Biomechanics
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Lecture-61
Kelvin Model

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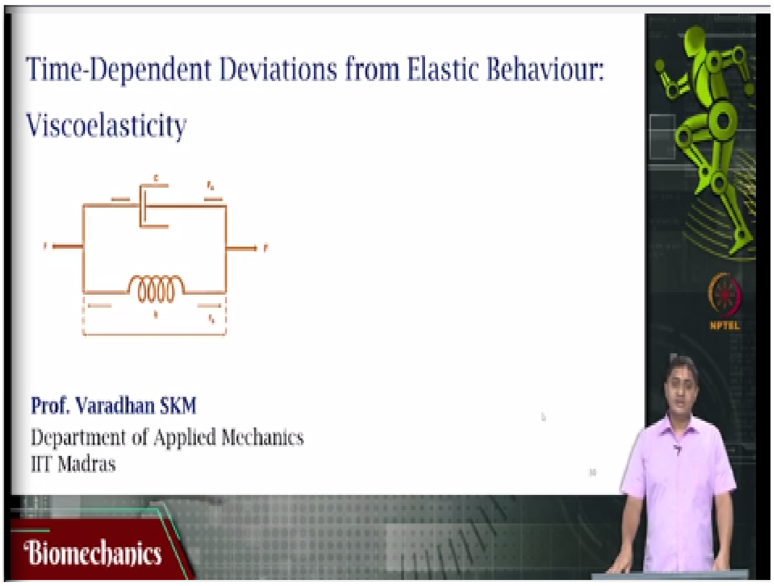

Vanakam, welcome to this video on biomechanics. We have been looking at a time dependent deviations in elastic behaviour of biological materials. That is viscoelasticity; specifically we looked at a couple of models of viscoelasticity namely the Maxwell model and the Voigt model. And we looked at creep and stress relaxation functions for these 2 models in the previous videos.

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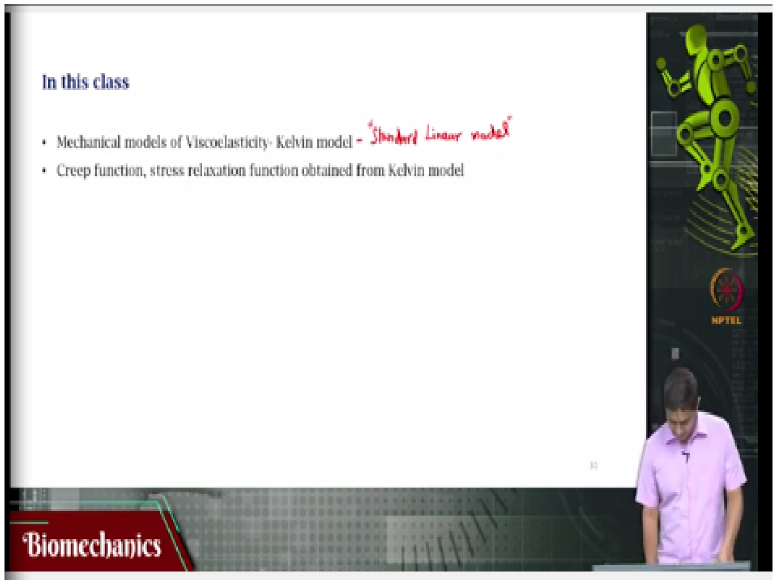

Now in this video we will look at another model of viscoelasticity called as the Kelvin model, this is called as the standard linear model. And we will derive the creep and stress relaxation responses for the Kelvin model, using the Kelvin model we will derive that creep and stress relaxation responses.

