Advanced Numerical Analysis
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Lecture – 02
Fundamentals of Vector Spaces

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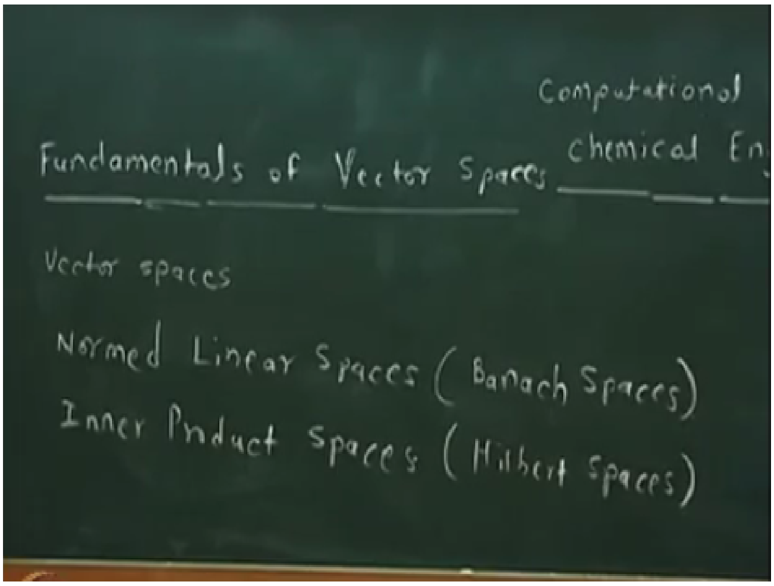

(()) (0:17) under this broad title, I am going to look at 5 or 3 or 4 sub areas, first of all, we will define probably what is the vector space, then we move to nonlinear spaces and also what are called as Banach spaces, then we have now inner product spaces and we will also catch up on Hilbert spaces, so these are names of these among famous mathematicians and then we look at Gram Schmidt process in the end.

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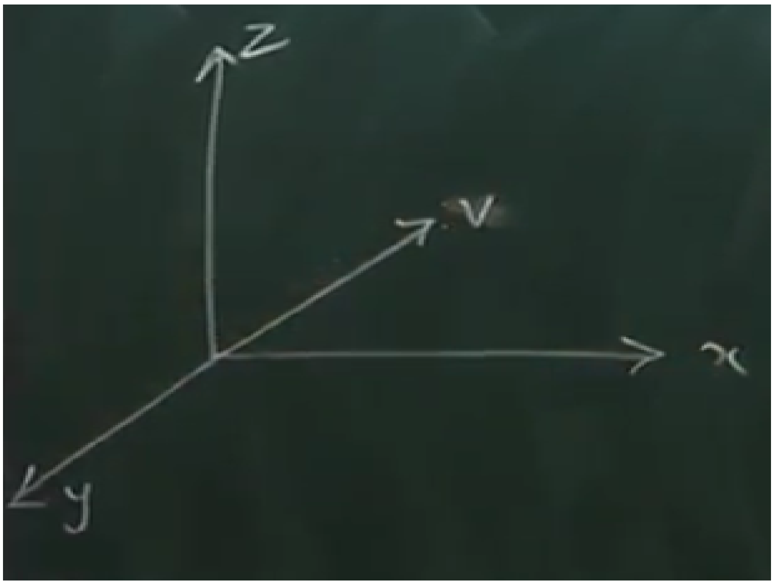

So, these are the 4 sub areas which I need to cover, now Gram Schmidt spaces as applied to any general vector space. Now, what is the vector space, what did we know about vector spaces, when we start thinking about vector spaces from what background we have; background at we have from undergraduate, we either look at; well, we normally imagine a vector in a 3 dimensional space.

So, this is the vector say x or let us call this vector v, let us 3 components; x, y and z and this is how we longer we imagine a vector space. Now, what is done in functional analysis? You should distilled essential properties of this vector space and then come with a new definition called vector space, which is more generic, which can be applied to any set of objects; any set of objects that are irrelevant to us when we do computational or analytical mathematics.

Now, actually the other course that we are doing in that also in the beginning there will be some introduction into these vector spaces, so now, what is it that we need to generalise and why we need do generalise? First of all, we can look at this vector space or set of all vectors in this vector space, so if I had said okay; as I said on which certain operations can be done okay, what are these operations?

Addition; you can have 2 vectors and get a vector and a nice thing is that you get a vector in the same space and you can multiply a vector by a scalar and you get another vector in the same space. So, these are e generic properties of any two vectors in the space and I could use this to generalize define a generalized notion of a vector space, it is not enough to just generalize this notion of a vector space, you also need something more to work with vector spaces.

We need to know about the length of a vector because that is a critical thing that we use when we actually work with vectors in 3 dimensions, so we need to have a generalization of that which is called as norm of a vector, we will talk about norms of vector, it is not enough to just talk about norm, we have notions of a sequence in one dimension, so a sequence which is converging to a something called limit.

So, what is the limit in n dimensions or 3 in dimensions, so we need to actually generalize the concept of conversions and limit that there are some funny things that happen when you start working with 5 dimensional vector spaces and that is where, we had; I had mentioned is Banach spaces and Hilbert spaces. So, these are some special category of spaces, which we are going to look at.

Now, it is not enough just to work with norm and conversions, we need something more, what else we use in 3 dimensions, what is the important geometry concept that you need? **"Professor – Student conversation starts"** Not coordinate; coordinate is of course that was define the space many times a coordinate system will come, it is not coordinates, its angle; angle between 2 vectors, so very, very important concept. **"Professor – Student conversation ends"**

Well, in 2 dimensions, everything comes together package, we do not really think of these things separately but when you generalize this concept to any other space, we need to make efforts to define what is angle between 2 vectors and also one of the most important concept that we used in 3 dimensions is orthogonal vectors, 90 degrees, 2 vectors and then well of course, the Pythagoras theorem which is used in many many ways.

So, what we are going to look at initially, is it possible to generalize these concepts and develop some notions of vector spaces on generic sets, which are useful in mathematical analysis. Why am I doing all this in beginning of the course, which is supposed to be computational methods and you would be starting with; there was a bit of recipes; well, if you understand these grand generalizations, which were probably done in the beginning of 19th century; beginning of 20th century.

Then it becomes very easy to understand the foundations of different memorial methods that we are going to study. So, this 6 or 7 lectures, which might look disconnected in the beginning are actually deeply connected with what we are going to do later okay, so this forms the foundation and you will understand basis of many many methods, if you understand this concept of vector spaces orthogonality and so on.

