Biomechanics
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Lecture – 79
Complex Numbers and Quaternion's

Vanakam, welcome to this video on biomechanics we have been looking at some practical examples practical applications measuring body segment kinematics using various methods. We looked at the rotation matrices we looked at Euler angles and we will continue our discussion of angle axis representation and quaternions in this video.

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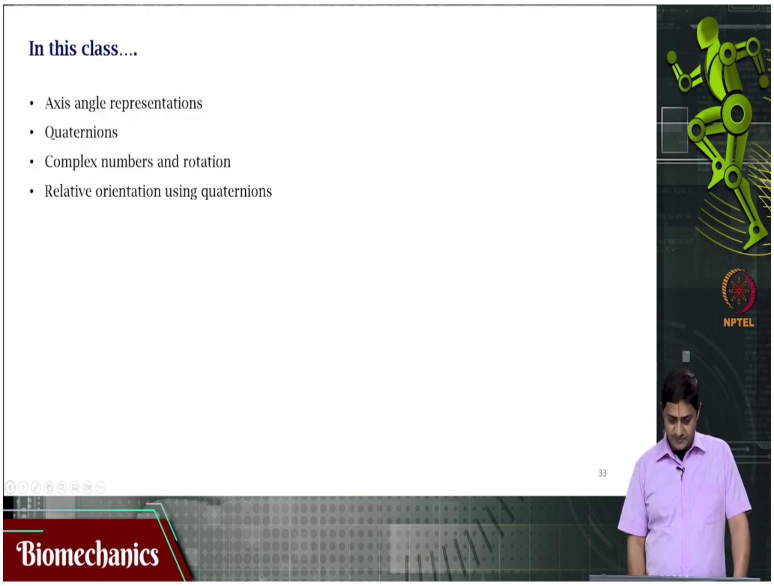

So, in this video we will discuss access angle representations quaternions, how complex numbers and rotations are represented and how to find relative orientation using quaternions.

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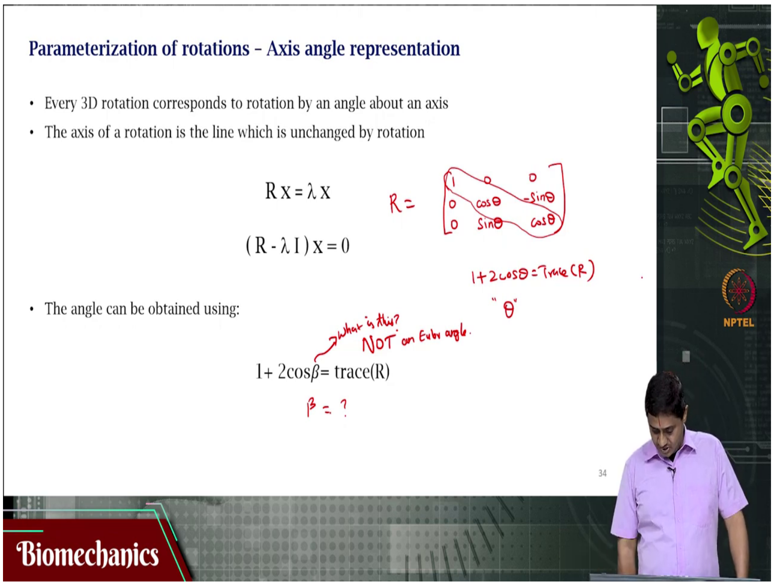

So, every notation in 3D essentially implies rotation about an angle or rotation of or by an angle about an axis. yesterday I gave the example of where this body let us assume that there are three principal axis under discussion for example that axis that axis and that these are the three principle axis. Now let us say this body is undergoing that rotation the axis about which the rotation is happening will have components in this axis this axis and the vertical axis.

And the rotation earlier it was vertical this. Now its horizontal rotation is about 90 degrees the net rotation is about 90 degrees. So, any rotation that happens in 3D essentially is rotation that is happening by an angle about a axis that does not mean that that axis is one of the principal axis and the angle is one of the Euler angles here I am talking about the net rotation that is happening the total rotation is happening about an axis and the angle that it rotates about is the net rotation.

