Advanced Numerical Analysis
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Lecture - 21
Projection Theorem in a Hilbert Spaces (Contd.) and Approximation using Orthogonal Basis

Okay, so good morning, we are looking at projections and projections general projection theorem in Hilbert spaces. so in the last class I talked about distance of vector or a point from a subspace in a general Hilbert space so we wanted to project just like we projected, in 3 dimensions we projected a vector onto a 2 dimensional surface, in the same way we could project a vector in general Hilbert space which could be infinite dimensional onto a subspace which is finite dimensional okay.

So this we derived a formula or we derived equations for computing the coefficients of the least square approximation, and so we are looking at what is called as the normal equation, I just go through over at very quickly.

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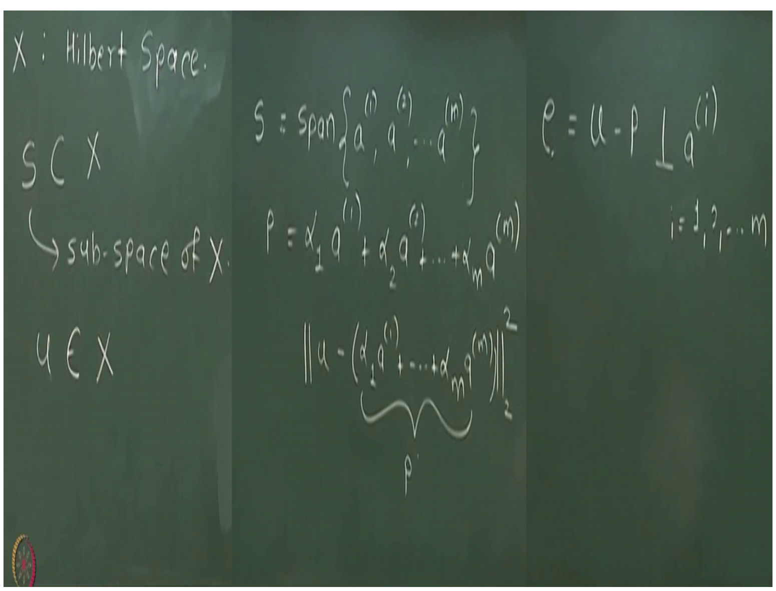

What we wanted was we have a Hilbert space, and we have these subspace S, S is a subspace of X, and then we are given a vector u that belongs to X, a vector u is given to you that belongs to X. Now X may or may not belong to these subspaces okay, I want to find out the point in the subspace which is at the shortest distance in the least square sense from u. So what I essentially want to do is, so let us say S=span of some basis vectors.

So these are the linearly independent basis vectors there are m basis vectors, and so this is a finite dimensional subspace S is a finite dimensional subspace, I want to find out so any vector that belong to this subspace will be of the form alpha 1 a 1 any vector that belongs to S will be a linear combination of these basis vectors okay. What I want to do is to find out alpha 1 to alpha m such that 2 norm of u- or let us say to norm square.

I want to minimize sum of the square of errors when it is finite dimensional, I want to minimize the integral I want to minimize the integral if it is infinite dimensional space, I want to minimize this integral over the domain this is 2 norm square that means inner product of this vector with itself, this is the error vector u is the original vector in X, this is the projection this P here is the projection, and then we want to find out a vector P that belongs to the subspace S which is at a minimum distance from.

So whatever I want to emphasize again and again it is not different from say this is the point okay, and this is my plane here okay, this is 2 dimensional subspace and this is a point the shortest distance of this point is obtained by dropping a perpendicular okay. So what classical projection theorem in Hilbert spaces says is that the way to go about the way to find the vector P which here is the shortest distance is to use the fact that this error vector e that is u-P this is perpendicular to ai, i going from 1 to m okay.

This error between the original vector and the projection, this error vector is orthogonal remember that orthogonality can be used only in the inner product space or Hilbert space, and that is why we work with 2 norm or least square approximation, so that is why this square approximation are so popular, because you can use orthogonality ideas okay. So this is the basic idea with this we derived what is called as normal equation.

Tis normal equation give as way to compute the least square estimate of alpha 1 to alpha m and that we constructed as follows.

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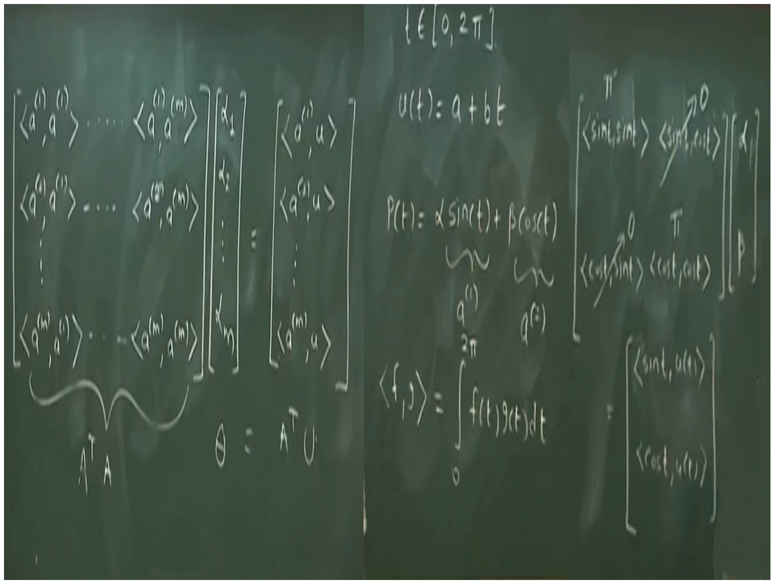