Many of these things are unknowingly used when you do undergraduate courses without you know being given a thorough explanation here will lay a systematic foundation of vector spaces. Now, let us; well, should I want to talk about you know 4 dimension or 5 dimension, 10 dimensional spaces and then I will also move to something called infinite dimensional spaces and well it is not possible to visualize.

In fact, it is not possible to visualize anything beyond 3 dimensions, you cannot visualize the 4 dimensions or 5 dimension spaces and obviously not an infinite dimension spaces, so these are word of caution before we move into this, is that it is enough to know your geometry; school geometry well, it is enough to know your undergraduate 3 dimensional word well. If you understand the concepts in the undergraduate 3 dimensional word or your school geometry, it is; you will understand everything I am doing, okay.

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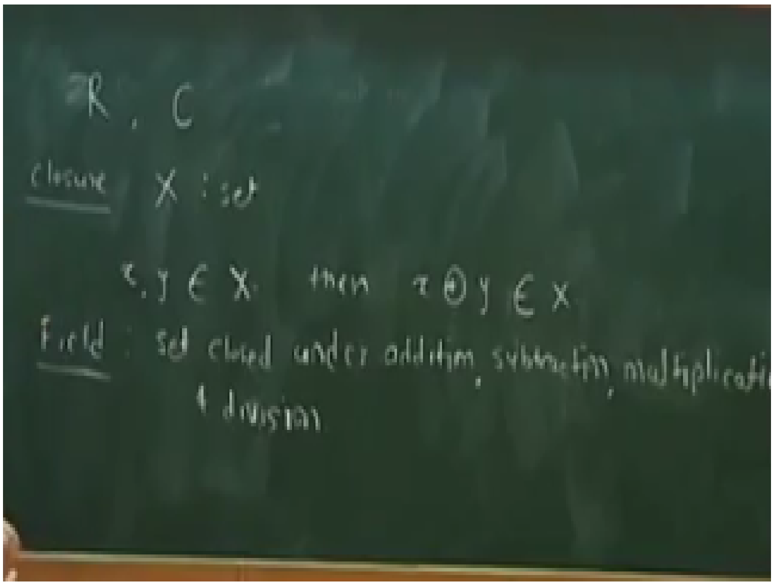

It is only a matter of generalizing these concepts, same concept which have been use in right, since our eighth standard, where one generalise into something very, very elegant. So, now let us begin with; let us begin with the concept of closure; a closure of a particular set for an operation is defined as if you take any two elements from the set; let X be a set, let X is a set and there are any 2 elements, say x and y belong to X okay.

Then x operation y also belongs to the same X, now this operation here I have written as plus, it could be any operation, it could be multiplication, it could be division okay, so given any set, if you take any two elements of this set and if you perform an operation, for example multiplication, okay and if the element that results after performing the operation also belongs to the set, then it is called a closed set, okay.

For example, set of integers is closed on addition, let us closed set will be division, right. The next concept that is important is a field, what is the field? and division, correct. So, field is the set of elements close under; so well, the well-known examples of field, set of real numbers, set of complex numbers and these are the two fields which have been they are going to use.

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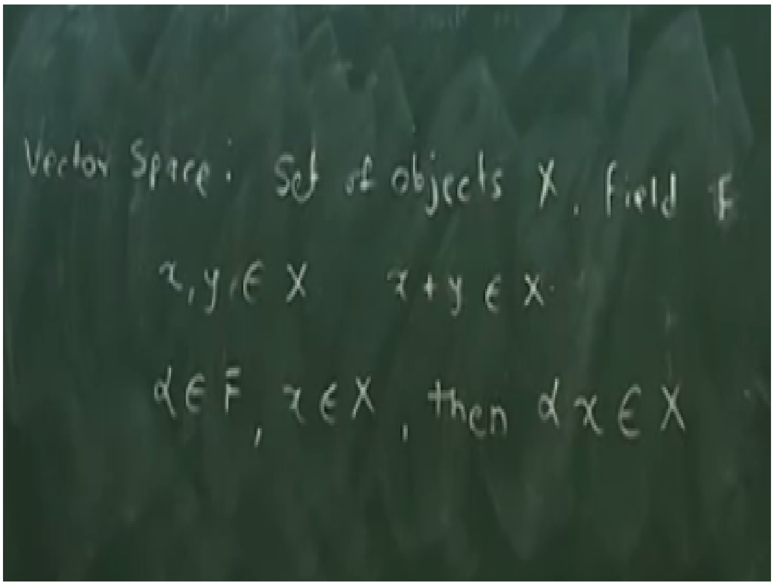

So, they are going to denote these as R; R we denote as the set of real numbers and C; let we put the set of complex sums that is closed under 2 operations. Now, a vector space is a set of objects, it is closed under addition, so if I take any element x and y belonging to X, then x + y also belongs to X but it is not enough to have this a set of objects, we need something more to define a vector space, we also need a field.

For example, a field could be R, so let us we call the field as F, so we need two things here, a set of objects X and a field F If I take any scalar alpha from F and any x belonging to X, then alpha times X is called a scalar multiplication, this also belongs to X. So, our vector space; a conservative generalization of 3 dimensional vector space is nothing but; nothing but a set of objects which actually satisfy these 2 operations.

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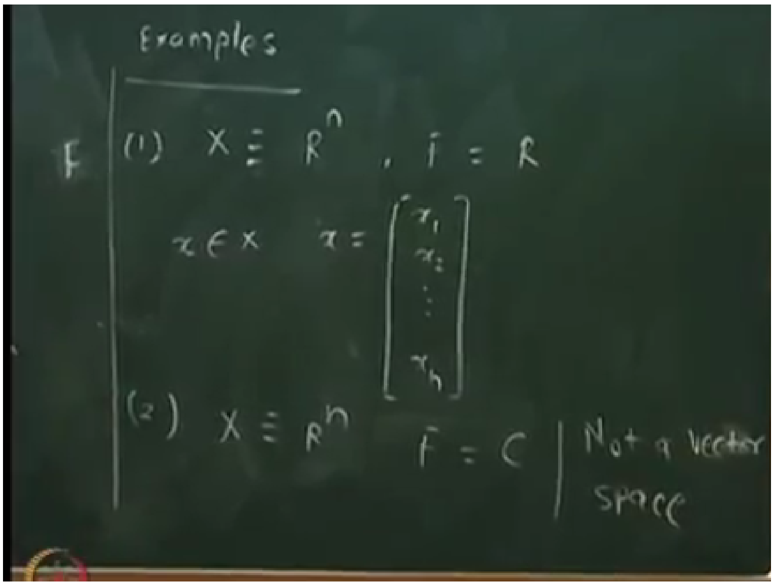

Or a set, which is closer under these two equations given the field F okay, it depends upon the combination, so these X and F, they are a combination, you can never separate them, you have to consider them together. Well, let me start generalizing and giving you examples of spaces, which are my first example is going to be X corresponds to Rn and F corresponds to R, what is Rn? Rn is n double okay or a vector which has n components.