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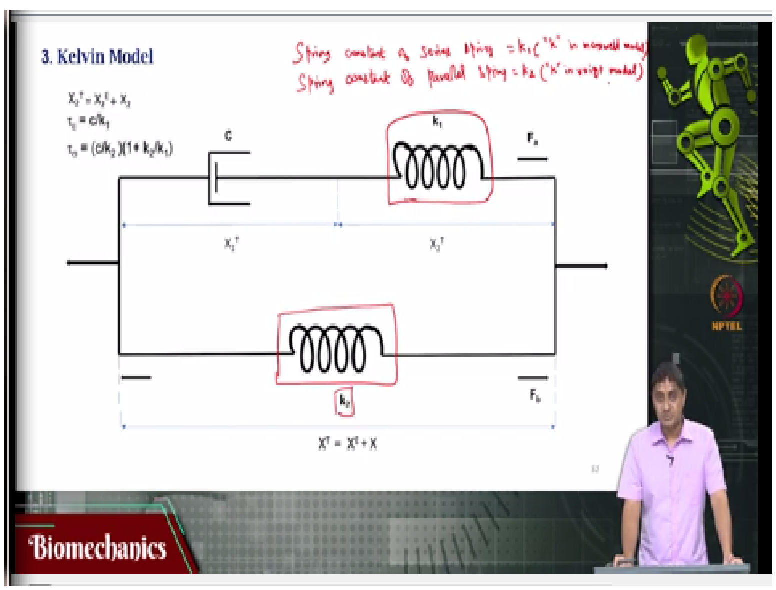

Now what is the Kelvin model? The Kelvin model essentially incorporates both the Maxwell and the Voigt model or it is a combination of Maxwell and Voigt model. So, there are 2 springs, one spring is the series spring that is that spring, I am going to call the spring as the series spring throughout this lecture I am going to call that spring as series spring and it has the spring constant k 1. And there is one more spring that is parallel to the dashpot pot that I am going to call as the parallel spring; this is parallel to the dashpot pot as in the Voigt model.

And it is having a different spring constant k 2, it is spring constant is not k 1 but rather k 2. And you have the dashpot pot that is having one series spring and one parallel spring, the combination of these together gives rise to this Kelvin model or the standard linear model. So, let us remember spring constant something that we need to remember throughout the lecture the spring constant of series spring k 1 and the spring constant of the parallel spring is k 2.

Let us remember this k 1 was called as k in Maxwell model and this k 2 was called as k in Voigt model. So, we are trying to relate the current Kelvin model with the Maxwell and Voigt model which we already know. What was called as k in Maxwell model is now called k 1, what was called as k in Voigt model is now called k 2 and there are these 2 springs one in series and one in parallel to the dashpot pot.

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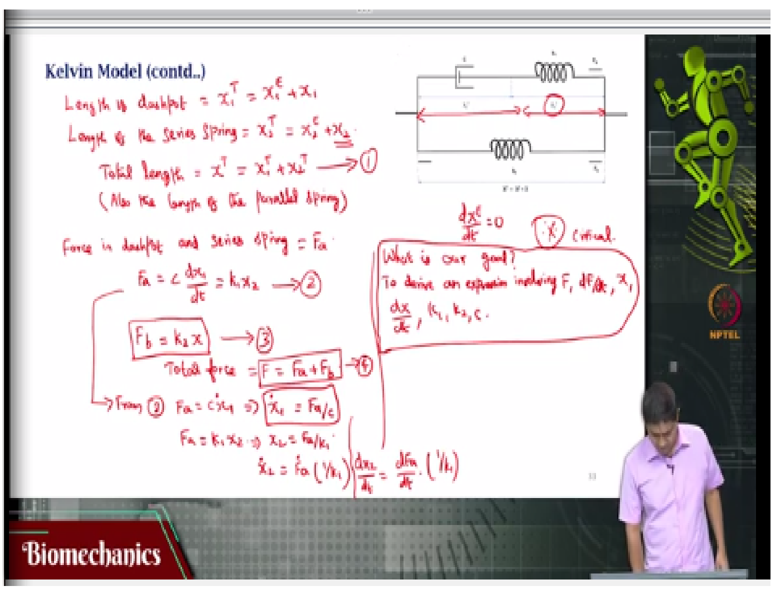

Now the length of the dashpot pot that is x 1 T, what is it? It is x 1 E + x 1. The length of the series spring is that x 2 T, what is it? That is x 2 E + x 2. Now what is the total length? The total length is sum x T and that is what is it? That is x 1 T + x 2 T, it turns out this total length is also the length of the parallel spring, this is also the length of the parallel spring, you need to remember this, I am going to call this equation as equation 1.