So we got this normal equation this was a 1 inner product of a 2 a 1, a 2 am likewise okay, so using the fact that the error is orthogonal to each of the basis vectors a 1 to a m we arrived at this equation, this equation is called as the normal equation in Hilbert spaces, this when I solve this these inner products these are the inner products here, once you compute this a 1 to a m, this basis vectors are known to you, you can compute their inner products okay.

So this will be a matrix, what is the dimension of this matrix this will be a m cross m matrix, there are m vectors this would be m cross m matrix, this came across a matrix is invertible in fact it is symmetric positive definite invertible matrix you can check that. And then you can solve for alpha 1 to alpha m, you can get a least square solution, what you will get here is the least square solution of alpha 1 to alpha m okay.

You will get the projection vector, once use this you get the projection vector onto this subspace S, which is at the least distance from u okay, so this is the best approximation of. Now I began in the working on this using finite dimensional spaces, I had derived a formula which said that a transpose a inverse theta, so this in the finite dimensional spaces this just go back and I will look at what we have done this will reduce to the same think will reduce to a transpose a.

This will reduce to theta, this will reduce to a transpose u, in the finite dimensions this equation will reduce to okay, the matrix equation which have derived earlier. So I am just extending the idea from finite dimensions to a general Hilbert space, here I could work with any of the. See for example I began my lecture last time by saying I have this vector u t which is a+b t and t belongs 0 to 2 pi okay, I want to find out an approximation P t which is alpha sin t+ beta cos t.

I want to find out approximations alpha sin t+ beta cos t okay, let us say we can so for the time being let us take only 2 vectors, what are the 2 vectors what is a1 a2 here this is my a1 vector this is my a2 vector okay. How do you find out alpha and beta, so what I have to do here to find out alpha beta, how do I set up the problem to find out least square estimates? So I have to find out sin t sin t sin t cos t, what is sin t cos t over 0 to 2 pi, can you do this integrals?

What is the inner product defined here? How is the inner product defined? Inner product between some f and g is integral 0 to 2 pi f t g t inner product is defined here like this okay, inner product is defined here like this, so now we know that over 0 to 2 pi sin and cos are orthogonal, so this is 0, this is 0 okay. If you notice what we are actually deriving are the first 2 coefficients of the Fourier series, I am doing Fourier series expansion of u t in terms of sin and cos okay.

The nice property of sin and cos over 0 to 2 pi with this as my inner product is that they are orthogonal, so this is 0, this is 0 I get a diagonal matrix, solving for this level matrix is very, very easy it is not so difficult to solve for, so what is integral sin t sin t it is pi inner product of sin t u t and inner product of cos t u t, so this is=pi this is=pi this is 0 this is 0 okay, the least square estimate the best estimate of in the least squares sense in terms of sin and cos of this function u t, u t in this particular case I have taken a+b t.

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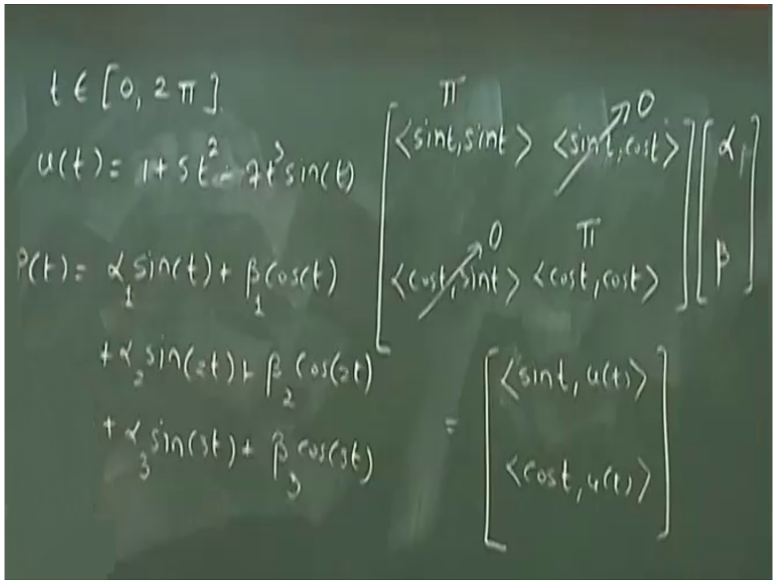

It could be it need not be you know this kind of function, I can take for example u t to be any other complex functions, I could take this as you know 1+5 t square-7 t cube sin t let us say this is my function does not matter, this is the function okay in the space of in the set of continuous functions over 0 to 2 pi okay. I want to find out best approximation to this function using sin and cos okay, I can go on if here I have included only 2 basis vectors.

