error between the guess solution and the true solution is governed by this difference equation error at iteration K + 1 so this tells you how iterations progress. You start with some error and so we come up with the analysis that ek is nothing.

But S inverse T raise to k e at 0, error at 0, initial error and what we logically deduced was if S inverse T raise to k this goes to null matrix as k tends to infinity then if S inverse T has a property you say it is a nice matrix that when you multiply S inverse T with itself multiple times the product travels towards a null matrix. It converges to a null matrix and as k goes to infinity you get null matrix.

If that happens then error will go to zero, error going to zero which means the guess converging to the true solution. Error going to zero is a guess converging to the true solution. So now we need to analyze so seems to be at the heart of this equation. At the heart of this equation and I am going to analyze equations of this type. I am going to analyze equations of this type. so I am going to abstract it, I am not going to exactly keep working with S inverse T will apply results specific to this particular problem. I am going to look at a generic problem.

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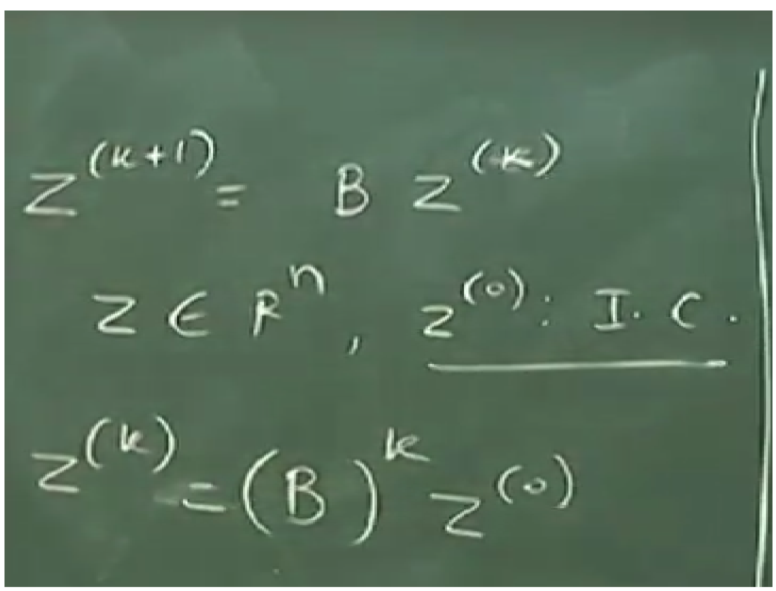

Say Z k + 1 = B zk. This is linear difference equation. Z belongs to R n and initial value is this is the initial condition. So this is my abstract problem for this I am going to generally in today's class I am going to analyze behavior of these kind of equations. This equation if you notice, this equation is a special example of this kind of equations. I want to analyze so what do I mean by I want to analyze? I want to come up with a judgment about the behaviour of this solution.

I want to come up with a judgment about how solution behaves asymptotically as k tends to infinity without requiring to compute. I do not want to actually compute the solution. If I want to compute the solution and then later realize that it is diverging, converging, I am not interested in that. I want to analyze without having to do explicit computations. Well you might say this equation you know I just I have done there if you start solving, if you know z 0 you can say that z at any point k is nothing but B to power k z o.

It is very easy to show that not a problem. Just apply it multiple times. This is the solution computational solution. If you go B matrix. If you know z 0 and this is how it will solve. In this particular case we do not know what is z 0 right here. If you map to this problem I do not know what is the initial error. If I knew initial error I knew the solution, then the problem of doing iterations would not arise.

So I want to analyze this without knowing z 0. I just want to relate this to properties of matrix B. If matrix B has certain nice properties, then I can claim that asymptotically this difference equation behaves to the particular way that is what I want to reach. So for the time being let us forget about these iterative methods. Let us concentrate on this equation and let us start insides into what happens, how do I analyze this?

This is a very, very fundamental equation, linear difference equation. It arises in many, many forms in computations. It arises in solving real problems, engineering problems and t is good to get inside into how this behaves. Let us first look at scalar equation.

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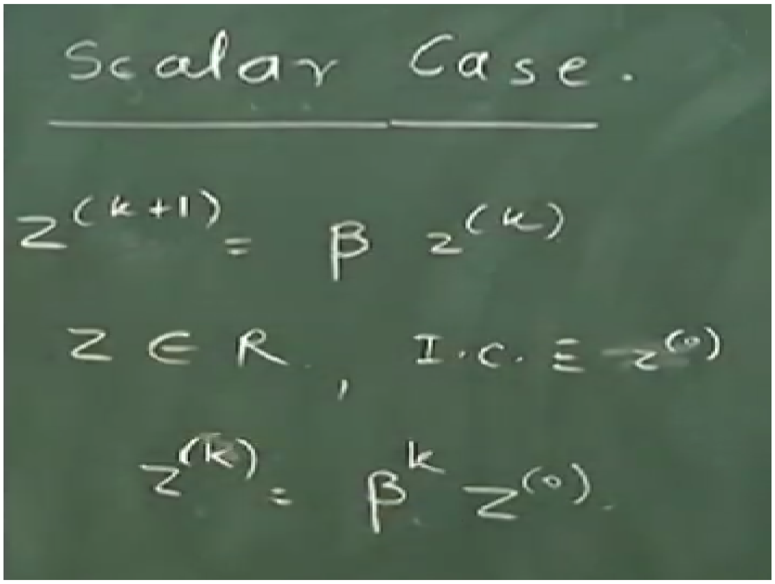

And let see whether the inside that we get from the scalar case how can we translate that inside into the vector matrix case. so my first analysis is z k + 1 = beta zk where z belongs to R, beta is a real number and this is my difference equation. I am staring from my initial condition corresponds to some z 0 well you can very easily show that zk = at iteration k this will be beta to power k z it is very, very easy to show repeated use of this equation.

You can get this is it not. I want to talk about how zk behaves as k goes to infinity. Will it depend upon beta? Or will it depend upon z 0. It will depend only on beta. Will it depend upon z 0 why I need support for that why should it depend only on so suppose z 0 is very large will it matters? What really matters? What is changing with k? Z0 is not changing with k. What changing with k is beta to power k.