Now what is the force in the dashpot pot and the series spring? I am going to call that force as sum F a force in dashpot pot and series spring. This force is the same in both the dashpot pot and the series spring that I am going to call as F a. What is the value of this F a? I can write that, how do I know? Because I know the characteristic of the dashpot pot and the spring I can write the expression for F a in terms of C and k 1, I can write this.

And remember F a is the same for both the dashpot pot and the series spring, for the parallel spring it is different but we will come to that in just a little bit. That is c dx 1 by dt, x 1 is the dashpot pot length, c dx 1 by dt. Remember, dx E by dt is 0 regardless of whether that is for a spring or for a dashpot pot dx c by dt is 0 that is why it is an equilibrium length. For the spring the equilibrium length is assumed to not change that is it is resting length for a given spring the resting length is a constant regardless of the passage of time.

So, if I differentiate that with respect to time I will get 0. So, essentially this is dx 1 t by dt but that is essentially dx 1 by dt, this is dx 1 by dt. And that is also k 1 x 2, remember x 2 is the deformation in the series spring and it is spring constant is k 1, so the force is k 1 times x 2. Please pause the video and check if your understanding or if what I am saying is clear, check if this is correct, I am going to call this equation as equation 2.

Now what is the force in the parallel spring? Is that the same as the top combination? Is that the same as the combination of the dashpot pot and the series spring? The answer is no, because it is the length that is the same not the force. So, that is some mother force F b and what is it? That is dependent on the parallel spring constant k 2 and the definition x, that is k 2 x not x 2, k 2 x because that is the total length.

Once again take some time and check whether this is correct. For the top spring or for the series spring the force is k 1 x 2 because it is deforming to x 2 value, whereas the parallel spring deforms to the value x, so the force developed in the parallel spring is k 2 times x, not k 2 times x 2. Please note this, there is a difference and this difference is very critical for the rest of the analysis, I hope this is clear. If it is not clear I request you to again watch both the Maxwell video and the Voigt video and check this video until this point, you will definitely understand.

It is not k 2 x 2 but rather k 2 x and I am going to call this equation as equation 3. Now what is the total force? The total force is the sum of the 2 forces F a and F b and that is F = F a + F b. Now it is somewhat confusing as to why are we doing this? What is our goal? Let us write it down, many times it is very useful to write down your goals, when you write down your goals you have made a commitment with yourself, it is like a contract that you have made with yourself.

And because there are so many things that are happening I would like to write down the goals of this derivation, not my life goals, what are my goals or what is our goal in this exercise? To derive an expression involving F dF by dt x dX by dt k 1, k 2, c. To derive a relationship between the forces and deformations involving only F dF by dt or F, dF by dt x, dX by dt k 1, k 2 and c, this is all I want, I do not want to have other things in my expression.

I want to have a relationship between force and the deformation, I want a relationship between F and x's, I do not want in particular F a F b, x 1 x 2, this is something that I do not want, let us remember this, this is our goal. Now let us take equation 2 and do some rearrangement. What it says is F a is cx 1 dot, remember what it says, it does not say c x dot but rather c x 1 dot, so this implies x 1 dot is F a by c. What it also says is F a is k 1 x 2.

Now if I differentiate this expression F a = k 1 x 2 or rather I can first write this as x 2 = F a by k 1. Now if I differentiate this with respect to time I will get x 2 dot is F a dot times 1 by k 1. For clarity let us write this in the expanded form as dx 2 by dt is dF a by dt times 1 by k 1. We have derived this or something similar to this in a previous video, I hope this makes sense, I hope there is no confusion about this part.