For example I can represent this rotation as a combination of multiple rotations as a combination of three rotations in the three axis. The question is how do I find the axis of rotation? Well that is the particular vector that is the particular line that does not change due to rotation right. For example if this object is undergoing that rotation right. Let us Mark a point here that will undergo a change due to the rotation.

But the axis the set of all points on the axis of rotation will not undergo any change due to rotation. So, suppose R is the rotation Matrix. So, I can perform and eigen value decomposition and represent this as R axis Lambda x. So, the characteristic equation then is

R - Lambda I times x is 0. And the question is what is this x it turns out that the rotation Matrix R will have an eigen value 1.

Now eigenvector that corresponds to eigen value + One is the axis about which the rotation is happening will take some simple examples and check if this is indeed true. Now the question is what about the angle of rotation how much or by how much the rotation has happened? Well that you can find by using this relationship 1 + 2 cos beta is Trace of R. From this find the value of beta that is not really too difficult to find.

So, if you have the rotation Matrix find the trace of that Matrix the trace of that Matrix is 1 + 2 cos beta and find the value of beta from this relationship that is the relationship between the angle of rotation and the rotation Matrix. Remember what is this beta is it the rotation about the x axis or y axis or the z axis. Remember this beta is not an Euler angle this is not an Euler angle this is the net angle of rotation.

This is a net angle of rotation. Now let us check if this is true for the simple case when the rotation is happening about the x axis by an angle theta. So, what would be the rotation Matrix we know what the rotation Matrix is from the previous class that would be 1 0 0 0 cos theta - sine theta 0 sine theta cos theta this is the rotation Matrix R right. In this case what is the trace of this Matrix.

The trace of this Matrix is the sum of these three elements is it not that is 1 + 2 cos theta equal to the trace of the Matrix R is it not. Now it is very clear that the rotation angle is the rotation angle is theta how do we know this? Just compute this and you will know that this rotation angle is theta because you just have to take the you just have to look at the rotation Matrix and it is immediately obvious that this is a rotation that is happening about the x axis and the angle of rotation is theta right.

Now of course this is the simple case of the rotation happening about one of the three principal axis right this is the case when the rotation is happening about the x axis. But the question is what would this be when the rotation is a composite rotation. That is the rotation has components in all the three directions in all the or about all the three axis there is a component of rotation that has happened.

Now what is the angle of rotation? Remember once again this angle of rotation that we are discussing for a composite rotation Matrix is the net rotation you cannot compute the Euler angles from this. This is not an Euler angle but the net rotation that has happened for example I gave the example of this phone being rotated right. So, this is the phone and I am rotating it. Now, because the three axes have different components for this rotation or for the axis of rotation.

The net rotation might be some angle say 80 degrees or 90 degrees in this case but the Euler angles for each of the three are the Euler angles about each of the three axes will not be that value. So, you cannot find the Euler angles by looking at the net rotation. Here what we are looking at is the net rotation and the axis of rotation this axis of rotation may have components about the three principle axis something to keep in mind that there is a fundamental difference between this axis and this angle and the Euler angle.

So, right and the principal access so, this is a simple concept that you can use to find the actual axis about which the rotation is happening and the angle of the net angle of rotation something to remember and keep in mind.