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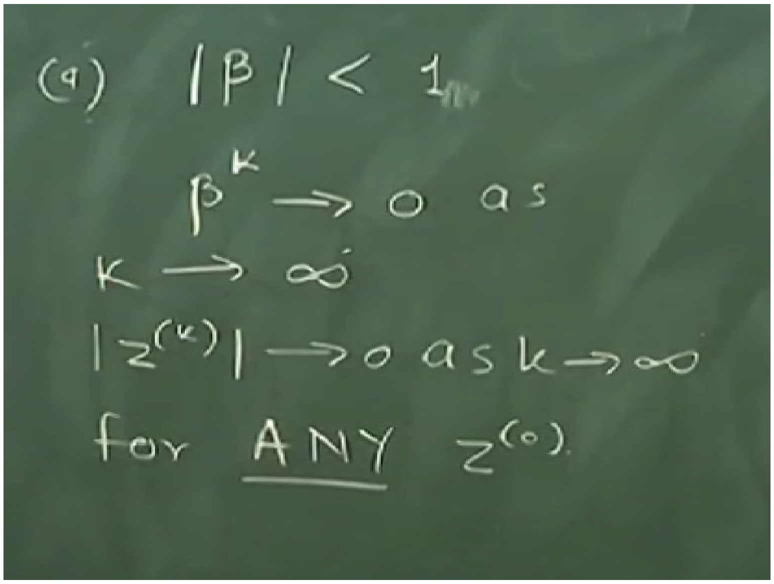

Well if I say that mod beta is strictly less than 1 suppose you have this additional information then what will happen is if so 1 k is mod beta is strictly < 1 in that case beta to power k goes to 0 as k goes to infinity. So what happens to mod zk tends to 0 as k goes to infinity. What is important is for this is very, very important. For any z0, z0 is 0.1, z0 is 1000, z0 is million, z0 is billion whatever.

This has to happen why because mod beta is < 1. Every time beta k shrinks and then it goes to zero for any z0. So I can look at this equation look at just beta and talk about how asymptotic behaviour is going to be.

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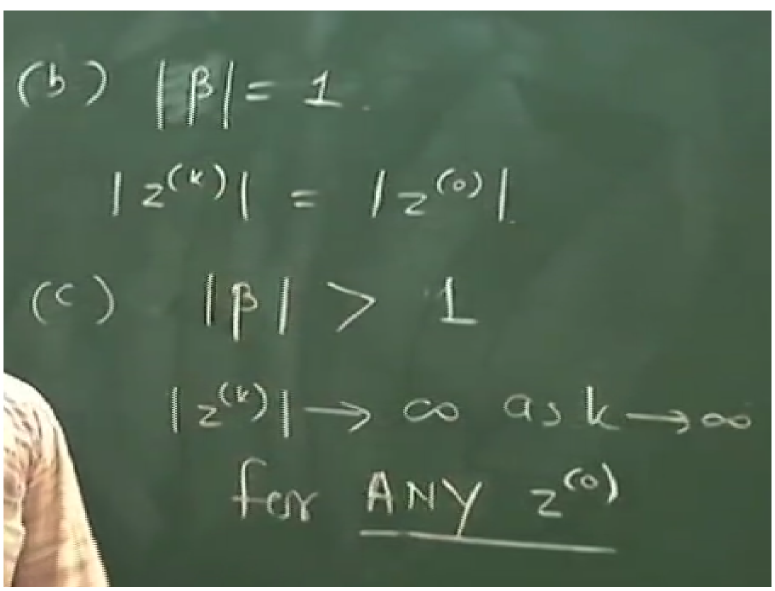

My second case is beta = 1 or mod beta = 1. So beta could be - 1, beta could be + 1. “Professor - student conversation starts” What will happen in this case? (()) (12:20) So at any point in time mod zk will be equal to mod z0. So zk will neither grow nor it will decay and my third case is mod beta is > 1 what happens here. (()) (12:58) It will go to infinity whatever is z0. That is another important thing that even if I start again this is very, very important. “Professor - student conversation ends”

For any z0 even if I start with z0 to be 10 to the power - 10. After sometime this equation will actually diverge. Zk will go to infinity even if you start with z0 which is a small number your equation is not going to converge to a small number just because you are starting from small number same thing is here. Even if you start from a very large number, if mod beta is < 1, zk is going to 0 after sometime.

If mod beta is > 1 even if it is slightly > 1 even if it is 1.0001 it might take longer time to go to infinity, but it will go to infinity. As k goes to infinity, beta to the power k will go to infinity when mod beta is > 1 and the solution will what we say is that solution will diverge. So this qualitative behaviour of the solution for this particular case I did not actually I need not solve the equation for a particular z0.

I can just look at beta and say that well if beta is like this, this is what is expected to happen. So that would be my basis. Now how do I extend this analysis to a matrix case. The tough part is that matrix b is a full matrix and then we have a trouble there. We cannot do straight forward extensions because what is matrix being less than 1 well we do not know. What is matrix being > 1 we do not have such notions.

Matrix is a array of numbers and then we have to work out some tricks to come to use this kind of analysis. So what I am going to show next is that in the attempt to solve this problem, to analyze this problem in the matrix case what pops out surprisingly or not so surprisingly is the Eigenvalue problem. In fact, you can see what is the origin of Eigenvalue problem when you start solving these linear difference equations.

So probably when it was taught to for the first time you wondered why you know some matrix * vector = scalar * vector. Why did somebody think of this? What is the logical way of arriving at such an equation? Where do you hit upon such equation? See for example we talked about positive definite matrices. So when you have first introduced positive definite matrix you do not know I mean why somebody call about positive definite matrix.

But when you understand the sufficient condition for optimality you see that these are naturally there is a need to define a matrix which are the special property which is x transpose a x is always > 0. Then you know the local point with the Hessian is positive definite then you will get a minimum and so on. So the same thing is here. You will see that what will pop out in the attempt to solve this problem.

You will get Eigenvalue problem. So now I am going to move to the vector case and I want to keep these ideas. These ideas are very nice. I want to use them to analyze my vector case.
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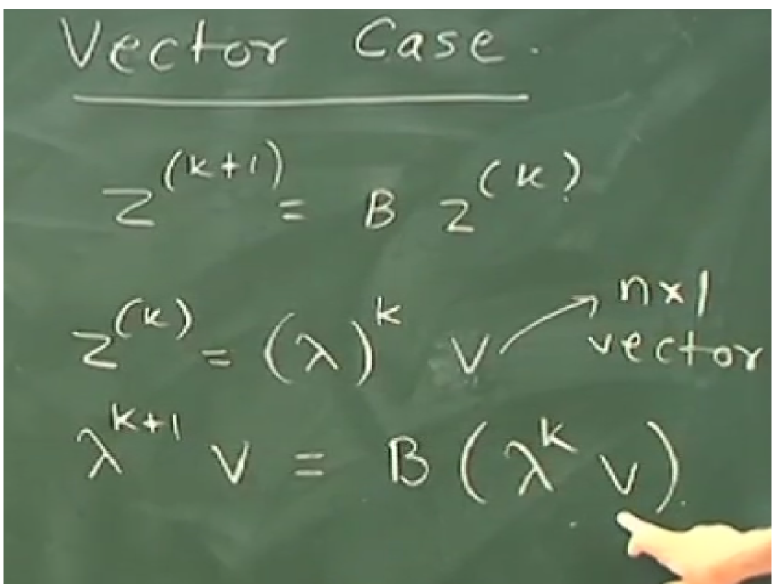

Let us start looking at a vector case. Here the solution in the scalar case was beta to power k * z0 some scalar. Let me take a motivation from my scalar case and I propose a solution for the vector case. So my problem now is z k + 1 = Bzk and I am proposing the solution z at kth iteration is some lambda, some scalar to power k I am just taking motivation from the scalar case I do not know what lambda is right now some scalar * some constant vector.