So, my general vector here x that belongs to X will be represented as it has n components x1, x2, xn okay, it may have n components, can you give me an example there you need a such a thing? Let us say I am dealing with a; I am dealing with some chemical reactor and I decide to associate a vector space that defines different variables, so here x1, x2 to xn could consist all; for example, the Xn could be temperature in the reactor, x2 could be pressure and x3 to xn could be different chemical species that are present inside the reactor.

So, this is a vector of that represent the state inside the reactor or let me a take a distillation problem. Let us say distillation problem will have different trace and on each tray, you have a temperature, pressure composition, right. So, if there are 20 trays and it is binary distillation problem, how many variables you will expect to have? 20 temperatures, 20 pressures actually; pressure will be varying across the column, when compositions about 60, well there are correlations between y and x.

So, there would be 80 elements in a vector or 60 elements in the vector that defines all the variables and are associated with a distillation problem which has 20 trays, final distillation problem, which also; I can think of examples from chemical engineering, which would actually have, where you deal with vectors at a higher dimensional vectors, where an example that will tell you what is; which combination will not be a vector space.

See, for example if I take X to be Rn and F to be C, set of complex numbers, **"Professor – student conversation starts"** Will this form a vector space? No, Why? **"Professor – student conversation ends"** The scalar multiplication will break down, if I take a scalar which is complex multiplied to a vector, which is the real value I learned it element from x, so this is not a vector space when my next example is; my next example is little unconventional.

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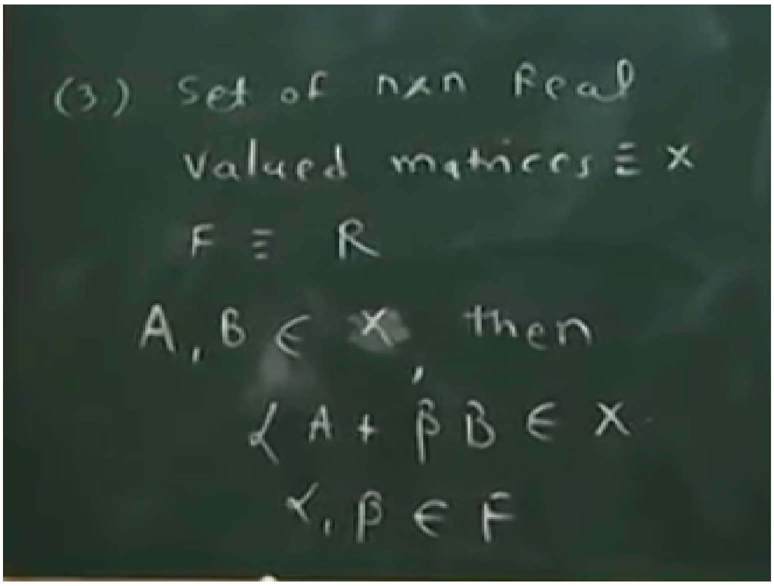

So, now I am going to move to set of real value matrices, so this is my x, this corresponds to my x, okay and my F, is set of real numbers; set of real value matrices, **"Professor – student conversation starts"** Let us this form matrices? Why? **"Professor – student conversation ends"** If you take any two n cross matrices, if you add them, you still get an n cross n matrix. If you take a scalar and multiply it to an n cross n matrix, you still get an n cross n matrix okay.

So, if I take any two matrices, say A and B, which belong to F, then I can say that any alpha times A + beta times B, belongs to also X, sorry this is X here and alpha, beta belong to F, so for any matrix X, for any matrix A and any matrix B, which are both n cross n, so these are elements, these are vectors in this space; vector space. If I take a scalar multiplication of A and a scalar multiplication of B, then this sum should also belong to this space, which is true for any n cross n vector, okay.

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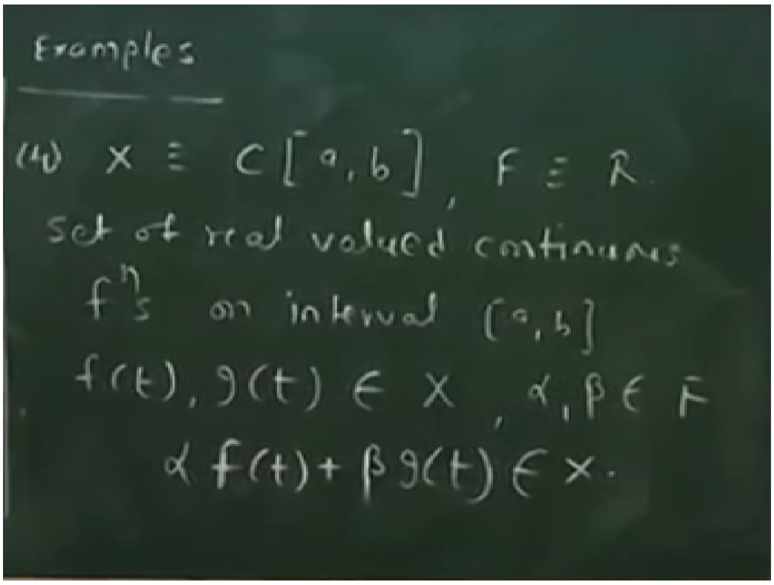

So, and my fourth example is something that I am going to use quite often in this course, so my example number 4 is; this is denoted as C; a, b. So, C a, b is set of continuous functions, set of real valued continuous functions, set of all the real valued continuous functions on interval a, b. So, if I have a function say, ft which and a function gt both of which belong to X, then and if I take any; if I take any two scalars say, alpha and beta that belong to F.

Then alpha times ft + beta times g of t also belongs to X, set of real value; have you come across this kind of functions, where? Where did you study this assumption? Fourier series not ever (()) (18:57), what happens in Fourier series? You talk about; you talk about functions on -pi to pi or 0 to 2pi remember something; rings a bell; and b are 2 constants, have you going to look for Fourier series much more detail in next 2 lectures, okay.