Well, somebody might say just using 2 is not sufficient you want to use 4 okay, in which case I would use sin t cos t sin 2 t cos 2 t and so on okay, so the approximation could become more and more complex if you want, so I can have alpha 1 beta 1+ alpha 2 sin 2 t+ beta 2 cos 2 t+ alpha 3 sin 3 t+ beta 3 cos 3 t, let us say I want to develop approximation like this to this problem, how will you do it? We just methodically apply this formula okay.

Now there are 6 vectors the subspace, what is the subspace spanned? Subspace spanned by 6 vectors sin t sin 2 t sin 3 t, cos t cos 2 t cos 3 t we want to find out best approximations of the given function okay best in the least square sense okay, in the subspace spanned by this 6 vectors. What I do is, I find out these inner products okay I validate these inner products between 6 vectors, this gives me this matrix on the right hand side.

I have to compute inner product of the given function with each of the vectors okay, and when this vectors are orthogonal actually what we recover is the Fourier series expansion the Fourier coefficients okay, when these are orthogonal you know only the diagonal elements of this matrix will be non 0 okay only diagonal elements will be non 0, all the of diagonal elements will be 0 because orthogonal vectors inner products are 0.

If I take an orthogonal set, why do we actually want to approximate something using an orthogonal set? That is because the projection problem approximation problem gets very, very simple okay, also there are some other advantages like actually what we are doing here is when you do these orthogonal projections you are expressing a vector in terms of its orthogonal components, why do we work with x i y j+z k in 3 dimensions.

Because x is a component along unit vector in the X direction, y is the component along okay, they are orthogonal they are not related to each other okay. The same way what we are doing here is that we are expressing a vector in terms of a basis okay, we are projecting onto orthogonal basis, now the complete vector you can write if you take all when you see if I take a vector x which is given by 3 coordinates x, y, z okay.

When will I get the entire vector if I take all the 3 components together, in the infinite dimensional space when will I get entire vector if I take all the basis vectors and take you know projections along each one of them okay if I take projections along each one of them, then I will get the Furious series I will just write down this whole thing it will become more clear, but there is one more thing I just want to tell you.

See this vector right now I had chosen them to be orthogonal I have not chosen them to be orthonormal, if I choose them orthonormal what will happen suppose this vectors are orthogonal, what is the property of orthonormality? Inner product is =1 right, and inner product between a i a j is 0 and with itself is=1. What will happen to this matrix, if I take orthonormal vectors this will be identity matrix.

If I take orthonormal vectors to find out the projection all that I have to do okay is to okay.

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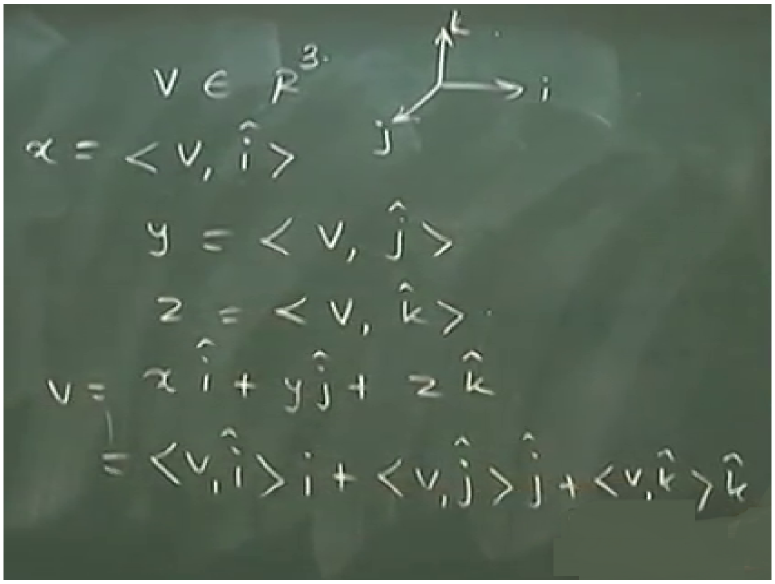

So the concept is not different from I will just leave this and come back there. So the concept is not different from see if I have this 3 dimensional vector, if I have 3 dimensional vector V belongs to R3 okay. How do you find out the component of V along x-axis, I take inner product of V with i, i is the direction, so I am talking here about this is my i, j, k orthogonal basis for 3 dimensional space okay.

If I take inner product of V with i, what will I get? I will get x component right I get x component inner product of V y component is inner product of V with z, and z component is inner product of V with k right. How do we write this vector? How do we express this vector? We express this vector as V=x i+y j+z k, where i, j, k let us put this cap here let us say they are unit vectors i cap, j cap and k cap are unit vectors okay.