If you look at the scalar solution it was a scalar raise to k * some number that was z0 it turned out to be z0. Well we still have to worry about how z0 here will come into picture, but this is my guess solution. If this solution has to satisfy the linear difference equation if this solution is to satisfy linear difference equation, I will just substitute I will get lambda to power k -1 * v.

V is vector n * 1 vector, lambda is scalar and this is my solution at any iteration k. So lambda to power k - 1 should be the left hand side = b * lambda. You agree with me. See look at this. If this has to be a solution to this problem, then it should obey the difference equation that is the first criteria. It should obey the difference equation. This equation I am just going to rearrange and write.

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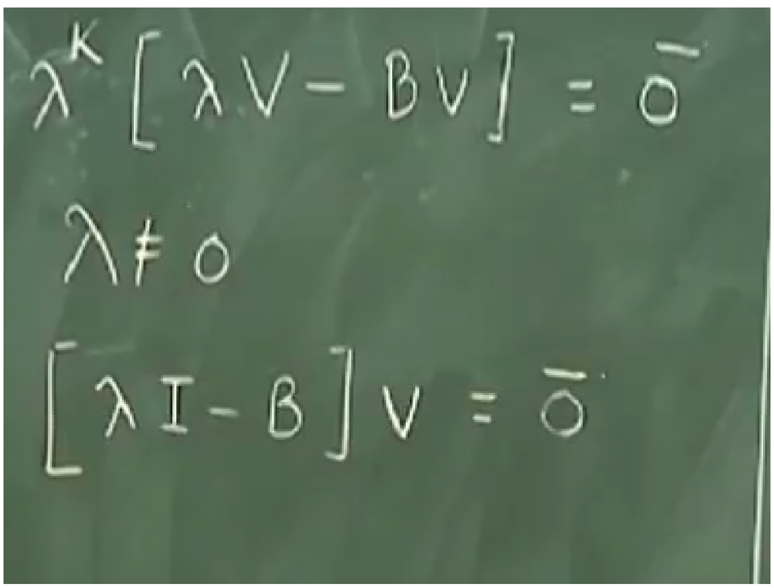

So I am writing this as lambda to power k, lambda v -Bv = 0. Lambda is a scalar. Lambda to power k * lambda will be lambda k + 1, the first term is covered. B * v, lambda is a scalar so I can take it on this side not an issue. So lambda to the power k I am taking it outside and. I am not interested in the trivial solution of lambda = 0 because then I will get 0 = 0. I am not interested in the trivial solution obviously.

So we rule out lambda = 0, next in general lambda is not = 0. Well there will be some situation when lambda will be = 0 but we will talk about it little later. Right now, we are interested in the nontrivial solution. In general, a general nontrivial solution will be obtained when this equation holds. A nontrivial solution to the problem I have just read it in this equation. Lambda cannot be = 0 it will give you a trivial solution.

I am interested in nontrivial solution of this particular problem. So this is lambda I - B * V = 0. Have you seen equations of this type where well not Eigenvalue problem I want to relate in some other context. Matrix * vector = 0 when does this happen. A = 0, Ax = 0. So when does this Ax = 0 has a nontrivial solution.

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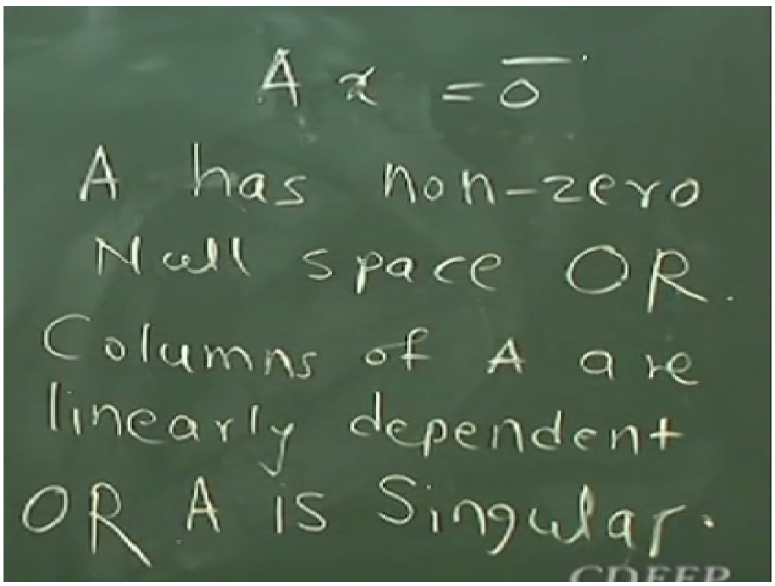

So you have seen equations of this type Ax = 0 when you get nontrivial solution when do you get a solution x is not = 0. (())(21:38) You get nontrivial solution for this when A has nonzero null space OR. What can you say about a rank of this A? it should be full rank. If it is full rank, then only solution you will get will be zero right so it should not be full rank. A should not be full rank. What should happen when you get nonzero solution?

Columns of A should be independent? Is everyone is getting this, everyone with me on this. If columns of A are linearly dependent what will happen, you will get a nontrivial solution x. If columns of A are linearly dependent then only you will get a nontrivial solution x, nontrivial means nonzero solution. You will get a nonzero solution for x = 0 only when columns of A are linearly dependent.

When the columns of A are linearly dependent what do we call the matrix A to be? Singular matrix. When A is singular matrix. Now just compare this equation with this equation.