Now let us reiterate our goals, what are our goals? Our goal is to have an expression between F and x involving only k 1 k 2 c dF by dt and dx by dt but not F a, F b, x 1, x 2. k 1 and k 2 are okay but x 1, x 2, F a, F b are not okay, this is our goal remember, this is our goal, very, very critical to keep your goal in your mind as to why you are doing, what you are doing? This is very, very critical, always keep moving towards your goal, this is our goal. We may have to revisit this slide from time to time because we have to take this some of these equations to the next slide.

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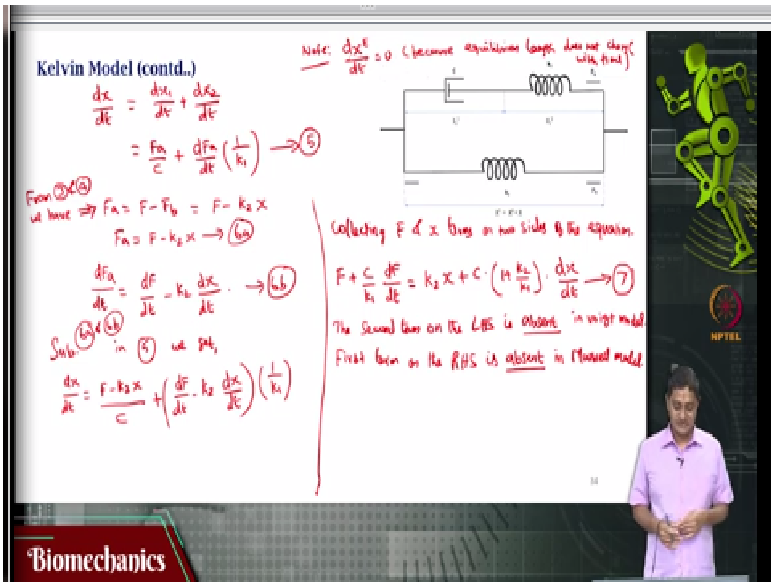

Now what is dx by dt? Because x is x 1 + x 2, so what is dx by dt? dx by dt is dx 1 by dt + dx 2 by dt or x 1 dot + x 2 dot, in the previous slide we derived what is x 1 dot and what is X 2 dot, let us write it out. Once again let us remember; let us note dx E by dt is 0 because equilibrium length does not change with time. Now let us go back and find out what is dx1 by dt and what is dx 2 by dt and write it up.

Dx 1 by dt is this, that is F a by c let us write it up, that is F a by c and dx 2 by dt, so this is equal to x 2 dot is F a dot times 1 by k 1 which is equal to dx 2 by dt or which implies rather that is a line between them that is dF a by dt times 1 by k 1, is this clear? I hope till now things are clear, if it is not clear again I am requesting you to pause the video and check the flow of the math, so far I have been presenting this a little patiently, so that you will be able to catch up.

But if you are not, pause the video go back and check because this is a very critical topic, not just from your exam point of view but also from your understanding point of view, for everything this is a critical topic, so do check this. Now a question is what is the relationship between F, F a and F b? Because in this equation it seems to me like I am only having F a, what is the relationship between F a, F b and F? That is the question.

Well, I know that, that is F a is F - F b, how am I saying this? Well, that is that is here that is here, I am going to call this equation as 4. F is F a + F b that means F a is F - F b but what is F b? We know what is F b, F b is that k to x, it is not k 2 x 2, it is k 2 x. So, I can now write this as F - k 2 x, F a is F - k 2 x, how do I know this? This is from equation 3 and 4, I have this equation, F a is F - k 2 x.

Now I can substitute for F a in the above expression for dx by dt, I am going to call this as equation 5. Now I can substitute for F a in equation 5, what is it? But before that let us try to find out what is dF a by dt? Because here I have F a then what would be dF a by dt? That would be dF by dt - k 2 dx by dt, this is dF a by dt, I have F a and I also have. Because if I want to substitute for F a I also need to find out what is dF a by dt because both F a and dF a by dt are there in equation 5.