But this this writing is same as inner product V i+ inner product V j, I could as well write this like this right, this and this is same you agree with me okay.

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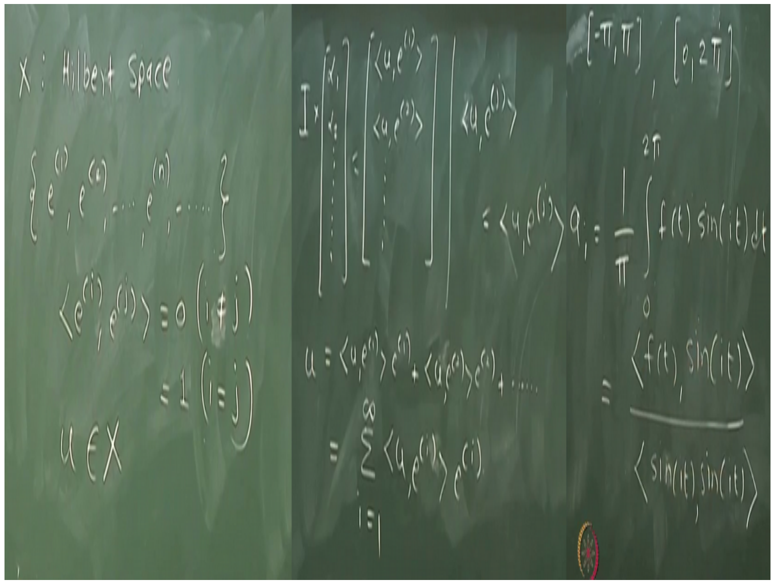

Here, what we can do is that in a general inner product space, if you are given a basis now this X is my inner product space or Hilbert space okay it is a Hilbert space and I give you a basis, so this Hilbert space could be any of the Hilbert spaces that we have looked at okay. So not be just finite dimensions it could be any of the infinite dimensions Hilbert spaces that we have looked at. And then I give you a set of orthonormal basis in this.

See for example I can create if it is 0 to 1 you know I can create shifted Legendre polynomials with which are orthonormal okay, I can create a basis which is shifted Legendre polynomials which is orthonormal and you can use that to you know define the subspace. So in general you are given the basis set you are given a basis, I will call this as e1, e2, en this could be a finite basis, this could be an infinite basis depending upon what kind of inner product space you are looking at okay.

If you are looking at set of continuous functions twice integrable continuous functions then this will be infinite basis okay, so I may have finite number I may have infinite number of elements in my basis set, these are orthonormal vectors. What is the meaning of orthonormal vectors? Orthonormal vectors which means e i e j=0, if i !=j, and this is=1 if i=j if i=j then this is =1, these are orthonormal vectors their magnitude is 1 and they are orthogonal to each other okay.

So this is the orthogonal basis, which orthogonal basis is same as i j k that we considered in 3 dimensions, why do we like i j k in 3 dimensions? It is orthogonal basis it is very nice okay. I can express any vector in terms of components along each direction okay, the same thing is true about any inner product space. Now how do I express an arbitrary vector u that belongs to the inner product space? How do I express this vector in terms of this basis?

How do I get those components? I can use the I want to project, what I want to do? I want to project this vector onto the space spanned by all possible linear combinations of this okay, let us say there is an infinite set okay, so if you start writing the normal equation, what will be the matrix? Matrix will be I because you know because of orthonormality this is 1 if it is diagonal elements will be 1, of diagonal elements will be 0 okay.

So by virtue of this okay you will get the coefficients, if you start writing the normal equation you will get coefficients I*this vector say alpha 1 alpha 2 and so on this is=inner the product of u with e 1, inner product of u with e 2 and so on, because this is I on the left hand side you have orthonormal vectors okay, the inner products are between i !=j or 0, inner product of with itself is=1, so the left hand side is all I, and I which is of infinite dimension let us say if there are infinite vectors I of infinite dimension okay.

So these are the coefficients how did we write the vector here, we wrote the vector V as V component of V along i times i, times in the sense this is the direction i+ component of V along j times j and so on okay. Same thing I can do here in the inner product space, I can write the vector u as inner product of u, e 1 e 1+ inner product of u, e 2 e 2+ is everyone with me on this this is clear? I am just writing this vector u now I am not approximating when I take the basis.

When I take this e 1 e 2 e 3 e 4 e n or e infinity whatever it is when I take this as a basis okay, I am not approximating it is equality okay, if you have infinite basis you will get infinite sum here, so this is=sigma i=1 to infinity u, e i e i okay, actually what I have written here is generalized Fourier series expansion of any vector u in terms of orthonormal basis in fact okay.

So actually when you write this just remember this when you are writing this you are actually writing Fourier expansion of V you are writing Fourier expansion of V okay in terms of orthonormal basis that is what you are doing. I am just using the same idea in any other space, probably when you studied Fourier series for the first time in your second year of engineering? You start wondering why I mean somebody comes and says take this function and it write as sin and cos, what do I gain out of it okay.