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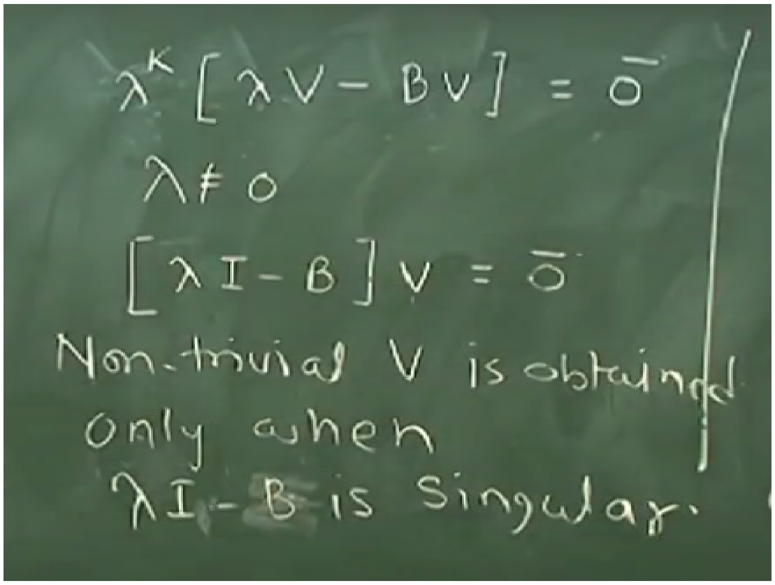

I want a nontrivial solution V, a nonzero solution V. If V = 0 then 0 = 0. I am not interested in that solution. I am interested in the nontrivial solution. So when will I get a nontrivial solution? What is singular? Lambda I - B only lambda I -B is singular then only I will get a V which is nontrivial which is nonzero and you will give me 0. Only when lambda I - B is singular, only then you will get a solution to this problem which is nontrivial.

You will get a vector V which is not = 0 vector. Only when this is singular. What is the algebraic condition for this matrix to be singular? Determinant = 0. That is the origin of your Eigenvalue problem.

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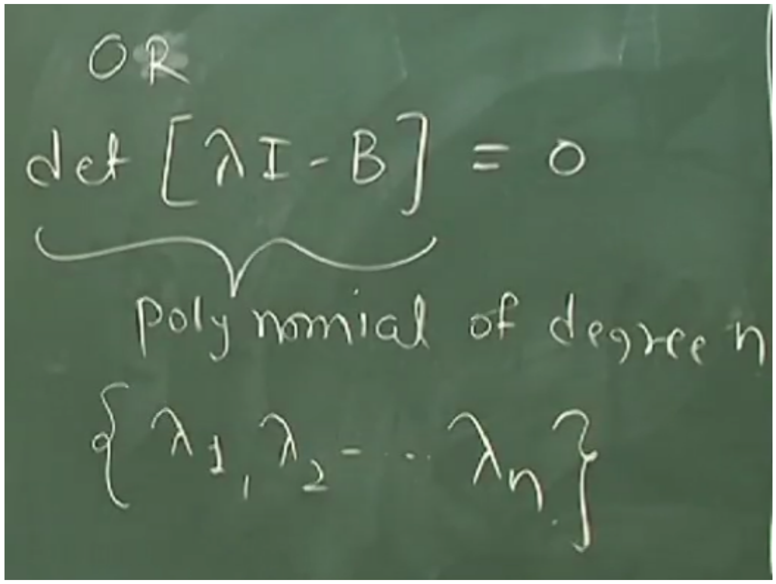

So I am just continuing from there or determinant of lambda I - B = 0. Normally when you start studying Eigenvalues and Eigenvectors we start with the point which is actually the end point. So normally your thought that Eigenvalues of a matrix B are defined as this. So why this? This comes out because you are trying to solve a linear difference equation and this equation pops out. If you want to get a nontrivial solution for the linear difference equation actually same thing happens if are trying to solve linear differential equations.

These equations pops out when you are trying to solve linear differential equations. Look at Strang's book equation, it gives a beautiful derivation for how it happens. So this equation is a fundamental equation it pops out and that is why we keep studying Eigenvalues Eigenvectors. Let us look at this equation little more detail. There is one more thing here. At this point how many unknowns are there? Another view point. The same equation.

How many unknowns are there in this equation? Three unknowns. V is a n cross 1 vector. You do not know V, you do not know lambda. So how many unknowns are there? B is a matrix which is known. Lambda is not known and elements of V are not none. We do not know what is V? How many equations I have? I have n equations. These are n equations, there are n + 1 unknowns to solve it exactly how many equations you need n + 1 equations and n + 1 unknowns.

Is this is a linear equation or a nonlinear equation. linear? Think what are the unknowns? Lambda and V. Does lambda multiply V is it a linear equation? My unknowns are V1, V2, V3, Vn and lambda. Lambda multiplies you have n nonlinear equations in n + 1 unknowns. Here n nonlinear equations in n + 1 unknowns and you have to solve it. To solve it somebody would have said I will use (()) (27:38) or something.

We have used very, very intelligent argument we said a solution will exist only when this matrix is singular. How many solutions we should get here? You will get a nonlinear equation in general has multiple solutions. It will become evident here that this nonlinear equation will have multiple solutions. So you have this additional equation now this one more equation those are n equations and n + 1 unknowns.

This is additional equation n + 1th equation, this and that together you can solve. What do you get here? Now this you know. When you put this you get a polynomial of degree n. So this is actually it is a polynomial of degree n and this has roots lambda 1, lambda 2, lambda 3 this has roots lambda 1, lambda 2, up to lambda n. It will have n roots some of them could be repeated, some of them could be whatever. We do not worry about it right now.

So it has n roots, it is the nth order polynomial. It will have n roots. Now corresponding to each root you get one V vector because this equation will hold for every lambda. If you pluck in lambda 1 here, you will get 1 matrix which is singular. Corresponding to that matrix there will be a null space.

Corresponding to that matrix is a null space and that V will belong to that null space. If you pluck in lambda 2 will be one more similar matrix and V2 will be in that null space and so on. So for every Eigenvalues this root of this equation well this is called a characteristic equation of matrix B and roots of this equation are called Eigenvalues all that you already know. Now so what happens is?