I need to find F a and I need to find dF a by dt to substitute in equation 5, I am going to call this as equation 6. Now substitute equation 6 in equation 5, also substitute the expression for F a, what do we get? For convenience I am going to call this as 6a and 6b, I am going to call this as 6b, now substitute 6a and 6b in equation 5. That is substitute the values of F a and dF a by dt in the expression for dx by dt, let us write it out, what is it?

Dx by dt is F a by c that is F - k 2 x divided by c + dF a by dt, what is dF a by dt? That is there in equation 6b. That is dF by dt - k 2 dx by dt the whole thing multiplied by 1 by k 1, is this clear? If it is not clear, you need to go and check the algebra, please pause the video again and check whether the algebra works out, it is correct but can you check, you write this in your own notebook and check whether this is correct and whether you are convinced, you need to be convinced.

I do think that we have reached our goal, what is our goal? Let us go back and recheck our goal, my goal is to have an expression connecting force, force rate, deformation, deformation rate k 1, k 2 and c only these I want, I do not want F a, F b, x 1, x 2. I want F dF by dt, x dx by dt, k 1, k 2, c this is what I want. Only these things I want such that I am able to get a relationship between the force applied and the deformation or the deformation applied and the force, this is what I want, I do not want F a, F b, x 1, x 2.

This is our goal, it seems like we have reached our goal. Because in this expression I have dx by dt is F - k 2 x by c + dF by dt - k 2 dx by dt times 1 by k 1, there is no x 1, there is no x 2, there is no F a, there is no F b. So, I think I have reached my goal except that there is some challenge. The challenge is F is present on everywhere and in particular x is present everywhere, dx by dt is present both on the left hand side and the right hand side, x is present on the right hand side.

So, it seems like I have to do a little bit more Algebra, I have to collect terms; I have to collect F terms and x terms separately and then write it out. It is better that I do it in this slide itself, so that I can take that to the next slide otherwise it will be a little confusing, so let us try to do that. So, collecting F and x terms on 2 sides of the equation, so that is what I am doing. I have achieved my goal but it seems like a very clumsy confusing equation, so I am trying to make it a little bit more need by collecting terms on 2 different sides of the equation.

So, that means that I am going to do some algebra. So, after some algebra what I get is the equation 7, this is what I get. Once again please pause the video and check if the algebra is correct, I have simply rearranged the previous equation such that F and dF by dt are at the left hand side of this equation and x and dx by dt are on the right hand side of the equation. It is simple algebra but I have skipped some steps or I have just written the answer.

Please check if this is correct, I know that this is correct but please do check. Now it seems like there is some similarity between the Voigt model and Maxwell model with this expression. It seems like there is some familiarity, when I see this there is some familiarity, what is the familiarity? Well, the second term on the left hand side is absent in Voigt model and the first term on the right hand side is absent in the Maxwell model.

So, that is some familiarity, it seems vaguely familiar to the Voigt model and the Maxwell model but it is not the same, there is some similarity but it is not the same. What is absent? The second term on the left hand side is absent in the Voigt model, the first term on the right hand side is absent in the Maxwell model, that is the similarity.

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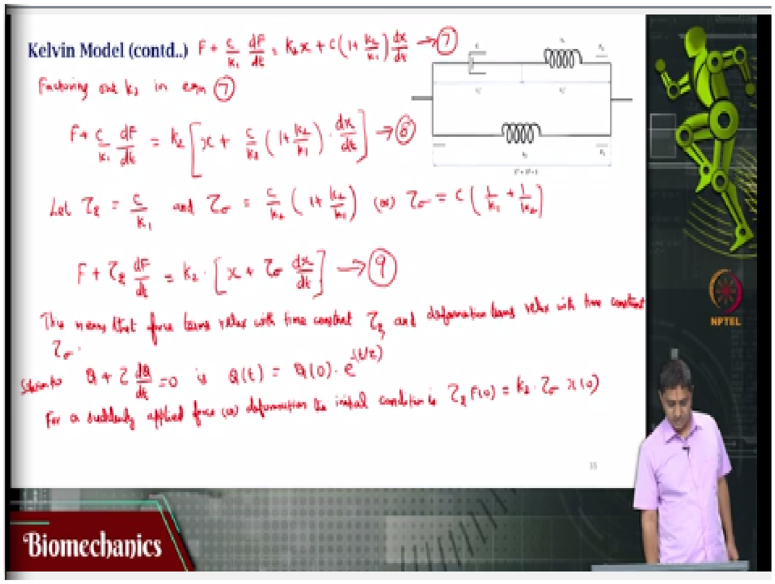