Actually, what you are doing is the same thing as writing x i+ y j+ z k what you do in 3 dimensions, it is nothing different, same idea extended to any other Hilbert space any other general face. What is nice about orthogonal basis? You know you can look at individual components separately okay, you can look at into your individual components separately, it is very, very easy to work with orthogonal vectors.

We know that in 3 dimensions that is what we want in any other 3 dimensional vector space that only we want to do it any other vector space which is a Hilbert space, it is possible only when you have inner product. You have definition of you know angle generalized, otherwise it is not possible to do this that is why Hilbert space that is why least square approximation are so important know, in engineering or most of the applications.

Why we look at least squares? Why we use 2 norm all the time? Why not 1 norm why not infinite norm okay, because 2 norms comes tied up with angle orthogonality, you know everything that is nice in 3 dimensions okay. So this is generalized Fourier expansion if I give you a general function say or specific example of this would be the classical Fourier series which you studied in second year of engineering is for the space X for this in a set of continuous functions over either -pi to pi.

Or we study over 0 to 2 pi right square integrable functions over -pi to pi or 0 to 2 pi, this is where you look at this expansions, and then we are given this basis vectors unfortunately sin cos are not orthonormal they are orthogonal okay. And if you remember you get when you are thought this Fourier series you get this you know ai=1/pi integral remember this formula ai=1/pi, then 0 to 2 pi f t sin i t dt something like this right.

Where does this pi come from? This pi comes, why does this pi come? I mean I used to have this problem when I studied this why suddenly put pi there, first of all remember that this is nothing but inner product of f t sin i t/inner product of sin i t sin i t okay, this is the normalizing factor 1/pi which comes because sin i t is not an orthonormal vector. I want to get an orthonormal direction, so this is what it is okay.

In this Fourier expansion just look at, how will you find the coefficients? See this is my Fourier expansion okay if I take inner product of u if I take u, e j, what will I get? u, e 1 inner product of u, e 1=0, u, e 2=0, u, e 3=0 right, what will I get? You will get u, e j back because inner product of e j with e j is 1 right, and inner product of e j with e i i !=j is 0 okay, so all the terms in this series will cancel only one time will remain that is projection along e j okay.

So this is very, very nicely ties up with your x i+ y j+ z k, which you know from 3 dimensions just remember that Fourier series is nothing but extending this idea into a okay. So far nice we have little bit diverted because we are not going to use Fourier series in this course, but Fourier series will be useful in the mathematical method course, you will be looking at all kinds of Fourier series you know in terms of Bessel's function.

And in terms of some other familiar names will appear, but now they should fall in a place you know you should be able to see them in light of this general development of what is projection onto orthogonal basis. **"Professor - student conversation starts"** yeah, (()) (33:01) not subspace in R3, 3 dimensional subspace in R4 or 3 would be, (()) (33:08) perpendicular will lie in R4, perpendicular will lie outside R3 okay.

So actually you are splitting, you are splitting the vector into 2 components 1 along R3, 1 orthogonal to R3, which will of course lie in the general space. **"Professor - student conversation ends."** So this Fourier series is just a side note, it is important here of course but it is more important when you develop analytical solutions okay. So before we move on I know I still not coming to applications to boundary value problems or PD.

Before I move on I want to explain something which is very, very important, well I have been telling you that you know using polynomial approximation okay or polynomial interpolation high-dimensional binomial interpretation is the problem, same thing is to actually polynomial approximation, if you start developing a polynomial approximation which is even if you have large data set, if you have a polynomial approximation of this form.

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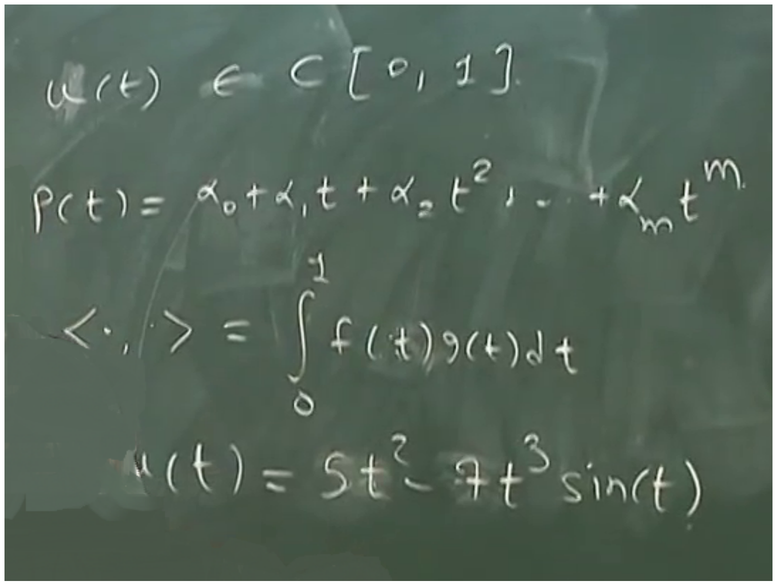

I have a function f t in which belongs to set of continuous functions over 0 to 1 okay, and then I want to write I want to develop an approximation P t which is you know alpha 0+ alpha 1 t+ alpha 2 t square+ alpha m t to the power m, I want to develop this approximation okay. So how do I go about doing this? What are the basis vectors now? 1 t t square up to t to the power m okay, I should set up the normal equation.