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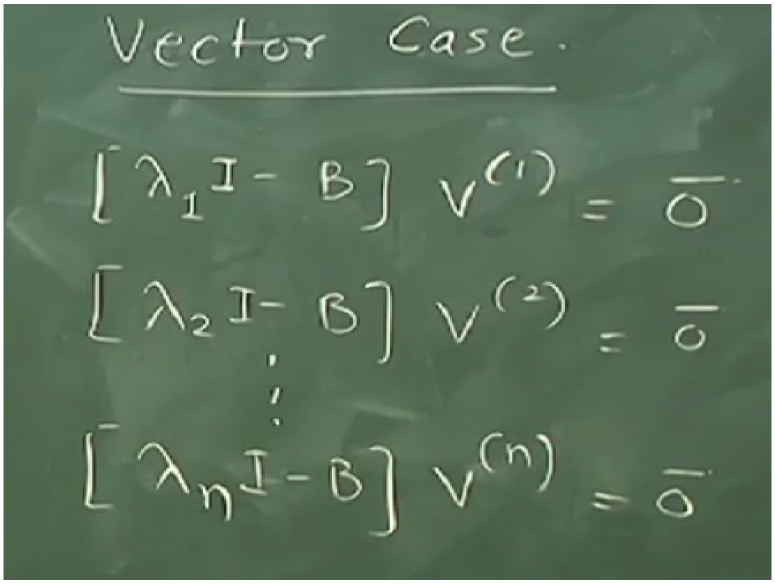

Lambda 1 I - B V1 = 0. So V1 is the first Eigenvector V we call this. This is actually a vector in the null space of lambda1 I - B. This is the singular matrix for this value of lambda. So there are n different numbers which make this lambda I - B singular. For each of those cases we have a vector V1, V2, V3. So I have this lambda 2 I - B V2 = 0 and so on. So I can write this n Eigenvector I get this n Eigenvectors.

Now what did I start with? I started with solving my linear difference equation and I took a guess solution.

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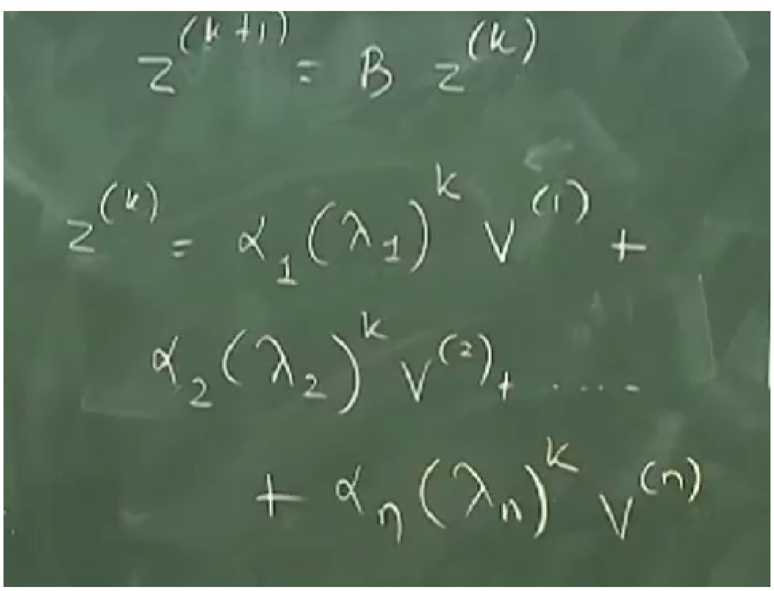

I started solving this z k + 1 = B zk. This is my linear difference equation. I wanted to solve it and then I said I have a guess solution zk = lambda to power k * vector V. It turns out that there are multiple such lambda. There are multiple such lambda and there are multiple such vectors V not just one vector. So I have multiple solutions I can call them as fundamental solutions or Eigen solutions so I have a solution which I would call as let me call this as solution number 1 which is lambda 1 to power k V1.

This also obeys the difference equation. Then let us call this z2k which is lambda 2 to power k V2. This will also obey the difference equation. What is the first fundamental criteria? The solution should obey the difference equation. So this will obey, this will obey. There seems to be n such solutions which obey the difference equation. In fact, this is the linear difference equation.

You can show that any linear combination of this Eigen solutions are fundamental solutions will also obey the difference equation. Any linear combination so I can construct a solution which is like this. I can construct a solution which is linear combination of fundamental solution. So I will let me use the same notation that is here. So linear difference equation you can very, very easily show that if each one of these fundamental solutions obeys the difference equation just pluck in and see that a linear combination of these also will obey the linear difference equation.

So the first criteria satisfied. What is the second thing that we need to do? There is one more thing that we need to do is to relate this solution to the initial condition because a solution is found for a particular initial condition z0. Z0 has not come into picture till now, but do you agree that this guess solution will exactly satisfy the linear difference equation. Next thing is to match it with the initial condition.

Now when I am writing an arbitrary linear combination this is the general solution what is unknown here? Alpha 1, lambda is known. Once B is known matrix B is known I can write I can get Eigenvalues lambda 1 to lambda n are known. Lambda 1 to lambda do not depend upon initial conditions z0 they only depend upon B. They only depend upon B. What about V1 to Vn. Will not you have a matrix? Eigen space is fixed. Eigen directions are fixed.

She might choose a different Eigenvector. She might choose a different Eigenvector, but they have same Eigen directions. Values 1 - 1 and - 1 1 are same Eigenvectors in the sense that same Eigen directions except. So Eigen directions are fixed. So I need to known alpha 1, alpha 2 to alpha n and what we will see is that alpha 1 to alpha n are determined by the initial condition z0.

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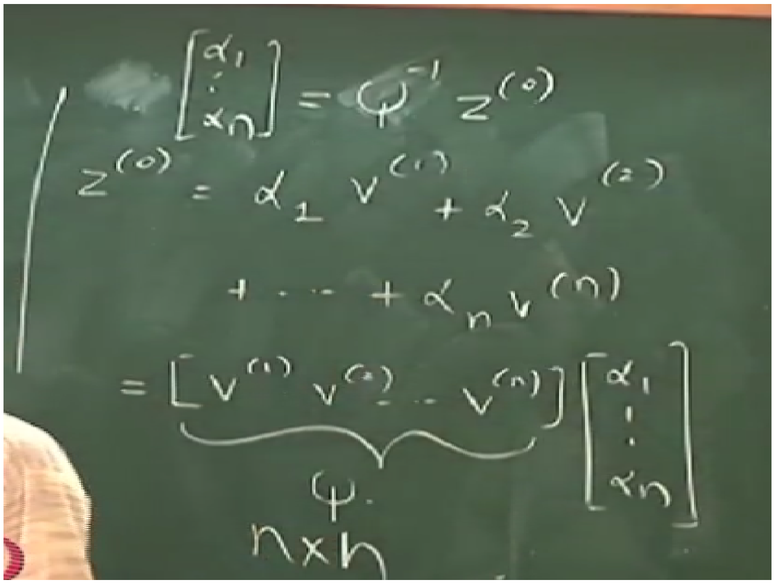

So this solution is if this has to be a solution, then z0 should be = alpha 1 V1 + alpha 2 V2, alpha n Vn. Everyone with me on this? I am just going to write this as V1 V2, Vn alpha 1 to alpha n. This is the first column. This is the second column. These are all vectors. These are all column vectors I am just putting them next to each other forming a matrix. Let us for the time being assume that Eigenvectors are linearly independent let us assume for the time being a simplified case.

