Advanced Numerical Analysis
Prof. Sachin Patwardhan
Department of Chemical Engineering
Indian Institute of Technology – Bombay

Lecture – 10
Weierstrass Theorem and Polynomial Approximation

Okay, given the element from the range space and the operator T, I want to find out x. This is called as inverse problem, okay and these are the problems which we normally have to solve. So, the core of this particular course is dealing with these kind of problems and a next class which is identification problems. So here, what falls under this? So, solving Ax = b, given A and b, classical problem, which right hand side given operator, you want to find out x, right.

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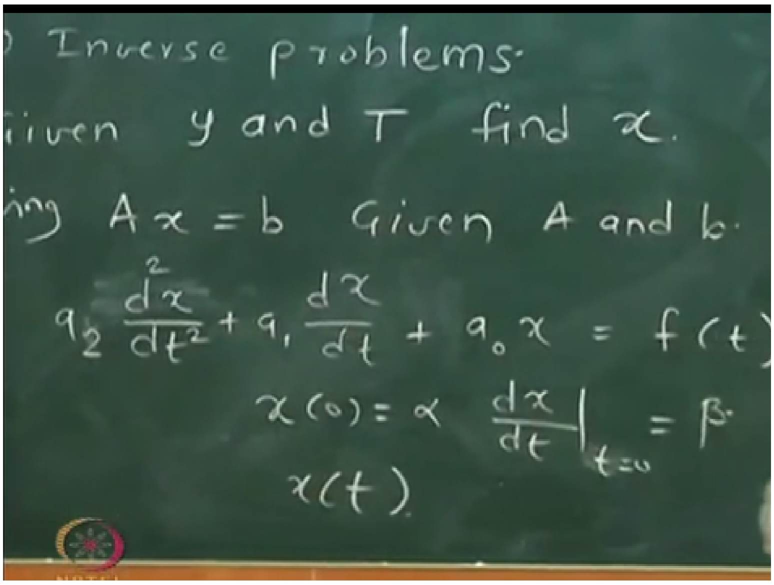

This is probably the problem which we will solve most often in this course. So, we are given some differential equation, so the other classical problem is ODE initial value problem. We are given a differential equation, so this is the operator, okay. We are given ft, ft is equivalent to b or ft is equivalent to the vector in the range space. I want to find out solution xt which satisfies the condition that initial value = alpha and initial rate = beta.

I have given 2 initial conditions, I have given the operator, I have given the vector in the range space, I want to find out solution xt, okay inverse problem. Operator is known, the range space vector is known, okay. initial conditions are known, I am going to find out xt, inverse problem. So, likewise ODE boundary value problem is an inverse problem or solving a partial differential equation that we encounter in engineering mostly inverse problems.

We are given the vector in the range space, we are given the operator, we have to find out x that satisfies the differential equation boundary conditions and solution gives you the vector in the range space, okay. So, these 2 problems are conceptually similar Ax = b or this operator operating on xt giving you this vector ft, okay and then the solution should satisfy these 2.

So, these kinds of problems are inverse problems. The third class of problems that you encounter in the engineering mathematics is identification problems. So, you are given x and y and you are asked to find out operator T, okay. The classic problem here is model parameter estimation. Suppose, I want for some particular material, you want to find out cp as a function of temperature.