So, you are agreed with me, if I take a continuous function and multiplied by a scalar, it is simply a continuous function, let us see the continuous function, right. If I add 2 continuous functions, will the addition be continuous? If f of t is continuous and g of t is continuous, add 2 continuous functions, I still get a continuous function, so scalar multiplication, vector addition both properties hold in this abstract set, it is the difficult to visualize.

How this set looks like, you know we are used to visualize it in 3 dimensions, nevertheless the property is that hold; the fundamental properties that hold in 3 dimension also hold in the same that is very, very important okay. So, if you have a vector; if you have a vector with which holds this side, multiply it by a scalar you get a vector in the same side, it is very important and g of vector which is; well, this is not enough to just define vector spaces.

We also have to talk about subspace, so very, very important concept in 3 dimensions, what is the subspace in 3 dimensions, what are the sub spaces that we go in the dimensions? **"Professor – student conversation starts"** Line passing through a; yeah, correct, it is only subset; a plane, so whether the plane not passing through a; will it be a sub space? Right, why? Good. **"Professor – student conversation ends"**.

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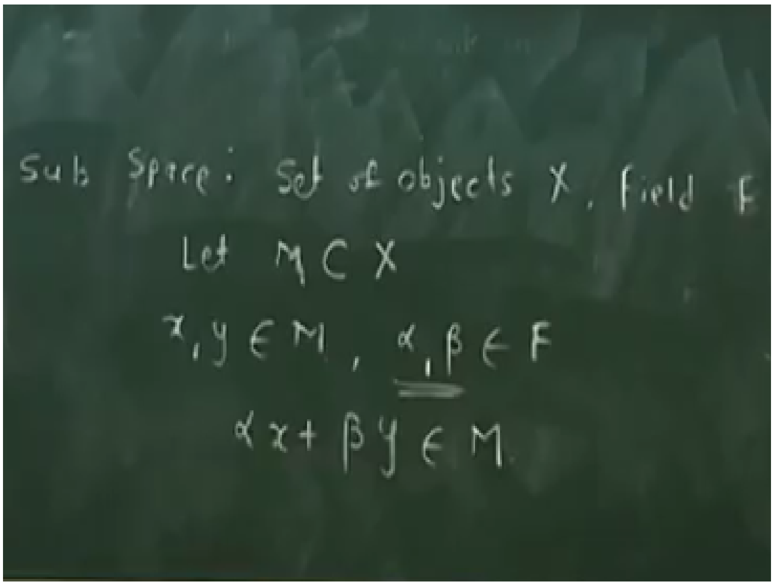

Let us define what is a subspace? if I want to define the subspace; so this is I have a set of vector X and field F and then let M be the subset of X and with some subset of X and nonempty subset of X and if I take any 2 elements in M, so x and y they belong to M and alpha beta belong to F, then alpha x + beta y should belong to M and the way we define subspace is a nonempty subset; a nonempty subset of original space X okay like example that he gave us now.

A line passing through origin or a plane passing through origin okay but why was origin important, that the origin is required in this space is hidden in this definition, can you denote? Any word is that you know; the main thing is any alpha beta belonging to F, so what about 0, 0, if I choose also alpha to be = 0 and beta to be = 0, then 0 times any vector + 0 times any vector give me 0 vector; 0 vector should be contained in the space okay.

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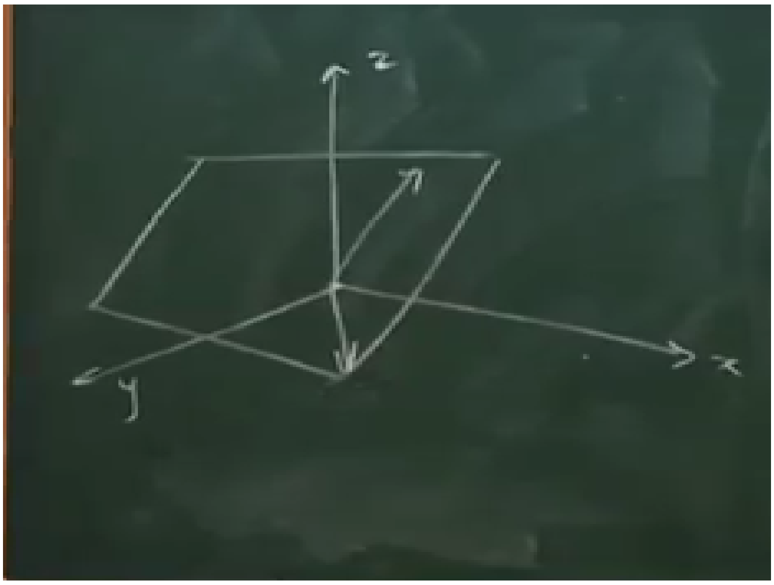

So, if I have a subspace, then it follows from this definition that the 0 vector, the origin should be contained in the space, so only those sets; only those sets; which contain the origin qualified to be subspaces okay. Now, let us understand this little more; if I take okay, let me try to draw a subspace, imagine that this square; the plane which passes through this; this is a plane that passes through the origin okay, now that 2 situations; 2 scenarios.

One is; this is a finite; this is a finite set, it is only like a piece of paper okay, let us look at this piece of paper, which is passing through the origin it is finite size, it is passing through the origin, will it form a subspace? Just because it passes through the origin, will it form a sub space? I can take 2 elements, this is passing through the origin I can take 2 elements such that x + y; not for all x + y, correct.

The word all is important, it should happen for every x + y, okay, it may happen that if I take a finite set like this and not the infinite set, it may happen that for 2 vectors, say this vector and this vector the addition may not belong to this small set. Yeah, but vector spaces, origin is included, it is obvious, the subspace is a small set; subspaces are a smaller set; does every smaller set qualified to be a subspace, is the question; I will answer.