How should be the normal say here my inner product is defined as 0 to 1 f t my inner product is defined like this okay, well there are 2 ways of looking at this problem, let us say I know the continuous function over the entire domain, then I can approximate. The other problems that we looked at earlier is you know if you know this function at the finite number of points okay, if you know this function at finite number of points, let us say n points.

Then it is the we will take this not f t this is u t, if I know u at finite number of points, then I found that formula write a transpose an inverse that least square approximation formula which is which can be used to okay. This is knowing the function at every point, if I know f t let us say f t is some function like t2 +5 sin t or whatever okay, so this is the function which I know over the entire domain everywhere and I want to do an approximation.

Well, so just to give you an insight, why polynomial approximation are ill-conditioned? I kept on telling you that you know polynomial approximation are ill-conditioned higher order polynomial approximation that is why you have to do in orthogonal collocation, we do piecewise polynomial approximation we have this spline functions, so spline is fitting a low order polynomial by dividing the entire region into smaller segments okay.

Why we do all these business? So that is why now I want to explain you through this okay, so the question is now here a classic problem, I want to develop a mth order polynomial approximation. Now what I know from Weierstrass theorem is that I can develop an approximation arbitrarily close to the function, see this only tells you is there exists a polynomial which is it does not tell you how to reach that polynomial, how to find is a different story okay.

Weierstrass theorem only gives you existence, now when it comes to actual computing there can be in trouble, unless you do some smart tricks okay, so what are the smart tricks? We will come to that okay. So let us first look at this problem mth m is let us a large I want to have some 10th order polynomial fitted here okay, and my u t is let us take some u t which is u t=5 t square-7 t cube sin t, this is the u t I want to develop a 10th order polynomial approximation of this function in the least square sense.

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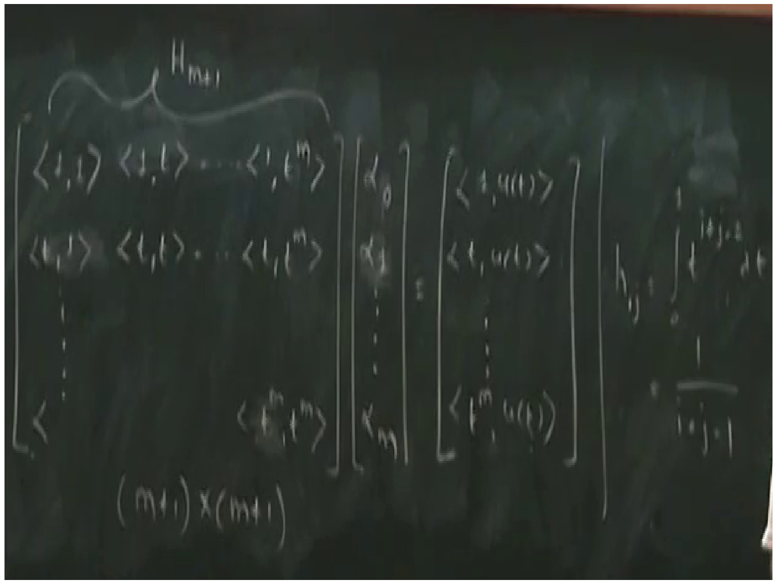

What is the meaning of least squares sense? That means the 2 norm of difference between u t and P t should be smallest okay, we know that this can be found by using normal equation. So we go about you know finding out this inner products, so my inner product is 1 with 1, then 1 with t, 1 with t to power m right, I am constructing the normal equation okay. I want to estimate least square estimates of alpha 1 to alpha m okay.

Then sorry t with 1, t with t and t with t to power m and so on, so this times alpha 1 alpha 2 alpha m, and then how do I get the right hand side? Inner product of 1 with my u t, inner product of t with u t and so on inner product of t to power m with u t. And you know classical projection theorem tells us that if you solve this you will get alpha 1 to alpha m okay. Now let me tell you what is this integral, but these integrals may not be that difficult to evaluate right.

Because you know binomial integrals 0 to 1 okay, you can actually write a general formula for this, so if this matrix in my notes I have call this as H matrix, this is m+1 cross m+1 matrix, there are alpha 0 alpha 1 right. We start with alpha 0 alpha 1 not alpha 1 alpha 2, so this is the matrix which is this matrix is m+1 cross m+1 okay, what is the element of this? So ijth element of this is given by 0 to 1, t to power i+j-2 dt this is=1/i+j-1, well why I have called this h will become clear soon.