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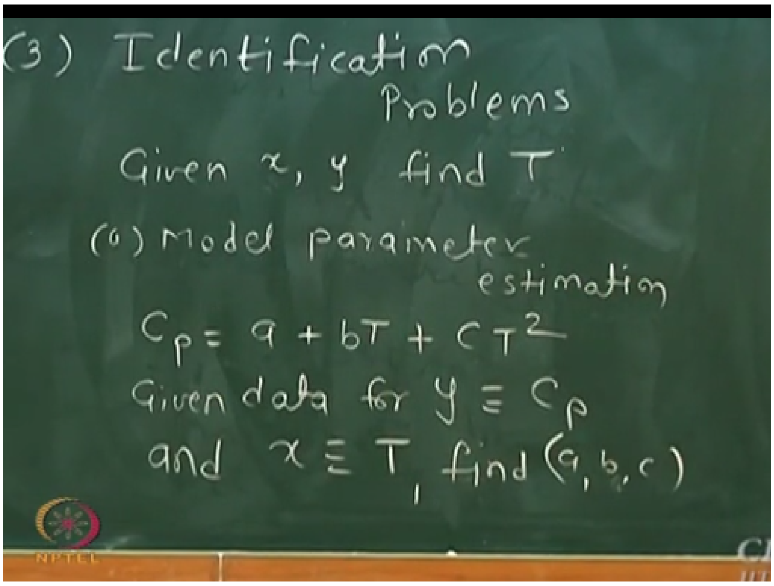

So, you have this a + bT + cT square. I do not know a, b, c, I have been given values of cp, I have been given values of temperature, okay, so I have been given x, I have been given y, y here is cp, okay, x here is temperature, what I want to find out is the correlation, is the operator, I want to find out the operator. Finding operator in this case it is to finding out a, b and c, okay.

So given data, parameter estimation problem. What are the other parameter estimation problem? You have seen in chemical engineering, reaction rate expressions. You have measured the rate of change of concentration of particular spaces and then you have a proposed expression, okay. You do not know the parameters, okay. You have rate values, you have concentration values, you want to fit.

Find out the parameters of the rate expression or you know you are trying to fit some thermodynamic correlation, PVT correlation, you have data for PV and temperature and you have a proposed model, we do not know the parameters you can fit, estimate the parameters from data. So, you are trying to find out the operator, okay knowing I mean if you look at y as a effect and x is a cause, so operator T operates on x gives y, y is the effect.

So, you know cause and effect, you want to find out the operator. Another example is estimation of transfer functions in process control, okay. You give an input perturbation, you measured the output, you tried to fit the transfer function into the data, okay all these are examples of identification problems, okay. So, bulk of our work in this courses going to be inverse problems and then, we will also look at identification problems to a large extent.

Direct problems are not going to be focused, I am not saying the direct problems are not important, but relatively easy to deal with these 2 problems are more difficult and we should get the better understanding of these problems. Now, the main problem in most of the cases, not in every case, in most of the cases is that once you have formulated a problem, it may not be possible to construct analytical solutions to the problem, okay.

Particularly if a operator is nonlinear, okay. So, I can say in general when operator T is nonlinear you cannot construct analytical solutions. Well, there are of course many exceptions, but the cases where you cannot solve are far more than the cases where you can solve, so in general you can say that when the operator T is nonlinear, so lot of numerical analysis is all about transforming a problem which is not analytically computable to a form which is numerically computable, okay.

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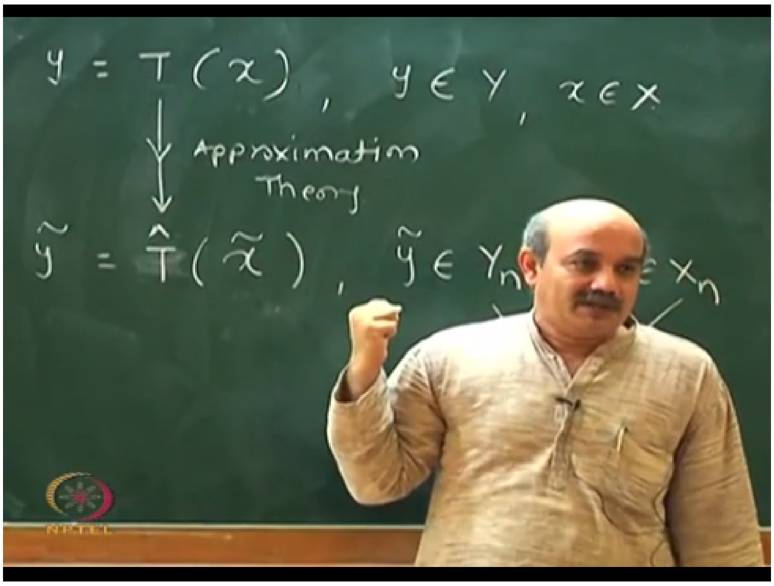