So, for convenience I am going to write out the previous equation in this slide, so that I can continue with the derivation. So, that is our expression, equation 7 that we got in the previous slide I have simply rewritten that for convenience. Now I would like to factoring out k 2 in equation 7. That is why I wrote equation 7 because if I do not have equation 7 I do not know how to factor out k 2. So, that will give me, now if I factor out k 2, this is the expression that I am getting.

Now let me define some time constants tau epsilon and tau sigma. What these are? Let us see, define or let tau epsilon be c divided by k 1 and tau sigma be c divided by k 2 times 1 + k 2 by k 1 or tau sigma is c times 1 by k 1 + 1 by k 2. Now substitute this values of tau epsilon and tau sigma in the expression force and deformation are in equation 8, what would I get? I would get F + tau epsilon dF by dt = k 2 times X + tau sigma dx by dt, I am going to call that as equation 9. What this means?

This means that force terms relax with time constant tau epsilon and deformation terms relax with the time constant tau sigma, this is what it means. Also I note that this equation is similar to the linear differential equation, the following linear differential equation which is Q + tau dQ by dt = 0 is Q, what is the solution? Solution to this equation is Q of t is Q of 0 times e power - t by tau.

So, for suddenly applied force or deformation, what is the initial condition? That is tau epsilon F of 0 = k 2 times tau sigma x of 0, this is the initial condition, k 2 times tau sigma x of 0.

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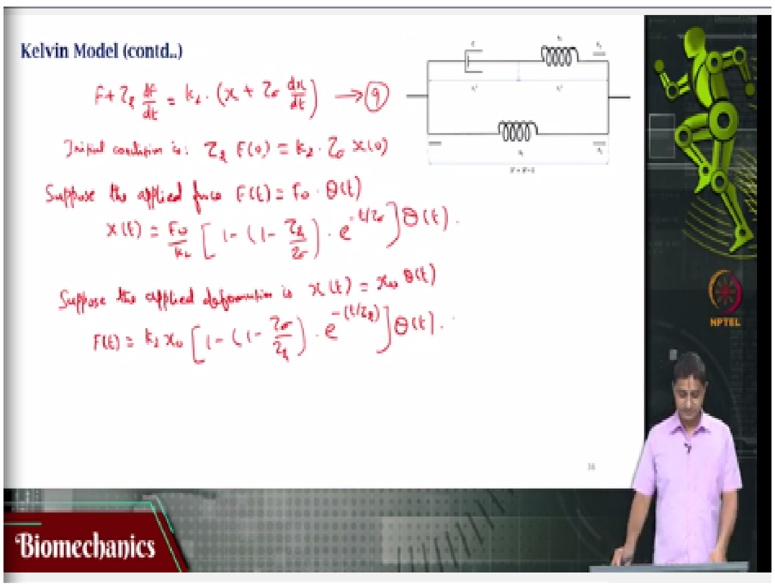

Now let us write out this expression between force and deformation again, so that we can continue the discussion. That is F + tau epsilon dF by dt = k 2 times X + tau sigma dx by dt, this is our equation 9 that we got, this is the expression between forces and deformation that we derived in the previous slide. And we also said that initial condition is tau epsilon F of 0 is k 2 times tau sigma X of 0.

Suppose I apply a step change in force, applied force is some F of t is what? Sum F 0 times theta of t, I am applying a step change in force, what would be the definition? That would be X of t is F naught by k 2 times **-** 1 - tau epsilon by tau sigma the whole thing multiplied by e power -t by tau sigma and this whole thing is multiplied by theta of t. So, this will be the deformation when a step change in force is applied.