So this matrix, this is a n * n matrix each one of them is a n * 1 vector, Eigenvector is n * 1 vector n of them are placed next to each other. This is a n * n matrix and this alpha 1 to alpha n so you have to solve you know z0 when you are solving the linear difference equation I know the initial condition I know z0. So it turns out that alpha 1 to alpha n are determined by the initial condition that is you have to solve this equation psi times = z0 or I could rewrite this as if Eigenvalues of or if Eigenvectors are linearly independent I can simply write psi inverse.

Is everyone with me on this? See what I wanted to say is that look we stared solving this equation then we came up with that guess solution. We found that there is not just one guess solution. There are n guess solutions. What are this n guess or fundamental solutions lambda 1 to power k V1, V1 is the first Eigenvector. Lambda 2 to power k * V2, V2 is the second Eigenvector and lambda 1, lambda 2, lambda 3 are Eigenvalues.

These are Eigenvalues and V1 to Vn are Eigenvectors. So this seems to be a fundamental property of the given matrix. So if I give you a matrix for any initial condition this will be the general solution. I need to find out any linear combination of this it is a linear difference equation you can very, very easily show that if this equation is obeyed or this fundamental equation is satisfied by the difference equation any linear combination of this fundamental equations is also satisfied by the difference equation.

So general solution is linear combination of fundamental solutions. General solution is linear combination of fundamental solutions. Now this general solution should match with the initial condition. Initial condition will give me these unknowns alpha 1, alpha 2, alpha 3 and so on. How do I get the unknowns? If Eigenvectors are linearly independent psi inverse z0. I have completely solved the problem. Now next question arises. Do I get insight by solving it this way? You would say well I could have solved this always as.

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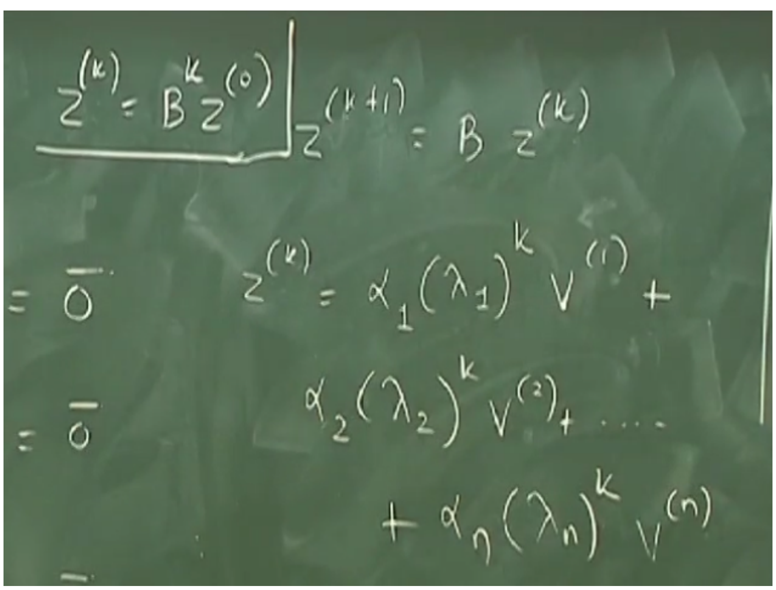

I know that this is the solution. What do I gain by writing it in such a complex way that is what I want to come to now that how does it relate to the analysis of this equation? Now let me see how do we translate the wisdom that we gain from scalar case to the vector case. This actually equation helps us in many ways than B to the power k because it gives you insight into what is happening internally.

First of all, if you look carefully here I have written the solution, a general solution of this problem as linear combination of some fixed vectors. Do you see this? I am saying any solution for any z 0 for any z0 the solution is given by this. So lambdas are fixed. (()) (40:58) you give me B matrix, lambda is get fixed. Eigenvalues get fixed. Eigen directions are fixed, so V1 to Vn is determined. So what is specific to initial condition is only alpha 1 alpha 2 how you combine the fundamental solutions is the only difference.

How do you create that linear combination is the only difference otherwise the solution always behaves according to this. Is that clear? There are fixed directions. Linear combination of fixed directions and then this lambda 1 to lambda n is fixed. It does not vary with z0. Only this multiplying constants change. Now let us start looking at what is changing with k? Lambda I to power k is the only term which is changing with k because once I have matrix B Eigen directions are fixed.

Once you give me initial condition alpha 1 to alpha n gets fixed by this equation because this depends only on the Eigenvectors. This depends on the Eigenvectors so alpha 1 to alpha n get fixed from the initial condition. V1 to Vn are Eigen directions only thing that is changing with k is lambda to power k. I need to now analyze how a scalar raise to power k behaves. I am very good at it. B to power k, B is a matrix. B is 100 cross 100 matrix difficult to analyze 100 cross 100 elements how they behave as a function of k.

Here I have reduced the problem to only analyzing n numbers. What are the n numbers? Eigenvalues. Now let us try to get inside into lambda I to power k.

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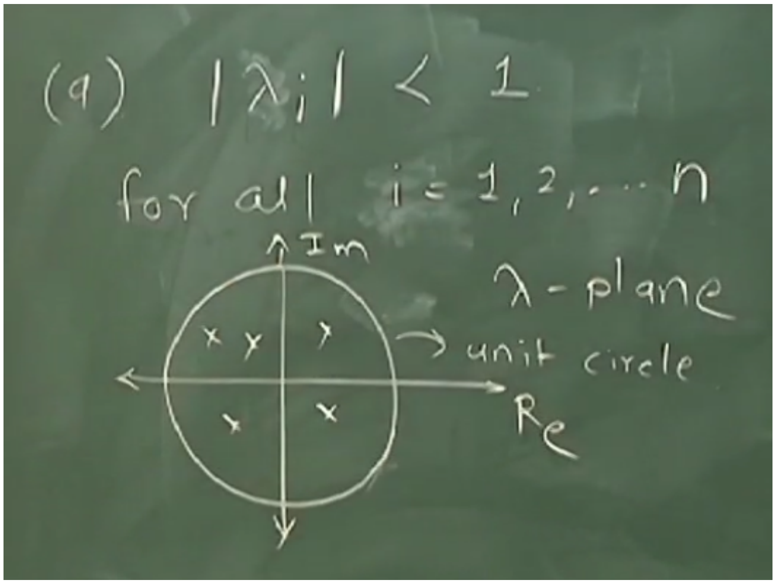

How does lambda I to power k behave? Can you tell me something? When lambda is < 1 so it - 10 also it will converge, - 10 is < 1. Modulus. So this seems to be important. Mod lambda I should be strictly < 1. If it is = 1 then you have trouble. So let us look at different cases. So my case 1 which is analogous to the scalar case is mod lambda I is strictly < 1 for all I = 1, 2, n. This is my first case. All Eigenvalues are such well there is one more additional things.