Because this what you get here is a famous matrix called Hilbert matrix okay, you can show that elements of this matrix, see these are inner products of t square, t cube, t to the power 4, t to the power 5, so I was just given a formula for general ijth element of this we can very easily verify this, you get this okay.

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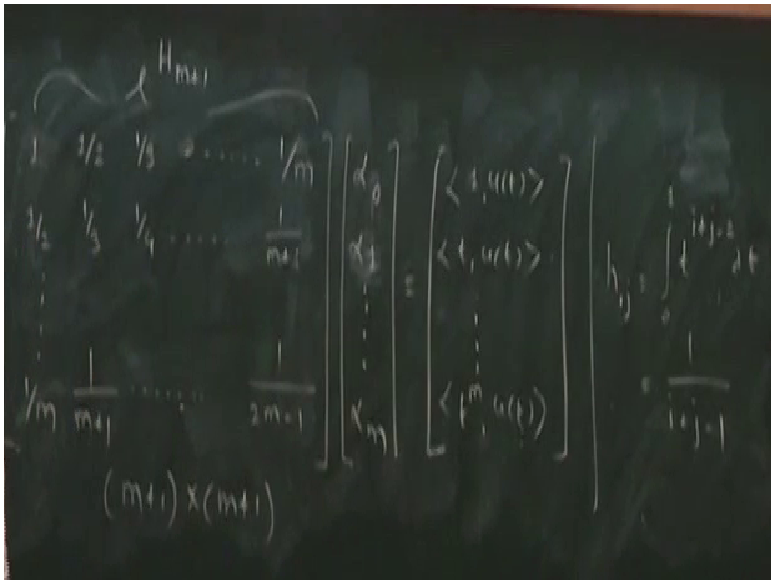

Now if I actually compute this, and then fill in the matrix okay, the matrix that I get here is like this, it is a very, very nice looking matrix 1/2 1/3 1/m, 1/2 1/3 1/4 1/m+ 1, 1/m 1/2m-1, this matrix is very, very difficult to invert it is highly ill-conditioned matrix. Now yet we have to define what is ill-conditioned matrix, you have to wait a little bit for defining ill-conditioning formally okay. But what you get here is a matrix okay just to preempt what it means is that.

This is basically first of all remember that this is symmetric matrix okay this is symmetric matrix, not only that it is positive definite matrix okay, this follows from the fact that this is actually you know what we have obtained in the case of projection, this is the projection matrix, this is obtained by you know projection matrix is obtained from projection matrix related to the projection matrix. So this is actually a symmetric positive definite matrix.

But it is highly ill-conditioned, because the eigenvalues of this matrix are very, very strange, the ratio of the highest eigenvalue to the smallest eigenvalue okay which is what for a symmetric positive definite matrix, which is what will define ill-conditioning, we will see this later systematically. It is so large that computations become impossible very, very difficult okay, so Hilbert matrix more than 4 or 5 m that is third or fourth order polynomial becomes difficult to invert okay.

If you ask mat lab to invert Hilbert matrix, it will give you some junk and say that do not believe the results okay, we will tell you that this matrix is ill-conditioned it will say it in a nice way not it will not tell you do not believe the result, it will say that the solution may not be reliable okay this matrix is highly ill-conditioned it will give you a number or conditioning is equal to something, when you do not know what when you are not done all this theory.

You just send ignore mat lab is giving something very, very important message that your results could be completely unreliable. Now to get alpha 1 to alpha m I need to invert this matrix okay, but if this matrix is highly ill-conditioned you know the inverse is unreliable, if inverse is unreliable alpha 0 to alpha m calculated are unreliable okay, and then you are fitting you know wrong polynomial, not because you know your formulation is wrong, but you just cannot compute properly know.

There is no way of computing, you are stuck because this is an ill-conditioned problem okay, this is an ill-conditioned problem. And but well if you are smart enough and done this course and still remember things that I have thought you, you will say well that is not the way to go, I do not want an ill-conditioned matrix here okay. So if I want to do polynomial approximation, what I will do instead is instead of using this raw polynomial like this I would choose to use orthogonal and orthonormal polynomial okay.

On 0 to 1, what is the orthonormal polynomial that we have constructed earlier? Shifted Legendre polynomials okay, so instead of developing this approximation. See this if you want an equivalent of 3 dimensions what you are doing here is you are trying to express a vector in terms of vectors which are not orthonormal or orthogonal. See a given vector if I give you one vector let us say this one okay.

I can choose to express this in terms of 3 orthogonal components or basis need not be always orthogonal, basis can be any 3 linearly independent vectors okay, so see the basis need not be like this, even basis is like this 3 linearly independent vectors okay, but in the 3 linearly independent vectors or something like this which are very close to each other we may have trouble expressing this any vector in terms of these 3 vectors.

Just because they are linearly independent does not mean they are convenient okay, orthogonal vectors are convenient you know because you can express them in a very nice manner.