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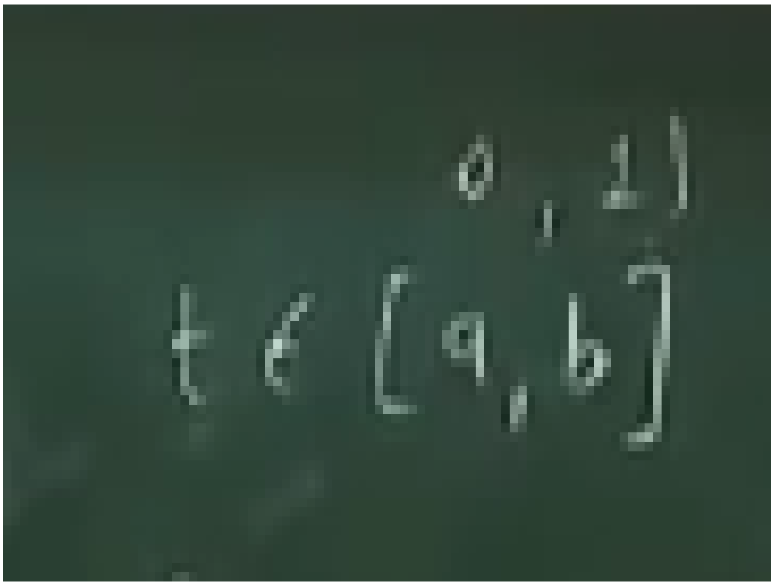

So, what is for example, what is zero elements in this, what is zero here? Yeah, constant function, ft = 0 over interval A to B okay. Here, it is important to remember that t belongs to a, b component. Well, do we get this kind of functions? Think of temperature profile in the heat exchanger okay; my, a would be 0 to 1, where it will not be time, t; do not associate t with time, it would be space, so z is my special variable okay varies from 0 to t okay.

And I can look at the temperature profile inside but distillation inside the heat exchanger, is it a continuous option? Yes, it is a continuous option, so this kind of vectors; these kind of vectors spaces are very, very commonly encountered in chemical engineering examples and of course the zero; the zero element would be; well nonzero temperature we often talk about perturbations and in steady state.

If you have a steady state and a perturbation the perturbation vector is zero, which means you have a steady state, so you may have; you will not have in the case of heat exchanger example, the zero element technically would be everywhere you have a zero temperature such as (()) (26:23) does that exist but the space; in the space you can of course define zero vector which is; so coming back to this subspaces, every subset does not qualify to be a subspace okay.

This thing is important, if I take alpha; any scalar alpha multiplied by vector, then the resulting vector also should be included inside the space. So, if I take this vector and if I multiply it by a large scalar, the new vector will be here which is not included in this small finite set, this is not a subspace. So, just going through a origin is not sufficient okay, you need to have closure; we need to have closure of this operation, alpha times x.

For any alpha, beta times y, this sum should be belong to; so can I give you an example, I will just give you an example, which is completely different but generalizes this concept. Let us look at this space, okay; set of polynomials; set of polynomials, are they continuous functions? Yes, right, set of polynomials is a continuous function okay, set of polynomials let us say define over a to b or 0 to 1, if you want to fix (()) (27:45) imagine 0 to 1.

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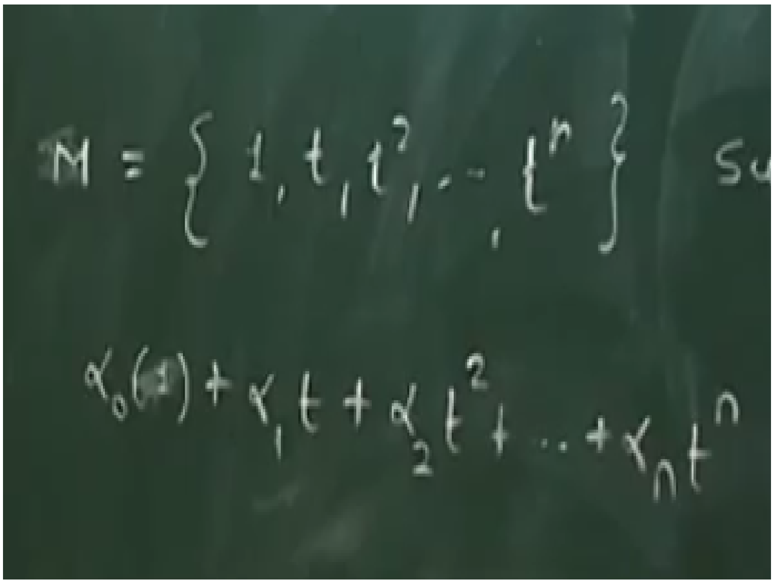

So, set of polynomials defined over 0 to 1, what will be the set? So, let me take this set; let me take this set s, we call it M here, let us be this side be M, which is 1, t, t squared, or t to the power n okay, will this combination be also polynomial, this is an nth order polynomial okay; this is an nth order polynomial, now when I am visualizing each one of them as nth order polynomial with some coefficient 0, okay.

So, this set of all possible polynomials with any alpha 1 to alpha n set of field, this particular set will form a subspace of this vector space because these are continuous functions, these are continuous function and then if you take a polynomial; if you take a polynomial; finite order polynomial adds to that polynomial and we will get a finite order polynomial okay. So, all those properties; all these properties with hold for set of polynomials and then you can show that this is subspace, 0 element will be there, zero function, zero polynomial okay.

So, all the things that you need will be there in this thing, okay and I will another example of subspace, can you think of a subspace for n cross n matrices? For example, let us take my set here, so you understand this if you understand these examples because just writing the definition is too abstract unless you associate with some real examples, it is not possible to understand these concepts.

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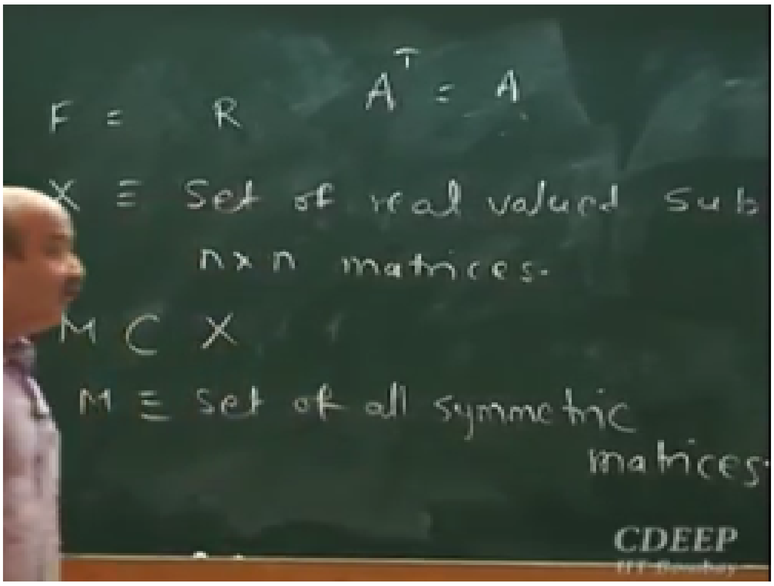

So, like by X be set of; and of course my field is R, okay. I am going to define a subset M which is a subset of X, which is a non-empty subset of X, so M here set of all symmetric matrices, is it a subspace? what is a sub space? If I take a scalar alpha, see when you call the matrix to be symmetric; A transverse is = A, okay. Let us test this you know; alpha times A transpose will it be equal to alpha times A, for any alpha, right.