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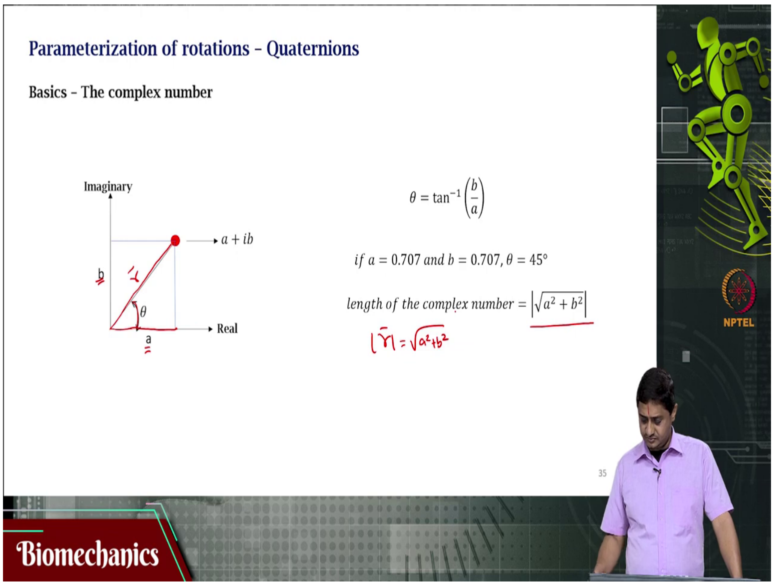

Now let us use quart onions let us discuss the concepts of Quaternians and how we can use quaternion s to parameterize rotations. Now this is something that we know from high school I have and I have a complex number say a + ib. Now this has a component a along the x axis under component b along the y axis. Now I can represent this also as a vector for example that vector and an angle that rotation it is as if I have taken this Vector along the real axis and rotated it by that angle theta right.

So, let us say I know a and b I know the values a and b how would you find theta from high school mathematics I know that that is actually tan inverse of b by a we know this. Now let us suppose a is 0.7n not seven and b is 0.7 not 7 then theta is 45 degrees the what is the magnitude of this that is the square root of the sum of the squares but not this is the magnitude that is R right. This is also something that we know from high school math.

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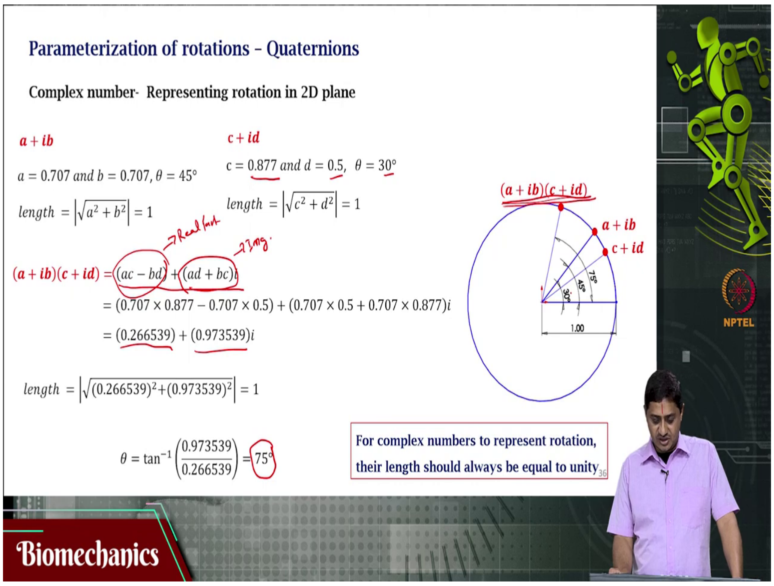

Now let suppose that I have 2 complex numbers a + ib and c + id where a is 0.797 b is 0.707 theta is some 45 degrees c is 0.877 and d is 0.5 and theta is sum 30 degrees. So, that is for this a and b the corresponding theta is 45 degrees that is when a and b are 0.707 the theta is 45 degrees and when c and d are respectively 0.877 and 0.5 theta is 30 degrees right. Now what is the length of these 2 complex numbers are these 2 vectors that is actually one because square root of a square + b squared is one in this case.

Now let us say that I am multiplying these 2 complex numbers a + ib times c + i d will give me a another complex number with real part ac – bd and an imaginary part ad + bc I perform this multiplication and I find this is actually the real part is actually 0.266 and the imaginary part is actually 0.97 4 approximately and the length of this multiplied complex number the new complex number that is the product of the 2 original complex numbers is also one.