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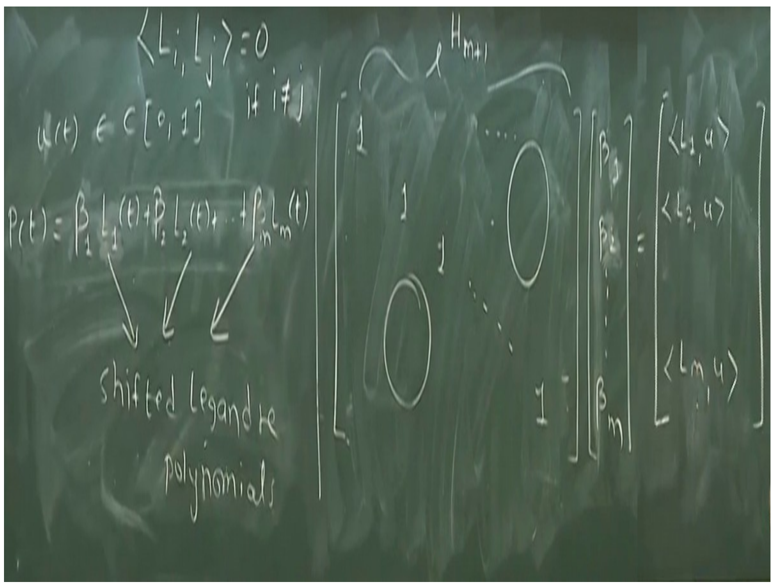

So what you would do instead of using this polynomial, you would say I will express this P t as alpha 1 say L1 t + alpha 2 L2 t+ alpha m Lm t, what are this Lm these are shifted Legendre polynomials okay. I do not have to worry here now about the order I take, why I do not have to worry? Because when I use shifted Legendre polynomials I am not going to get the Hilbert matrix.

By the way in mat lab if you want to play with Hilbert matrix just use command hilb and given to you know into bracket the dimension you want hilb 5 will give you Hilbert matrix of 5 cross 5, it will generate this matrix same matrix okay, I am just use inverse of that hilb okay it will crip that this matrix is ill-conditioned, try to multiply inverse into that matrix, mat lab is fairly good tell 12th x 12 it does a very nice job, but beyond x12 that is 12 cross 12 it starts breaking okay.

So amazing that even for so highly ill-conditioned matrices, it is able to do it now okay. If I used the Legendre polynomials what will happen to this matrix, this matrix will not be a full matrix because okay, let us call them instead of alpha 1 alpha 2, let us call them some other numbers you will get confused otherwise, say beta 1 beta 2 some other the coefficients are different because the basis has changed the coefficients are different.

So now if I want to find out beta 1 beta 2 to beta m, I have to take here inner products will change, this will be L1, u, this will be L2, u and this will be Lm, u, now what about this matrix what we know is that for shifted Legendre polynomials okay inner product of Li, Lj=0 if i !=j. What will happen to off diagonal elements? 0, what will be the diagonal elements if I take orthonormal 1 okay, so it should be just 1, 1, 1, this will be a 0 here, this should be here 0 here right.

I do not have problem of, so if I want to develop a high order polynomial approximation okay, it is easier to go through the roof of orthonormal polynomials then to use the raw you know 1 t square, t cube, t to the power 5 and so on, is it clear. Why we are obsessed with orthogonality? Why we want orthogonality so much in every application? That is because of this nice property. If I take shifted Legendre polynomials which are orthonormal this matrix which in earlier case was the Hilbert matrix of highly ill-conditioned I could not invert.

Now see this is the best matrix, there is no better matrix to invert then identity matrix right, so you get identity matrix okay you get the projections, and now there is no problem with what order you go okay. So it is not that you are not developing a higher order polynomial approximation, except you just turn around change the basis you get much better solution okay, you get much better solutions that is why we always want to work with orthogonal polynomials.

Now this orthogonal business will form, we looked at orthogonality, we looked at roots of the orthogonal polynomials in orthogonal collocations right, so this in this case here if I use orthonormal set approximations are you know just taking inner products with the vector. In fact, what you are done here is find out the Fourier series coefficient that is it, this is conceptually same, same as writing a vector V as x i+ y j+ z k that is all you have done that is all you are generalized to any other dimensional space.

If you remember this that Fourier series is nothing but x i+ y j+ z k extended to any other space that is enough okay. So now you know how to make approximations, not only that you know how to make good approximation, you can make good approximation to use orthogonal polynomials that is why we are concerned about generating orthonormal series or orthonormal functions or orthonormal basis and so on okay.

So the next class we will now start with the 2 things, one is how is least squares used for developing different engineering models, I will briefly go over that in the beginning. And then move on to you know using these methods for discretizing ODE boundary value problem or PD and so on. So this will lead to so-called you know the Galerkin method or finite element methods FEM you may have I am not going to 2 full of FEM.

Because that will consume rest of the semester, I am just going to touch the tip of the iceberg and say that this is what it is rest is for you to discover okay. So we continue in the next class about 4 applications of orthogonality.