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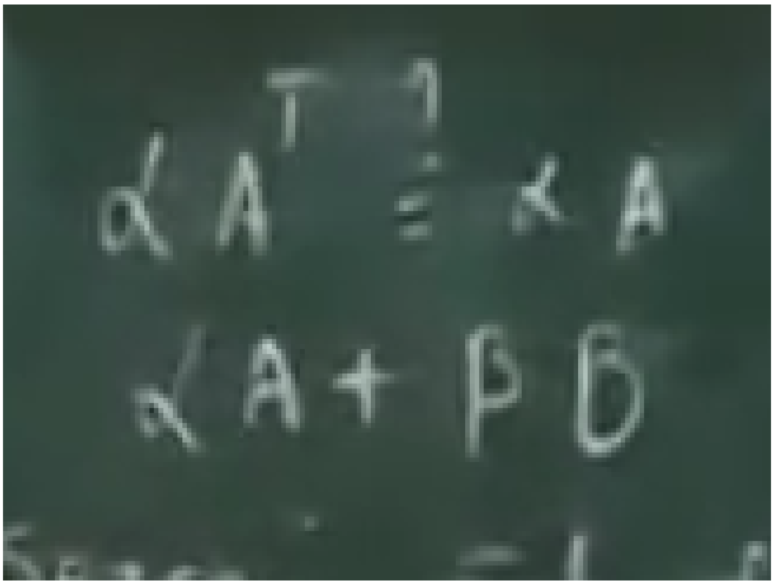

For any alpha; by the way what is the zero element in this space, what is the zero vector? Dull magnets, so if I set alpha to be 0, it is a symmetric matrix, right, it belongs to a subspace. What about alpha? If I take any of 2 matrices; A and B belonging to this subset M okay with their linear combination also belong to M, if this is symmetric and if this B is symmetric, will this addition also be symmetric? It is a symmetric matrix.

So, this is subspace on the subset defined by this; defined by this either you know, it is a set, which forms the sub space, not every set will form a sub space but particular set of set of symmetric matrices will form, likewise if you go to a set of complex valued matrices with field to be complex numbers, you can define with Hermitian matrices; set of all Hermitian matrices okay, will be a subspace of the set of complex valued matrices okay.

So, these are the generic example, what is the next thing that will be; when you start looking at vector spaces? What is the thing that I use most? Well, one of the most important concepts that we use is basis and dimension. What is the dimension of vector space? What is dimension? How do we define dimension? This is the common (()) (32:57) number of coordinates okay.

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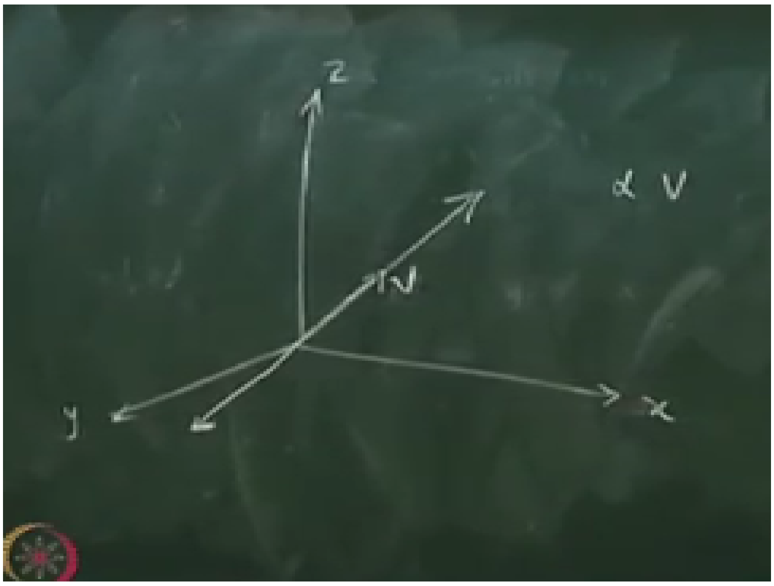

Let us look at this subspace, this is my; this is a line passing through the origin and all of us agree that this is a subspace okay, so line passing through the origin, any vector of this line will be represented by 3 components; does it mean it is 3 dimensional subspace, what is the dimension? Why? Number of independent vectors are only 1, so just because a vector has n components, does not mean that you know the dimension of the vector of the vector space or a subspace.

The dimension of this subspace is only 1 okay and there is only 1 independent direction, let us say you call this some vector x, so any vector, let us call this vector B, so any vector on this line will be alpha times v okay. If alpha is minus, it will go in this direction, so alpha is going in that direction, so but basically, it is alpha times; it is only one independent direction, okay. So, likewise, is a plane cutting or a plane passing through the origin; what is the dimension of the space? 2. Because there are only 2 independent vectors.

Two independent vectors can generate the entire space (()) (34:40), so we need to now generalize this concept of dimension. Well, to generalize this concept, we need; we analyse many other concept, we have to have notion of linear combination defined after that we will have to define what is called as a basis set and then move on to; so you should go through the notes, I have given many more examples of vector spaces.

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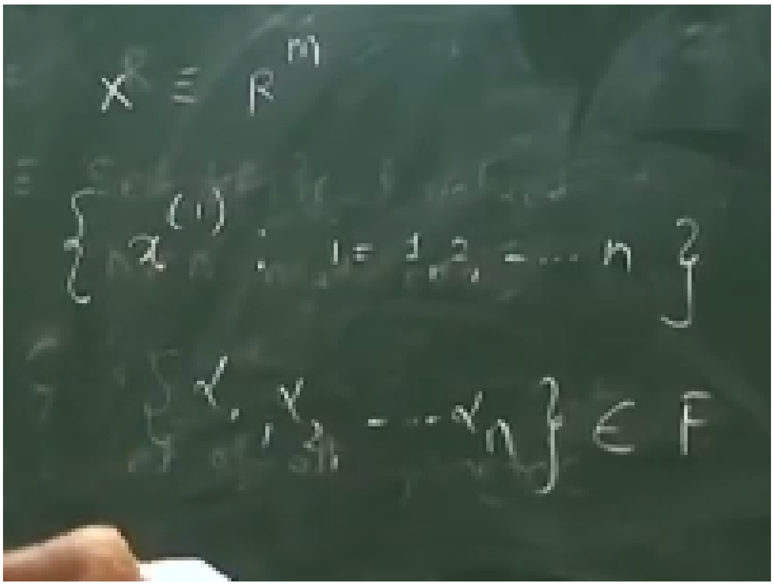

So, and now as I said, I am not going to write everything onto the board, we should look at the notes. Now, if I am given the vector; now I have to introduce one important notation here because they are going to work with set of vectors and each vector might be n double okay, each vector might be n double, so I have to introduce a new notation. Now, I am going to consider a set here; a set xi.