That is that number in the unit circle that is this number on the unit circle and you realize the theta for these 2 imaginary and real parts turns out to be 75 degrees. What this means is that multiplying these 2 complex numbers essentially implies that I am adding these 2 rotations one after the other first I am doing a + ib then I am doing c + id resulting in this composite in this rotation resulting in this rotation that is the sum of the 2 individual rotations 30 degrees and 45 degrees.

When you sum them the net rotation the total rotation would be 75 degrees for this to happen there is a condition there is a precondition that is the length should always be one only when the length is one you can actually represent or you can use complex numbers to represent rotations something that we need to keep in mind.

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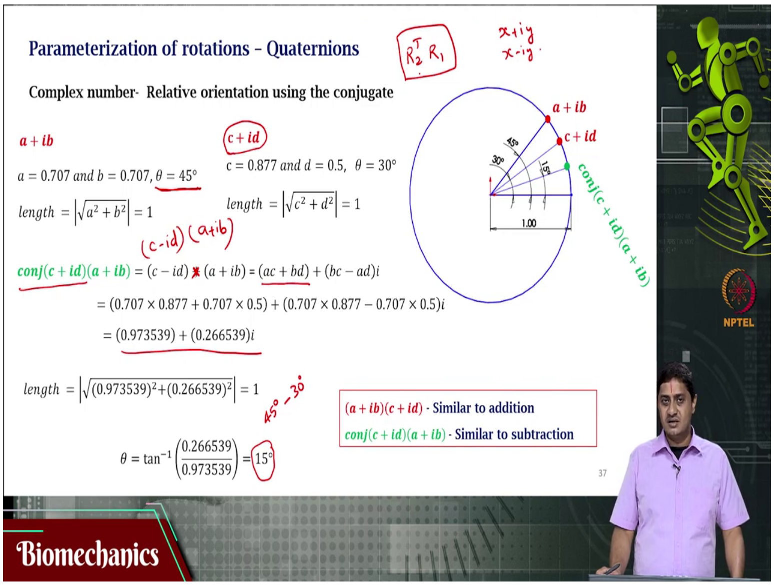

Now let us look at how we can use the complex number representation to find relative orientation remember in one of the previous videos we looked at finding the difference or relative orientation between 2 joints as for example something like R 2 T times R1 something like this right remember. Now what is this R 2 and R 1 these are rotation matrices. The question is, is there an equivalent in the complex number space for this transpose right yes that is there that is the complex conjugate.

Now if I have a complex number x + i y what is its complex conjugate we know that that is actually x - i y that is the complex conjugate. Now if I am interested in finding the relative orientation between these 2 points that is a + ib and c + id where a was a and b where 0.707

and 0.707 respectively that is what we saw in the previous slide that is representing theta as 45 degrees and c + id where c is 0.877 and d is 0.5 representing theta equal to 30 degrees.

Now remember both of these have length one suppose I find the complex conjugate of this I find the complex conjugate of c + id and I multiply it with a + ib here it shows actually as plus that is actually a multiplication. So, this is actually c - id times a + ib that that will actually give you another complex number with real part ac + bd and imaginary Part bc - ad remember that this is different from what we did previously.

Because here I am not multiplying a + ib and c + id rather I am multiplying c – id and a + ib when I do that I will get this complex number and if I find the theta as tan inverse of b by a will get 15 degrees. Now what is this 15 degrees this is the difference between the original the first angle 45 degrees and the second angle 30 degrees remember we performed this R 2 transpose times R 1.

Now the transpose is going to help you find the relative orientation remember. So, if I have a + ib times c + id then that is essentially similar to addition and if I multiply the complex conjugate of c + id with a + ib that is similar to subtraction. Of course this will work only when the magnitude is one or when all of these numbers are on the unit circle otherwise this will not work but this is an interesting formulation.