So, actually my original problem is y = T of x, okay and where you know y belongs to space Y and x belongs to space X, okay. I am not able to solve this original problem, so what I do is I approximate T using approximation theory and then I get let us call it T cap, okay. I get a T cap, which actually works on x tilde and gives me y tilde, okay. here, y tilde belongs to Yn and x tilde belongs to Xn.

So these are finite dimensional spaces typically and then we end up solving this problem, not this problem. We hope that the solution that you get suppose I solve an inverse problem which is I wanted to solve the original inverse problem, I end up solving an approximate inverse problem, okay and I hope that the solution x tilde is close to x, okay.

So this is generally the situation now, how do you get from here to here, next about 10 to 12 lectures are going to be how do I go from here to here. So this looks abstract right now, but keep this in mind in background that this is what we are going to do, okay. So we might start with the partial differential equation and end up with nonlinear algebraic equations, okay. See this might be a partial differential equation what do you end up here might be linear algebraic equations or nonlinear algebraic equations, okay.

So, what you start with and what you end up with can be completely different, okay. So, it is not that because I start with the differential equation, I will end up with the differential equation, okay. Now, when you go from here to here, there is no unique way of constructing T cap, there are multiple ways of constructing T cap, okay. Same problem can be approximated, discretized in multiple possible ways that is what we are going to see here, okay.

And each one of them has advantage and disadvantage. So there is nothing like though method to discretize, okay and as you go along doing numerical problems, you will develop your own preferences as to, so I am going to talk about not just one method. I am going to talk about multiple methods. So, you might wonder why I am talking about multiple methods because there is no one way to solve the problems.

Sometimes some methods are simple, but if they do not work, you need to go to more complex methods and so on, okay. So when you attack a problem, you should have a repository of tools or repository of you know approaches to deal with a problem and then you can go on you know simple method first, if does not work go to a more complex method, if it does not work, go to a more complex method.

What you mean by does not work, the proof of the pudding is x tilde close to x does not make sense. Now, you in the real situation, you never know what is x, true x, but since you are an engineer, okay if you look at the solution, you can make out whether this makes physical sense or not, okay. Whether the solution makes sense as a engineer, as a scientist, as a physicist, you can make a judgment and then decide whether your method is giving resemble results or not, okay.

So, there is a lot of subjective element here which requires development of expertise, okay. So, even though we are dealing with applied math which everything cannot be automated and that is why we are in business, okay. There is still scope for improvement you know for interpretations for doing it differently, getting better solutions and so on, so now let us begin, okay. So, what is the basic trick that is?

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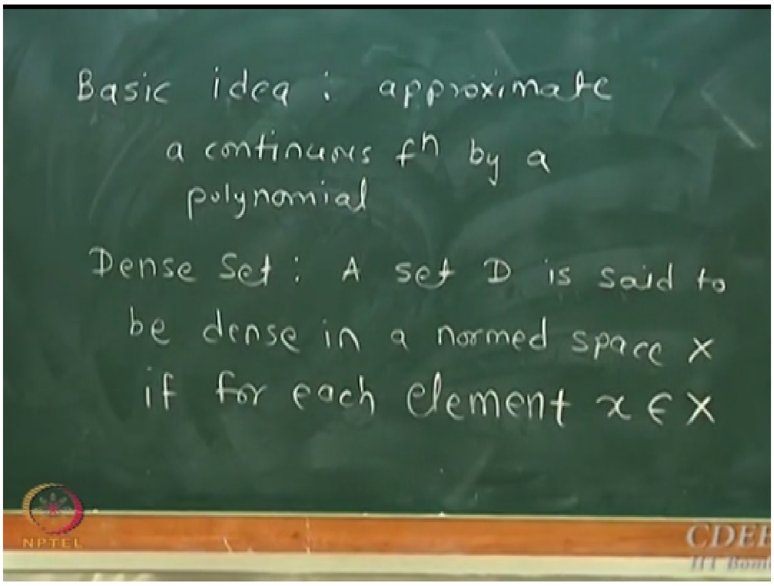