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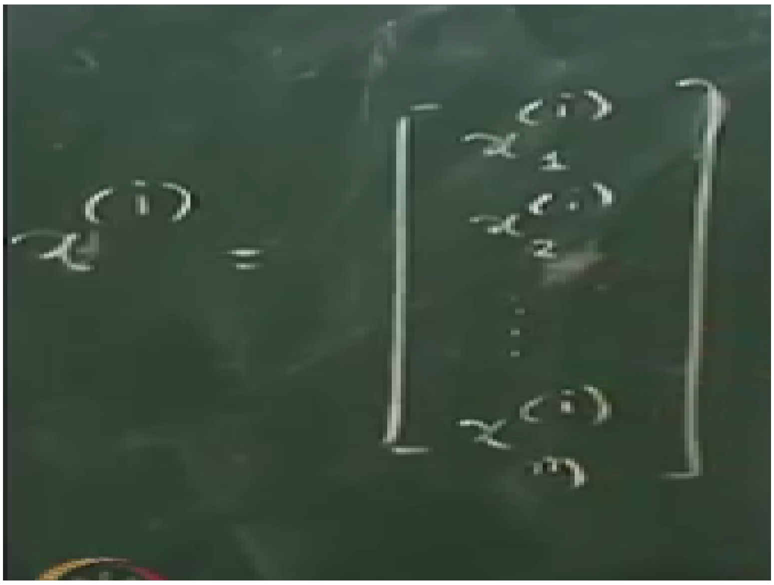

I have some space x here and this xi, these are vectors that belong to this set X where i was from 1 to n, okay. It is quite possible that my set is nothing but Rm, so m doubles, okay. So, an element here, there are n vectors and each one of them is an m double okay, so where I am going to define this is; xi, it corresponds to xi1, xi2, xin so this notation is central to our course, I am going to use it very, very often, then you have a sector.

So, superscript in brackets is used to indicate ith vector okay and the subscript is used to indicate the component of the ith vector, okay. So, xi2 is second component of ith vector and so on, so we will be using this notation very, very often. Now, if I choose any set of scalars; if I choose any set of scalars; say alpha1, alpha2, alpha n, then a vector which is defined by alpha1 x1 + alpha 2 x2, this vector is called as linear combination of this set of vectors.

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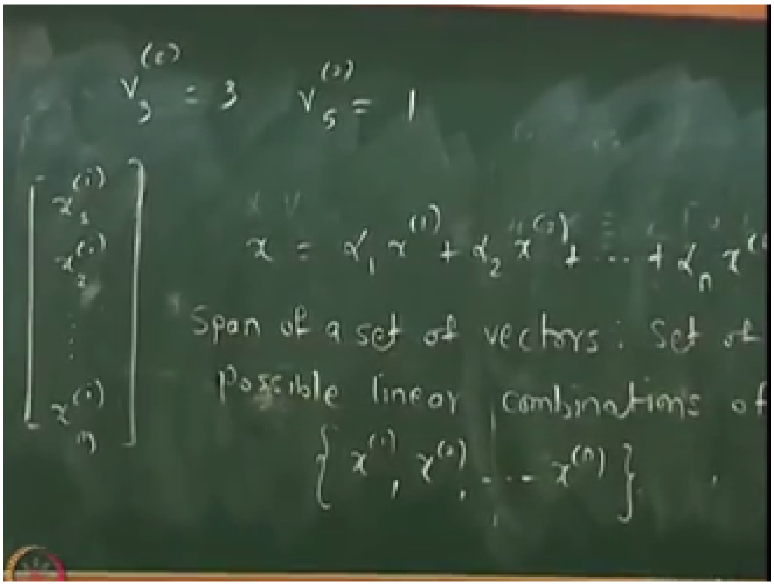

This is the set of vectors belonging to space X okay, alpha 1 to alpha n or some scalars; arbitrary scalars belonging to the field F and then the vector that you get by alpha 1 times x1 + alpha 2 times x2 up to alpha n times xn, this particular vector obviously we are dealing with a vector space, so or we are dealing with a subset in which this linear combination, we should talk about a subset alone, then it is a finite set.

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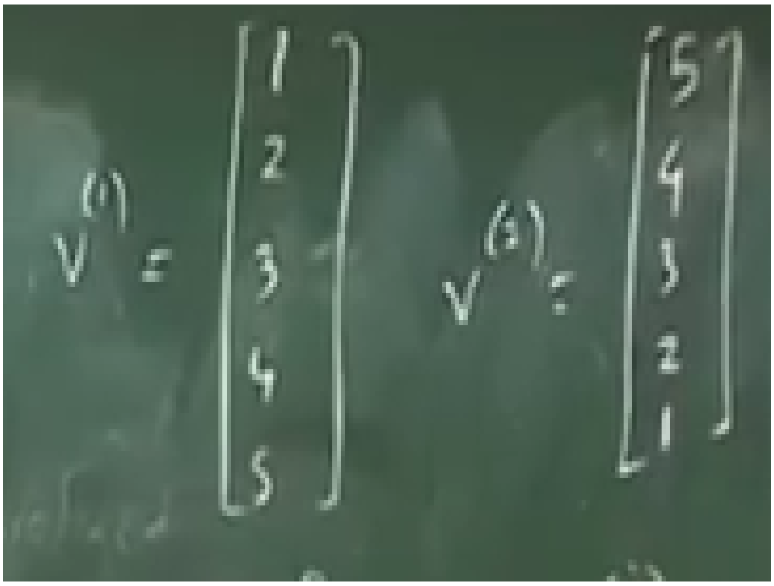

And if we take all possible linear combinations of these vectors, they give a special subset that is called a span. **"Professor – student conversation starts"** This is a vector, this is ith vector okay, see for example I may have 2 vectors, let me take a 5 dimensional space, so I have vector one okay; 1, 2, 3, 4, 5 and vector 2, which is 5, 4, 3, 2, 1, okay. Now, how do I refer to third element of vector 2?