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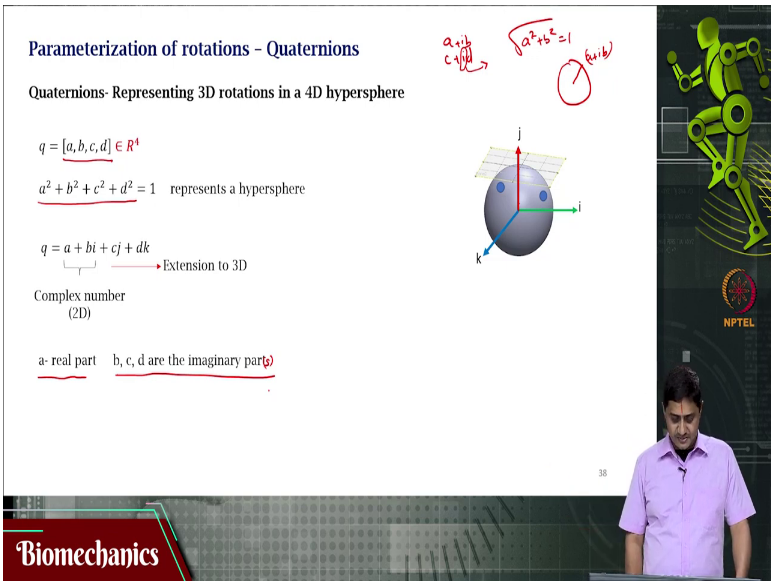

Now similar to what we had a + ib, c + i d what is this I what does this I represent it just tells you that what follows this I is the imaginary part this is a complex number with an imaginary part I that is what it tells you right. Now similar to that let us suppose I had a vector q a b c d like this right such that a squared + b squared + c squared + d squared is one remember in the previous discussion we just said that a square + b squared is one or rather the square root of s square + b square is one but the square root of a square + b squared if it is 1 then a square + b squared is one so, right.

Now this a squared + c b squared + c squared + d Squared is such that that it represents a hypersphere in 4D space immediately what we realized is that 4D that is not intuitive that is the issue with quaternions. That the moment the instant you say quaternions they are not immediately intuitive you are not having a feel for this unlike Euler angles for which you have a field you have an idea of what is going on.

But in quaternion s because this is in 4D hypersphere you do not have an idea or you do not have an intuitive understanding or feel that does not mean that this is not as useful or as critical this is a very useful method but with some difficulty or with a lot of difficulty in getting a intuitive feel for this. So, all these points remember when we represented the complex number we said that is we said that that is the number a + ib.

Now this is on the circumference of the unit circle right correct. In this case for a quaternion this will be a + ib + cj + dk each of this will be such that this quaternion will be a point on the surface of this 4D hypersphere like you are having this for the unit circle this is what would happen when you have a 4D hypersphere and all of these coordinates are points on the surface of this hypersphere.

So, in this case a alone is the real part and b c d are imaginary parts. So, this is the extension to 3D of this complex number representation.

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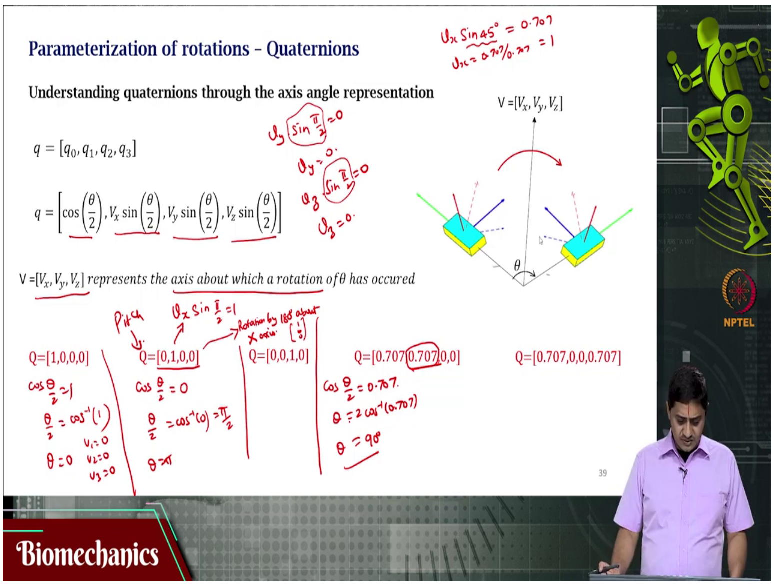

Now in the beginning of this video we started discussing axis and angle representation we said that that I can actually find the net rotation and the axis about which the net rotation is happening remember right this is called axis angle representation angle axis representation depending on how you call it. Now let us suppose I have a quaternion q naught q 1 q 2 q 3 I can represent this as cos of theta by 2 V x sin theta by 2 V y sine theta by 2 and V z sin theta by 2.