So, if you ask me to distill out one basic idea which is used to do these transformations from T to T hat, okay. Cutting across all the methods for boundary value problems or partial differential equations for all kinds of things, what is one trick that is used. Well, if you ask me to summit up, I will say that approximate a function by polynomial that is the trick that is the underlying trick.

You know if I cut across many of the methods and what I am going to show in next few lectures are that how this one trick is used to you know deal with variety of problems, okay. Starting from nonlinear algebraic equations to partial differential equations to boundary value problem, so all kinds of terms, we just use one trick in multiple ways. So, basic idea is approximate.

Now the question is, well, is it just an observation or is the basis why should I approximate a function by a polynomial function, why not cosine functions? You know why not exponential functions? Why polynomials? What is so get about them? Well of course, they are convenient when you do calculations, but not just that, there is something deeper into why polynomials are used for approximating functions and then developing different methods for solving the problems.

So, fundamental concept here is the concept of a dense set, see approximations is something that we very, very often use in mathematics, for example pi, you know when you start using pi, you start using that 22/7, right. But pi is not = 22/7, it is an okay approximation of pi for doing you know rough calculations, not exact calculations. When you start using for example E, we never use the true value E, right.

We use an approximation, a finite truncated approximation of E and do calculations, right. So, what allows you to do that? What allows you to do that is that you can approximate a real number using a rational number, okay, this property of rational number, there is something special about rational numbers. You can approximate any real number as close as you want by a rational number, okay.

And this particular property is expected by us when we do computations and the best example I said is pi being replaced by 22/7 or whatever. There are different, different rational approximations of pi, you remember something else, some 141 by or there is some other approximations also, which are not so popular in the school books. But what we use in school book is 22/7, right.

We are always, so why we can do this or I told you that when you are doing computing, all the number are finite procedure, right. No number in the computer, so there will be missing numbers if there is a finite procedure, okay. Not all numbers can be represented, particularly it is not possible to represent you know all real numbers, pi may not be truly representable, you can represent using some truncations.

Because when you do finite precision, okay if I write some expansion for pi, I can write that integer divided by some 10 to the power something and that will be a rational number, right. So, it is not the correct value for pi, it is a rational approximation for pi, okay. So, I am able to do this because of this denseness property, so what is a dense set? So, let us go over the definition.

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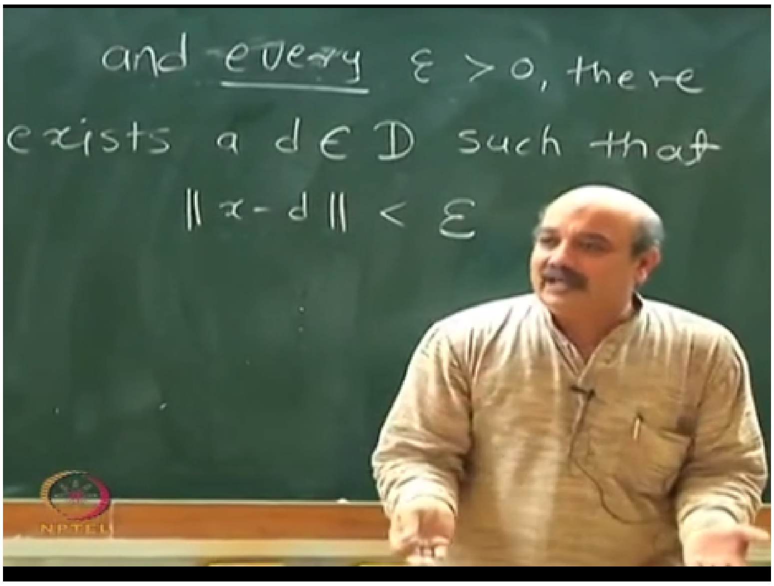

A set D is said to be dense in a normed space, so first of all you have to worked in a normed space, meaning norm, okay without that we cannot work both with these numbers or vectors. So for any element, if I give you any element x, in x I should be able to find and if I give you an epsilon, I should be able to find an element d belonging to dense set D, which is close to x, how close, there is an epsilon.