So, I will say v2, 3 that is equal to; okay vector 2 third element, okay. Similarly, v2, 5 will be; no, no this is just a notation; just a notation that which we are going to very, very often whenever you have subscript in brackets or superscript in brackets, it implies that ith vector okay and if I want to refer to this zth component of a ith vector okay, then I will use xi z, it is not a matrix, it is a rotation. **"Professor – student conversation ends"**.

Now, this kind of things do appear in numerical methods in computations because we do introduce procedures, okay we start from one vector and then you get another vector and another vector okay, so you have a sequence of vectors and that is where you need to know this, have this is a little complex notation, so sometimes we develop algorithms in which we need to worry about this superscript and in this subscript together okay, that is why we need to have this notation okay.

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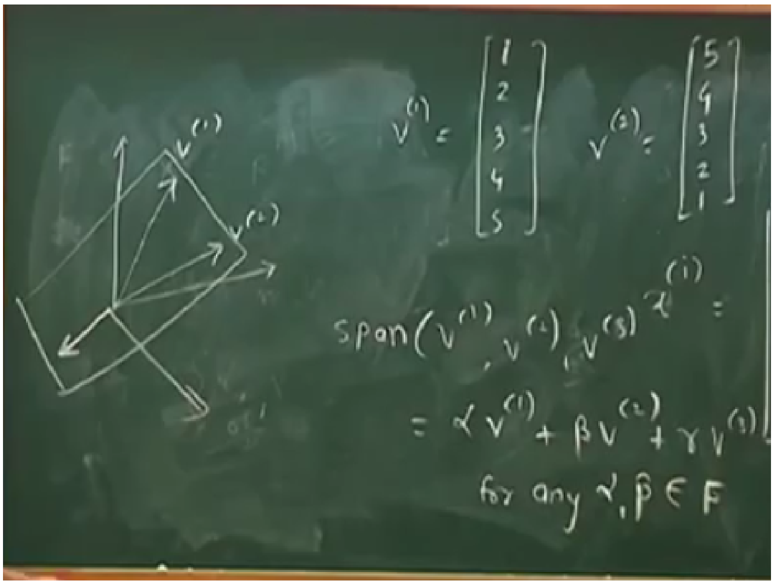

So, this span is set of all possible combinations understand; set of all possible combinations. If I give you 2 vectors in three dimensions; if I give you 2 vectors; okay, first start with two dimensions, if I give you any 2 witness; let us call them v1 and v2 are these 2 vectors in two dimensions. What is this set; so span of v1, v2, this corresponds to alpha v1 + beta v2 for any alpha beta that belong to F okay that belong to the field F.

What is this set? Field F is a real numbers, what is this set? It is a plane passing through origin okay because two independent vectors; two linearly independent vectors; you have to defined what is linear independence; so 2 linearly independent vectors, if I take all possible linear combination, then what I get is the span and the span is nothing but yeah, well, let us say that this is a subspace; this is the subspace, if I give you a third vector in the same subspace okay, which is v3 okay.

What will be alpha v1 + beta v2 + gamma v3, okay that is because this third vector v3 is linearly depend upon v2 and then you can and what you get here; so if I have some more vectors belonging to the same set here okay; same subspace and if I take all possible linear combinations, I am not going to get a different set; I am not going to get a different set okay. I will get the same set, which is this plane, I cannot leave this plane.

If I take linear combination of any two vectors in this plane, I cannot leave this plane. What is the minimum number of vectors that are required to generate this thing okay? So, the minimum number of vectors and are required to generate a subspace or space is called as the dimension of the vector space, okay. What is the dimension of this particular subspace? This is a subspace, right the span is a subspace, what is the dimension? Dimension is 2.

Because you need only 2; 2 linearly independent vectors to generate all vectors inside this plane which is passing through the origin okay, this is a 2 dimensional subspace of; it is the two dimensional subspace of; everyone will know this, okay. Now, let us consider these 2 vectors, what will be span of these 2 vectors? **"Professor – student conversation starts"** Correct, what dimensional (()) (44:45) okay.

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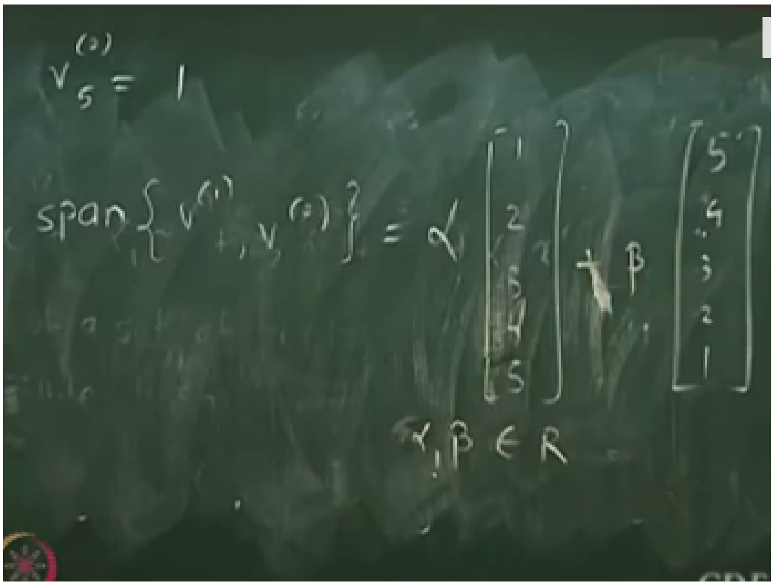

So, if I take span of v1, v2, it is alpha times 1, 2, 3, 4 and 5 + beta times 5, 4, 3, 2, 1, for any alpha, beta belongs to R, this all possible linear combinations of these two vectors is called a span of these two vectors and this span will be nothing but a two dimensional subspace of R5. What is R5? Space considering of vectors, each vector has 5 doubles, yes, 5 components. So, number of components in the vector does not define the dimension okay, this is the fifth; 5 dimensional couple.

Just because it has 5 components, does not mean this linear combination will define. If I take only one vector, say v1; what will be alpha times v1; it is a line and it is one dimensional subspace; it is a one dimensional subspace of R5; 5 dimensional space okay, may be one dimensional subspace of R5. **"Professor – student conversation ends"** Well, so far so good we follow and define what is basis and who want some more insights into why this is all required, where do I need this, we will do in our next lecture.