Now in this case it turns out the rotation the angle of rotation is actually theta and the axis about which this rotation has happened is V x, V y and V z this is the axis or these components in x, y and z constitute together constitute the vector about which this rotation has happened or this is the axis about which this rotation has happened similar to what we discussed. Now consider these five sets of quaternions five quaternions that are given.

Find the axis about which the rotation is happening and the angle by which the rotation has happened how do you do this well we just said how to do it. We will start with the first example what it says is cos theta by 2 is equal to one what does that say about theta well that says that theta by 2 is cos inverse of 1 is it not when is this happening this is true only when theta is 0 is it not.

What about V 1 V 2 V 3 well V 1 is 0, V 2 is 0 V 3 is 0. This is a rotation of 0 degrees about none of that that is there is no rotation essential about none of the three axis right because none of the three axis have any representation all of them are 0 0 0s right remember V x V y

V z represents the axis about which the rotation happens that is the last three components right the first one is the angle.

Now let us look at the second 1 0 1 0 0 0 1 0 0 what does that say cos theta by 2 is 0. Now that means theta by 2 is cos inverse of 0 what is that that is pi by 2 is it not that is pi by 2 R theta is pi it is a rotation of 180 degrees. But about which axis to find that I can find out that V x sine theta by 2 sine Pi by 2 is equal to one is it not. Now that will give me the value what is sin Pi by 2 sin Pi by 2 is 1 is it not.

So, this is the axis also is is there any component of rotation that is happening about these axis probably no because V y sine theta is Pi Pi by 2 is equal to 0 it says that means although sine Pi by 2 is 1 V y must be 0 for this to be 0 is it not likewise V z sine theta is pi Pi by 2 is equal to 0 how do I know because this is the quotient 0 1 0 0 is the quotient the third and fourth components are 0's.

So, V z sin Pi by 2 is 0 but sin Pi by 2 itself is one that means that V z must be 0 that is what it says. So, this quaternion 0 1 0 0 is a rotation by Pi radians are 180 degrees about the x axis about the x axis right is also called as pitch x axis that is 1 0 0 also called pitch or I will write here this is called pitch right. Now let us look at this case of q equal to 0.707 0.707 0 0 again these numbers are conveniently taken.

So, that it is not too complicated for you the actual problems that you will come across in real life and probably in the exam need not be so, simple ok but let us continue. So, q is 0.707 0.707 0 0 no cos theta by 2 is 0.707. So, what is theta by 2 or other theta is 2 times cos inverse of this value is it not we know what this value is cos inverse of 0.707 is actually 45 degrees 2 times 45 degrees is actually 90 degrees theta is the angle of rotation is 90 degrees.

Now what about this is V x sine theta by 2 what is theta theta is 90 degrees that is sine 45 or rather Pi by 4 is equal to 0.707 but we know this value is 0.797 is it not how do you know from high school math right. This will give me V x as you know 0.797 by sin 45 rather one something like that what this means is that the rotation is purely about the x axis only and the angle of rotation is 90 degrees right.

So, here we have taken and solved 2 three of the quaternions you can try on your own for this and this. So, essentially this quaternion what does it represent in this case the axis angle representation represents a rotation of theta about an axis of this body.