So, you know it is like saying if my x is pi and if I specify epsilon, you should be able to come up with a rational number which is close to pi such that the difference is < (()) (18:47). You know she might my epsilon is 10 to the power -3, okay I will come up with 1d, which is one rational approximation of pi, pi - d is < 10 to power -3 and he comes up when says no, no, no, I do not accept 10 to the power -3.

I want 10 to the power -9, is it possible. It is possible to find given pi, it is possible to find rational approximation such that pi - that number is < 10 to the power -9 and somebody does not accept 10 to the power -9 and you know he says 10 to the power -17, fine. I can find a rational number which is pi - that rational number will be < 10 to the power -17. So, any epsilon that is very important, okay.

So, what does it mean that on real line, these rational numbers are everywhere you know. I can use them as approximation of something else which I am not able to represent, which is very nice, okay. So, I can use rational numbers as an approximation of a real number and that is why I can work in a computer, okay. So, when I am working in n dimensional space, okay.

In Rn, even though I may not be able to represent you know all elements of the vector Rn because a real number may not be exactly representable, I can replace by its rational approximation, okay. See suppose I have a vector, which is like this. Suppose I have vector which is e, - pi, pi square, root 7. In a computer, can I really work with e, - pi, I cannot, right. I actually replace this by some approximations.

So some rational approximation of e, some rational approximation of this, some rational approximation of this and so on and why we can do this because set of no, no understand the philosophy, why we can do this, is because set of rational numbers is a dense, okay. Rational numbers are everywhere, you can just if you want to represent a real number, pick a very close rational number, you know you will be having a good approximation, okay.

Now, I want a similar result to this in set of continuous functions, okay. What I am going to work with, now we have seen that you know when you deal with partial differential equations, when you deal with boundary value problems, when you deal with ordinary differential equations, we will be dealing with set of continuous functions, okay. So, this is nice here that you know I can use rational approximations.

You know some q1, q2, q3, q4, so this is a rational approximation of this and in my computer, I can do calculations with this, in the same way I want analogy in set of continuous functions. This is a similar idea, conceptually a similar idea is that set of polynomials is dense in set of continuous functions. In the same sense, the set of rational number is dense in set of real numbers, okay.

Polynomials you know you can approximate anything by a polynomial, any continuous function by a polynomial, so these are the Lindemann theorem given by a German mathematician Weierstrass. I think somewhere in 1850s or 1860s and this is a celebrated theorem by called as, so this is a well known result, well this particular result is what is called as an existence result, okay.

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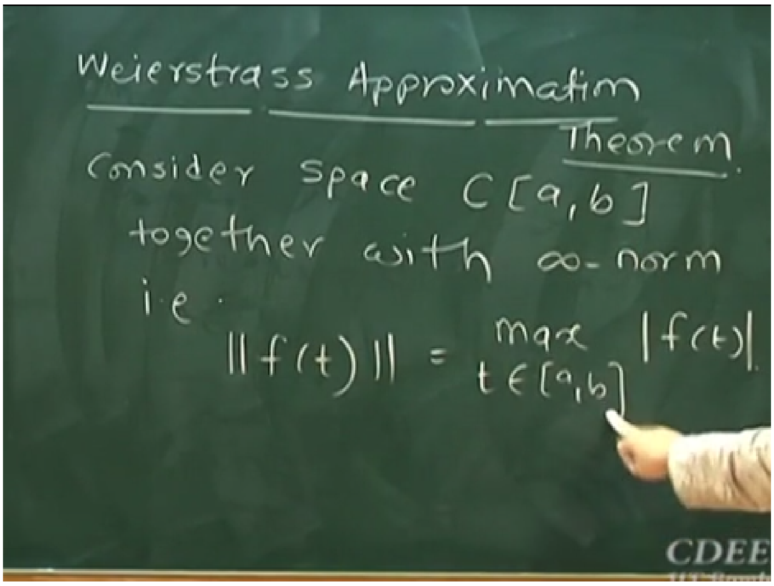