Now suppose the applied deformation is some x of t is x naught theta of t, that is if I am applying a step change in deformation what happens? That would be F of t is k 2 x naught times **-** 1 - tau sigma by tau epsilon, the whole thing multiplied by e power -t by tau epsilon and this whole thing is multiplied by theta of t. Now that would be the response that is the force that would be developed when the applied deformation is x of t is x naught theta of t or for a step change in deformation this will be the force that will be developed.

From this I can discuss the creep and stress relaxation responses. Now how do I check this? Substitute these values of x of t and F of t in equation 9 and it must be satisfied, equation 9 must be satisfied. I know that it is satisfied that if I am writing this down, that is why I am teaching this but can you take some time and check if this indeed works**.** For your conviction and for your understanding please spend the time and check if substituting x of t and F of t separately in equation 9 satisfies equation 9, it must satisfy.

Because these are the solutions for these applied force and applied deformations, remember that. So, solution means what? The equation must be satisfied, when you substitute the solution the equation must be satisfied, check if it is indeed satisfied, that is all.

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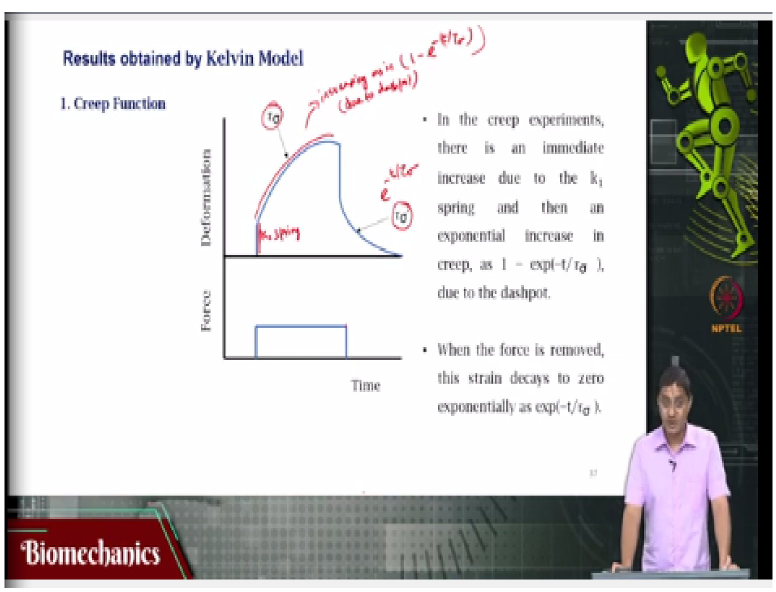

Now what about creep? When I apply a step change in force what happens? There is a sudden or immediate increase due to the k 1 spring, this is happening due to k 1 spring, there is a sudden increase, there is an abrupt or sudden increase due to k 1 spring. And then the deformation continues to increase as in 1 - e power -t by tau but which tau, tau sigma this is tau sigma, this is increasing as in 1 - e power -t by tau sigma. Why is this happening due to dashpot pot.

And then I am removing this force, when I remove this force what happens? The strain decays to 0 exponentially as in e power -t by tau, which tau? Again tau sigma, why is this happening because of the dashpot pot**.** So, the immediate response is due to the spring and the exponential response whether it is an increasing or a decreasing exponential function is due to the dashpot pot.

Now we can immediately see how this response is qualitatively different from the response that we plotted for the Maxwell model. This is qualitatively different from the Maxwell model, very different response, why? Because there are many things that we have included, that has complicated matters but has also resulted in this response, qualitative difference.