That crops in here which is different from the linear scalar case Eigenvalues they do not be always real. Eigenvalues can be complex. So actually when you are doing this analysis your space is not real numbers. They could be complex numbers. So in the complex plain, lambda plain what I am saying is that if I draw this unit circle this is my lambda plane and this is my unit circle this is imaginary, this is real this is imaginary axis, this is real axis.

What I am saying is that if all Eigenvalues of B are strictly inside the unit circle whatever is the matrix 100 cross 100, 10000 cross 10,000 or 5 cross 5. If all Eigenvalues are strictly inside the unit circle go back and look here. if all Eigenvalues are strictly less than inside the unit circle what will happen to mod zk? Norm zk now zk is a vector so norm zk. We need to look at now norm because what will happen to norm zk well you have to do the proper analysis.

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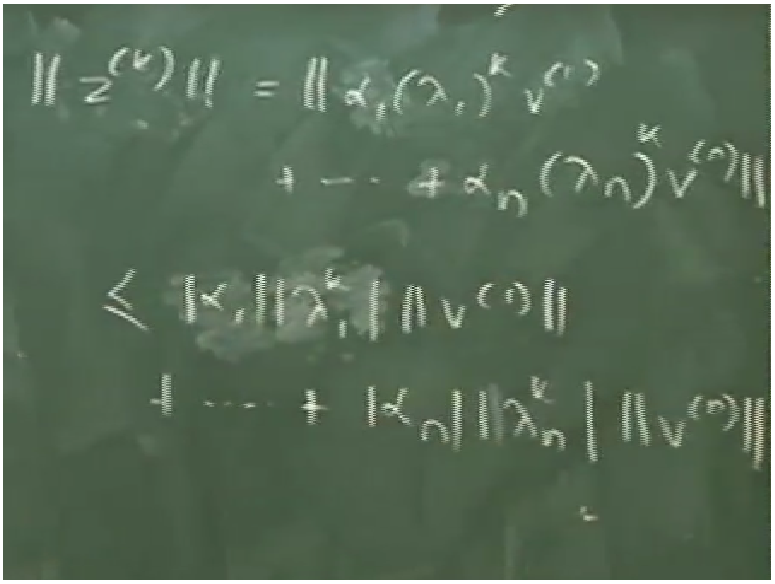

So norm zk will be norm alpha 1 lambda 1 to power V1 + alpha n lambda n to power k Vn which is < or = mod alpha 1 mod lambda 1 to power k norm V1 and these quantities this is not changing, this is not changing, what is going to change is lambda to power k if mod lambda I for all I is strictly less than 1. What will happen? The right hand side will shrink and this left hand side is less than the right hand side and this is of course > 0 right, norm is always > 0.

So if right hand side shrinks as k goes to infinity then the left hand side also will shrink so this will go to 0 because this will go to 0 because mod lambda I is. My next case so if all Eigenvalues are inside unit circle why unit circle Eigenvalues are complex numbers. If all Eigenvalues are inside there is one more step where it should show that this is < or = mod lambda I raise to k and so on. I have skipped one small step in between, but you can do that it is not difficult.

What if even if one Eigenvalue does not obey this. If you know there are 100 Eigenvalues 99 of them are inside to the circle one of them are on the unit circle what will happen if one Eigenvalue so the last Eigenvalue is on the unit circle what will happen? Other components all the other terms will shrink one term will not shrink. So this will not go to 0, this will be bounded it will neither grow after some time nor shrink it will become constant.

It is bounded above and it is bounded below because this is between 0 and some value.

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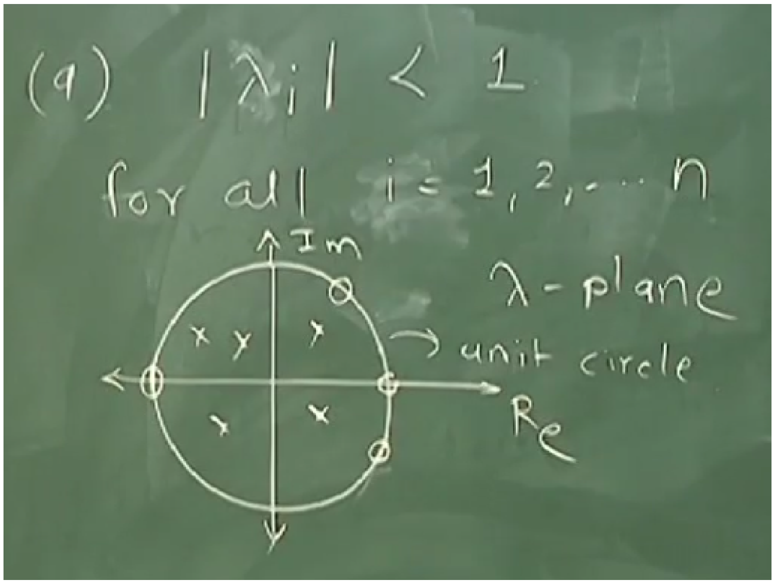

Even if one of them is on the unit circle so it does not help me if the Eigenvalue is lying on the unit circle somewhere. Even if one of the Eigenvalue is lying on the unit circle I have a trouble. The trouble in the sense well there is something that we have achieved her. Right now let us not worry about the convergence problem. What we have achieved is that just looking at the Eigenvalues I can tell you how the difference equation is going to behave.

I can talk about the qualitative behavior of the difference equation by knowing relative position of the Eigenvalue in the complex plane whether this is inside unit circle whether it is outside unit circle whether it is on the unit circle what if one Eigenvalue is outside the unit circle? effect of other Eigenvalues will go to zero but that one Eigenvalue will keep growing and zk will go to infinity (()) (51:01) See this fact for any initial condition it is what I want to emphasize.