I will tell you what I mean by existence result. It does not tell you how do construct a polynomial approximation. It assures that given a continuous function, there exist a polynomial which is arbitrarily close to the continuous function. Now, what is arbitrarily closeness? You need norm, okay. what is arbitrarily closeness? You need concept of norm. So, now what do we consider here.

We consider the set of continuous functions over an interval a, b together with infinite norm, okay. So this is the space, this is the norm defined on it, this is the norm linear space, okay. Now, Weierstrass theorem tells us that well I will move on to here to complete this theorem statement, so if I give you any epsilon > 0, any degree of accuracy, epsilon will specify how accurate you want the approximation, okay.

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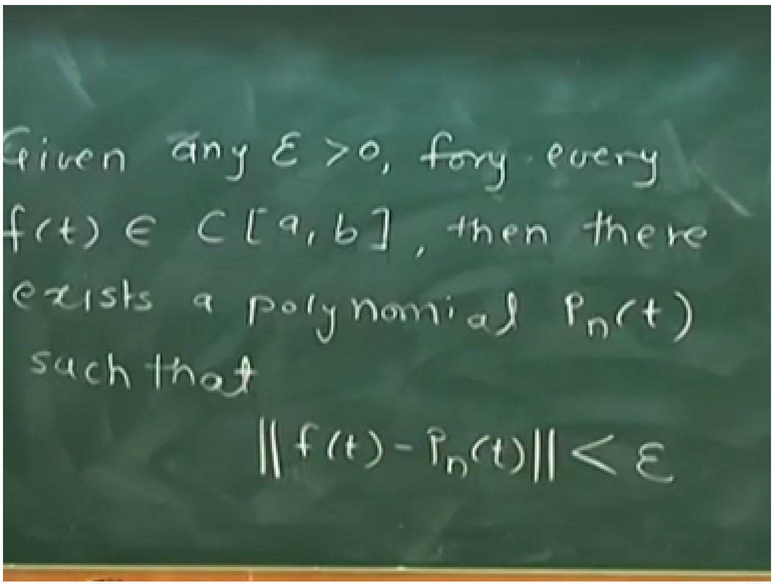

And if I pick up any continuous function ft from c a, b, set of continuous functions over a, b, then there exists a polynomial very, very important result, okay. So what does it say, given any epsilon, you give me the accuracy that you want. How close an approximation you want, okay? You can specify that epsilon a give me any function ft which is the continuous function, okay.

Then, there exists a polynomial approximation, I am going to call this as Pnt and we will be let us say order of the polynomial, okay. Such that ft - Pnt is < epsilon, okay. This is clear. So, what is this norm? This norm is absolute norm. We are finding out difference between maximum of the absolute value, see if I give you a function, let us say sine t, okay this theorem tells me there is nth order polynomial such that sine t - the polynomial absolute of this, okay.

Maximum is the interval will not exceed epsilon. Use specify epsilon, I will construct a Pn, okay. You give me an epsilon; how do you construct a Pn? Is not what is told by this theorem. It just says that there exists, okay. how do you construct that approximation? Well that is a different story. It only assures that there exists a polynomial which is arbitrarily closed.

How do we find out that particular polynomial is not given by this theorem, but it tells you that there exists a polynomial, so which means when I am approximating a transformation, I could use this basic idea could transform a differential equation or transform you know boundary value problem or partial differential equation into some simplified form. We will do it much more in detail.