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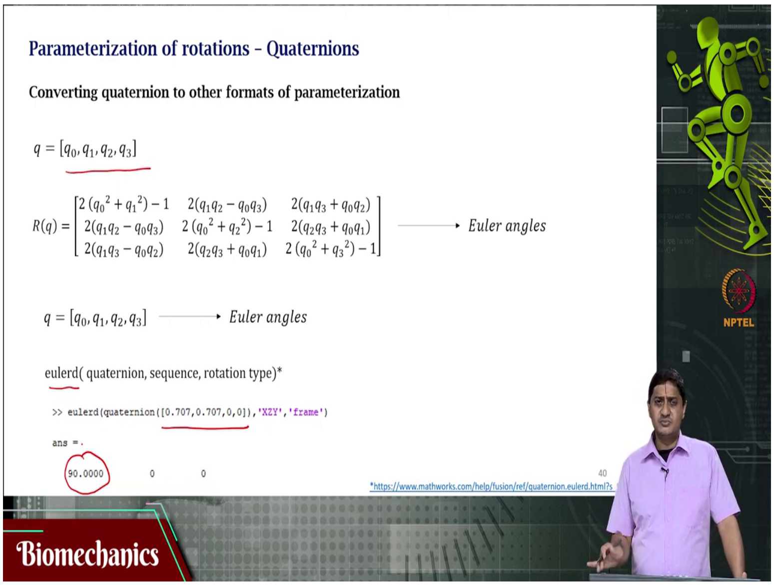

Now how do I convert this quaternion to other formats of parameterization such as rotation Matrix such as Euler angles why do I have to do that do I have to do that the answer is yes we have to do that. But why because as we just. Now discussed there is no intuitive understanding of quaternions I need to have an intuitive understanding. So, to have an intuitive understanding I need to convert it to things that I understand well and those are Euler angles and rotation matrices.

And so, suppose I have this quaternion q naught q 1 q 2 q 3 I can convert this into the rotation Matrix using this formula using this formulation. I am not performing the derivation showing that this is indeed true but you can check whether this is true this is true this is true. And then I can convert from this rotation Matrix to other angles there is a way to convert directly from quantities to Euler angles by using this function Euler d.

Of course I need to know the sequence of rotation I still need to know the sequence of rotation suppose this was the quaternion remember this is the problem that we solved in the previous slide suppose is it not that is what we solved here that is that problem. So, 0.707 0.707 00 right that is what we solved I am converting that to Euler using the Euler d formula right.

I will get the angle of rotation as 90 degrees as we found previously right of course I am taking some simple examples that are having some 0.s right the real quaternion that you measure are the measured quaternions from IMU's are from your electromagnetic tracking systems will not have such simple numbers and they will not be. So, intuitive you will need to convert this to Euler angles.

Of course for this you need the Matlab function of course this is the Matlab function there are maybe equivalents in other packages or languages.

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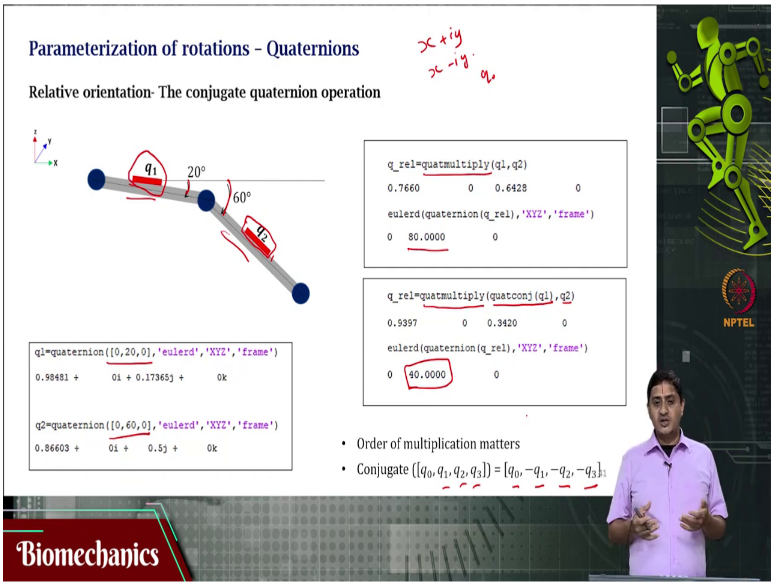

Now like we discussed previously how I can find the relative orientation we just showed an example of how we can find the relative orientation. We said that that you can do by finding the complex conjugate of the second one and then multiplying it with the first complex number something like that right. So, suppose I have 2 quaternions q 1 0 20 0 for example and another quaternion.