See this analysis tells us how the solution will behave for any initial condition. So initial condition will only determine alpha 1 to alpha n. Even if one of them is on the unit circle then and remaining are inside the unit circle then the solution will be bounded as to infinity. It will not go to 0 and even if one of them even if one Eigenvalue is outside the unit circle I am guaranteed so I can just look at the Eigenvalues.

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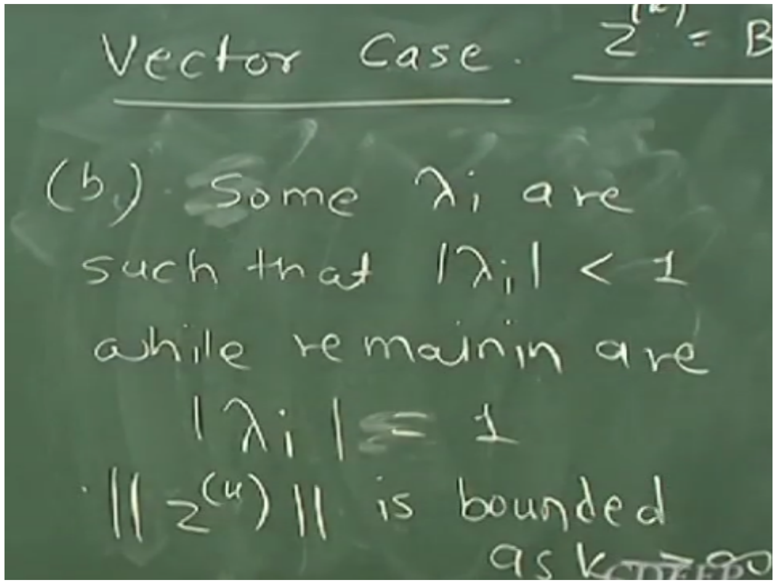

So my second case equivalently is that some Eigenvalues are strictly < 1 remaining are on the unit circle. All that I can say is that is bounded as k goes to infinity and my third case is obviously when.

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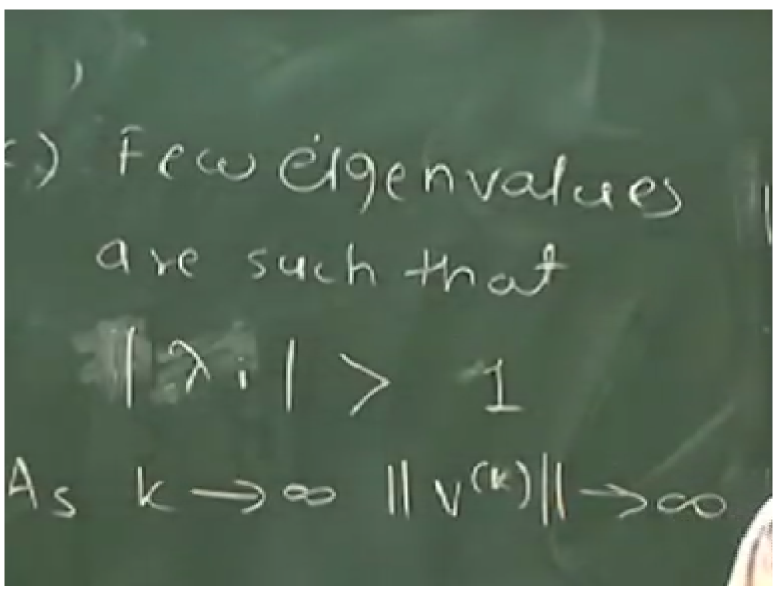

Some Eigenvalues are such that lambda I mod lambda I is strictly > 1. It could be 1.0001 does not matter. If it is outside the circle it is outside the circle. What we can guarantee is that as k goes to infinity norm vk will go to infinity. It does not matter few Eigenvalues are inside the circle. One Eigen outside the circle can make the solution go to infinity at sometime or the other as k goes to infinity.

So this will become unbounded as k becomes infinity. So this analysis we could do without requiring to solve for a given z0. I do not have to solve for z0. I just take matrix B, I look at Eigenvalues. When Eigenvalues have certain properties well I am done. I can tell whether solution is going to converge or to diverge so we started by looking at.

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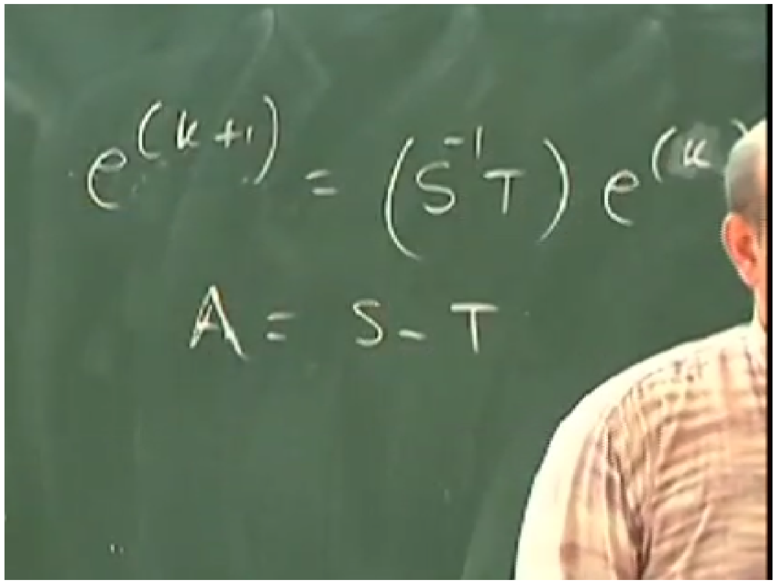

Ek + 1 = S inverse T ek. We started by analysis this difference equation. So what is the model of the story? Eigenvalues of S inverse T should be strictly inside unit circle. I should choose A = S - T. I should choose this split in such a way that Eigenvalues of S inverse T should be strictly inside the unit circle. If that happens I am guaranteed that convergence will occur. Wherever I stat from, I start from an absurd initial guess.

My iterations will converge to the solution. So this is the key. Now we will start developing from there because well you might say you are transferring the problem from one difficult problem to other difficult problem. If you have a 1000 cross 1000 matrix finding out Eigenvalues is also equally tough problem. Finding out Eigenvalue is not a joke you have to solve a polynomial of order 1000 well (()) (55:24) can do it but there is a limit.

The matrix size starts going, it also will hit into a it is not a trivial problem to get Eigenvalues. So we have a nice analysis but still we have problem because this still need lot of computations. Inside of the Eigenvalues you could have even solve for your z0 and tried to see how the solution is behaving. So is there some more something else? Some other properties I can use. So we will look at that in our next lecture.