But I will just give you a very, very simple example, okay. So, do you see parallels between here and here. We are talking about finding out a rational number which is arbitrarily close to a real number and using that rational number for calculations, this is not the real number, okay. Same thing, same idea we are going to do here, okay. The true solution would be a continuous function, okay.

I am going to approximate that continuous function by a polynomial function, why that will help me to solve the problem and you know by transforming the operator by solve the problem in a different way which is easier than the original problem. Well, one simple, this kind of things will hit on later. Let us look at a very, very simple demonstration. See, if I have dx/dt = some f of x, okay.

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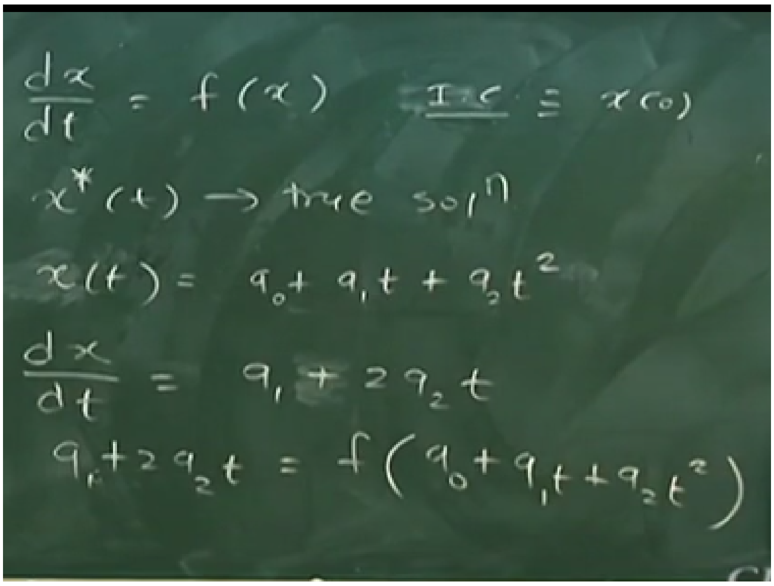

And I have given you initial condition corresponds to say x not, okay. Now what is xt, let us say x star t is the true solution, okay. Is this a continuous function? it has to be a continuous function. It has to be in fact differentiable function, not just continuous. It has to be a differentiable function, so it is a continuous function. Any continuous function can be approximated by a polynomial function.

I propose a polynomial solution which is say Pnt or I will call it xt which is a not + a1t + a2t square. Let me propose a polynomial solution. Now, this is a polynomial approximation. This is not a true solution, but I can substitute it here, I can substitute here and I can say that well I won the approximation solution such that, so what is dx/dt, a1 + 2a2t, right and then, I can substitute this here.

So for any t, I want this equation to hold that is a1 + 2a2t = f of a not + a1t + a2t square, right. I am doing something which is very lieu, we do it much more sophisticated mind there afterwards. I just want to carry some point here. Look, this is a differentiated equation, I started with. I approximated using a polynomial form with unknown coefficients. I do not know a not, a1, a2.

True solution is x star. With what boldness I can do this? I know at a continuous function can be approximated by a polynomial function, okay. I substituted this, what happen, what look like originally a differential equation, now looks like an algebraic equation with unknowns a1, a2, a not. The problem is transform from a differential equation to an algebraic equation, okay.

So, this idea of using a polynomial approximation of a continuous function will be used to transform problems, which are originally, so original operator t was a differential operator, t prime or t cap what you are getting here, looks like an algebraic, so I was talking about you know starting with an original problem, transforming the problem and solving the transform problem.

So, we might actually computationally this is easier to track than this. We might solve this as compare to this. What we get by this approach is the approximate solution, not the true solution remember that, okay. This is the approximate solution, but if we can accept 22/7 in place of pi, we can accept this approximate solution and as long as it is close and you know, your physics is you know preserved in some sense qualitatively, you do not bother too much about the difference between the 2, okay.