Quaternion corresponding to this Eider angle and quaternion corresponding to that Euler angle for example and I am interested in converting that is these 2 angles I am having 60 degrees and I am having 20 degrees and I would like to know the relative angle between these 2. This is the kind of problem that we will encounter in real life most likely this is the problem that we are interested in solving right.

Of course the numbers given are simple 20 and 60 obviously you know that the relative angle between this let us say this is the proximal segment and this is the more distal segment is 40 degrees I know this. So, I try to convert this to quaternions first and then I multiply the 2 if I simply multiply the 2 it is going to add like we did in the complex number case if I simply multiply the 2 quaternions using quad multiply of these 2 quaternion q 1 and q 2.

I am going to simply get the sum of the 2 angles but I am interested in finding the relative orientation to do that what we said in the unit circle case is you find the complex conjugate of the quaternion and then multiply right. If you do that if you do the quaternion multiplication by finding the complex conjugate of one of them and then multiply it with the other you will get the angle as 40 degrees.

Of course the order of multiplication matters right also remember that the like you have if I have a complex number x + iy. Let us its complex conjugate is x - iy. Now for the quaternion q naught q one q 2 q three the complex conjugate is q naught - q 1 - q 2 - q 3 why because the only real part in that is q naught q 1 q 2 q 3 are all imaginary parts right. So, if I use complex conjugates and multiply using quad multiply I will find the relative orientation between 2 segments right.

Now let us suppose that I use one IMU to find one quaternion that measures the orientation of this segment and another IMU that measures the orientation of the first segment both of them are outputting in quaternions if I want to find the relative orientation I you know find the complex conjugate and then multiply I will get the relative orientation this is that simple.

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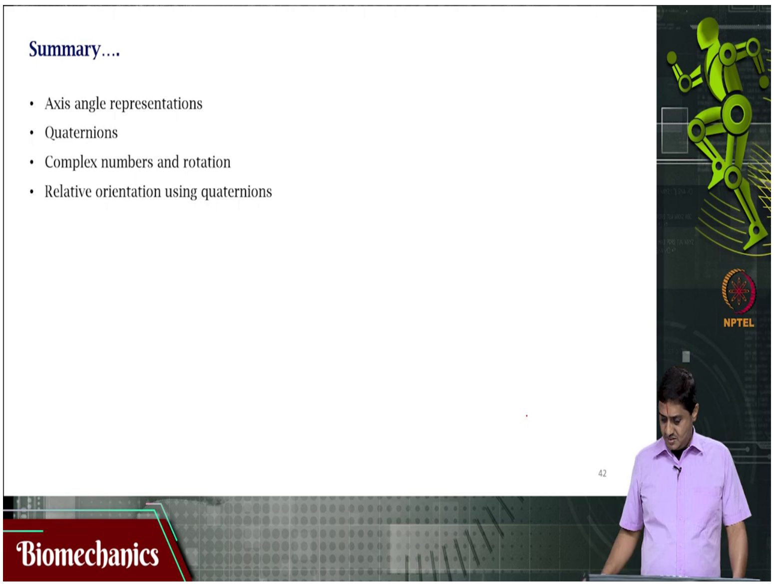

So, in this video we looked at axis angle representation we looked at how we can find the axis of rotation and angle of rotation from a rotation Matrix. That is the axis about which a composite rotation happens and the angle by which this composite rotation happens this is the net angle right that we looked at for the rotation Matrix case. Then we introduced the idea of quaternions give some simple examples of complex numbers and rotations and how to find relative orientation between complex numbers right.

And then we introduce the idea of quaternions and axis angle representation for a quaternion. And then we used quaternion principles to discuss relative orientation between 2 segments using quaternions right essentially to First find the complex conjugate of the quaternion and then multiply using quaternion multiply for example in Matlab right. So, with this we come to the end of this video thank you very much for your attention.