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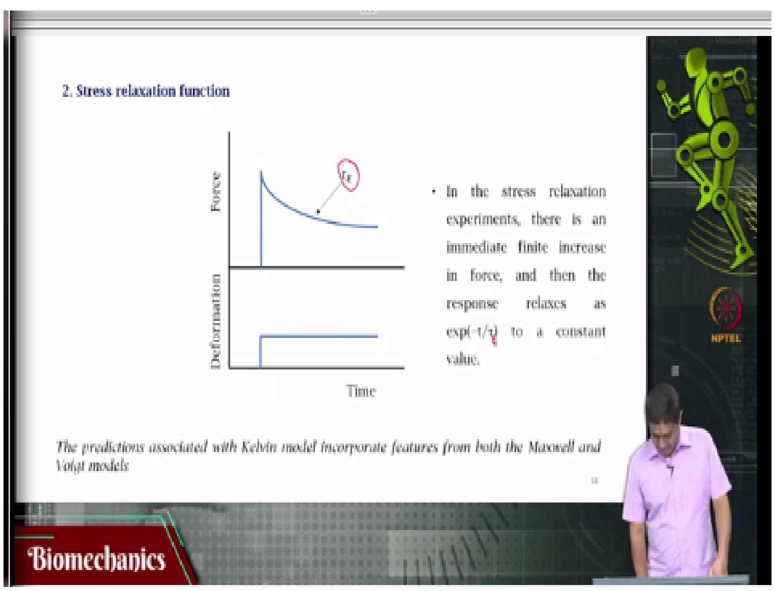

Now what about stress relaxation? Well, when I apply a sudden change in deformation what happens is that? There is an immediate increase in force that is expected, and then the relaxation happens with some time constant tau. But what is the time constant? That is tau epsilon and creep I had tau sigma and in stress relaxation I have tau epsilon and this is tau epsilon to some constant value, it decays and then it settles down at some constant value.

Once again this response is different from my expectation from the Voigt model. So, essentially the creep and stress relaxation responses are very different from what I would expect from either just the Maxwell model alone or just the Voigt model alone, qualitative difference between the Kelvin model and the Maxwell and Voigt models in the creep and stress relaxation. So, that means Kelvin model addresses some of the limitations of the Maxwell and Voigt models.

And it incorporates features from both the Maxwell and Voigt model. For example, it avoids the non-physical direct delta function found in one of these responses, please check that.

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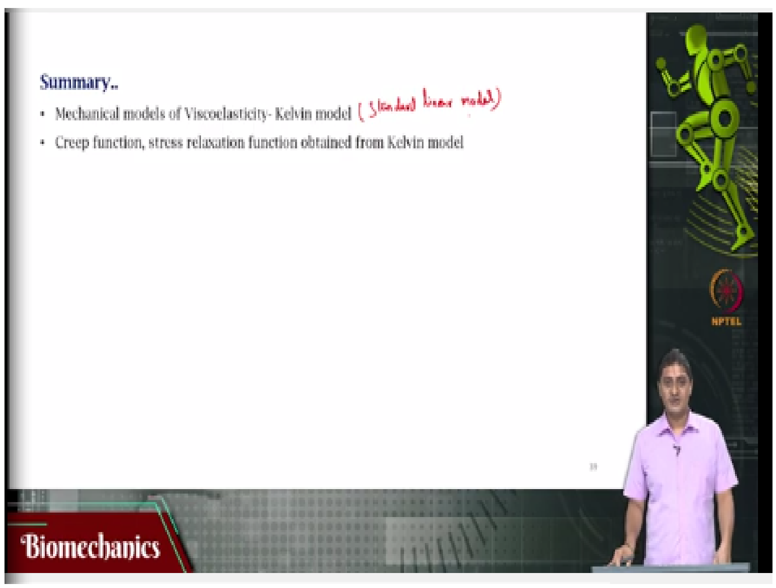

So, in this video we defined another model of viscoelasticity called as the Kelvin model or the standard linear model. And we derived the creep and the stress relaxation function for the same and we compared it with the creep and stress relaxation functions that we found for the Maxwell model and Voigt model. And we realize why the Kelvin model essentially incorporates the advantages from both the Maxwell model and the Voigt model. So, with this we come to the end of this lecture, thank you very much for your attention.