So this is how it is going to help us in transforming the problems. So what is used, so Weierstrass approximation theorem as such we never revisit again, but it is the foundation everywhere, you know Weierstrass theorem comes at in a hidden form, it is everywhere because we approximate continuous solutions using polynomials, so somebody asked what is the basis, why polynomials?

Because polynomials are dense, why should I be so much worried about a dense set? You know dense set is something which can pick an element, dense set and it can be as close as possible to the original, you know element in the original set, okay. So, dense set is a special set, so just like set of rational numbers is a special set in the real numbers, polynomials, set of polynomials is a special dense set instead of continuous functions, okay.

So this is a foundation, this is (()) (33:39) result, but it does not tell you how to construct a polynomial approximation. Now, we are going to use 3 different tricks for constructing polynomial approximations. So, there are 3 different ways by which we are going to construct a polynomial approximation. First is the Taylor series approximation, you are familiar with Taylor series expansions, we will just revisit them briefly in the next lecture.

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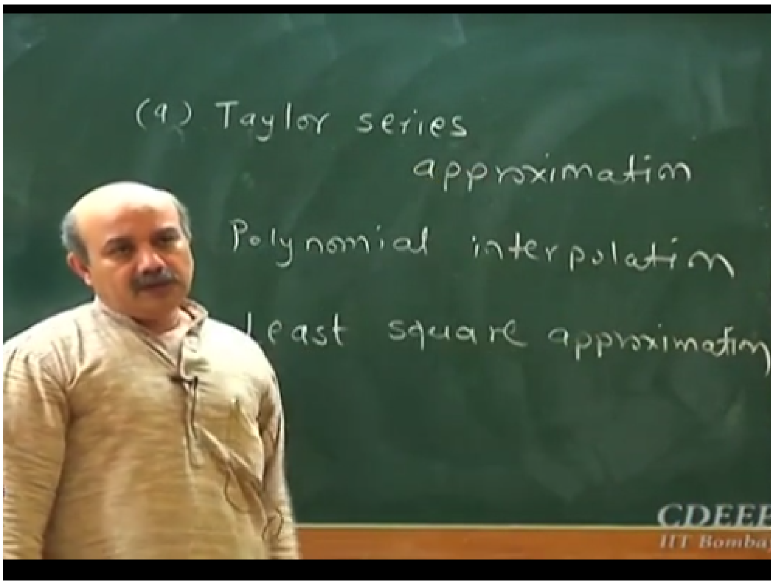

Then, we move on to polynomial interpolation polynomials and the third is the least square approximation. So, if you understand these 3 basic concepts, most of the problem transformations will be clear to you. How the problem is transformed to a computable form, okay. Then, comes how to solve the transform problem, okay, so that is the next part, so till mid sem will be actually working on now this.

Will systematically look at different problems particularly boundary value problems, partial differential equations, nonlinear algebraic equations and all kinds of things were we used these 3 ideas and then transform the problem to a computable form. So, next the 12 lectures are about problem formulation, okay.

You have formulated the problem from physics and you got some problem, which is coming from your courses and transport, reaction engineering, whatever, heat transformer, strength of materials whatever your specialization, so those original problem is coming from there, okay. I want to computer a numerical solution for these problems, so I use all these tricks to transform the problem to a computer bill form and then I solve that computer bill form, okay.

Construct a solution which is approximate numerical solution to the problem, okay. So in the next class, we will start with Taylor series approximations, okay. I will very quickly review Taylor series approximation, what is the basis behind Taylor series approximation, you are aware of only one variable Taylor series, we will move on to multivariable Taylor series, okay. Polynomial functions in n variables, okay.

And then we will look at for example one of the application of the Taylor series would be Newton-Raphson method, okay. Then, we will move on to show that this Taylor series approximation actually gives rise to the finite difference method of solving, boundary value problems, finite different method of solving, partial differential equations, okay and so on or the polynomial interpolations.

So we will develop in the class, method of orthogonal collocations and see how an orthogonal collocation arises from polynomial interpolations and so on. So, all these 3 different approaches give rise to different ways of problem discretization and that will be the center theme for next few lectures.