Cellular Biophysics
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Beam Theory Applied to Biopolymer

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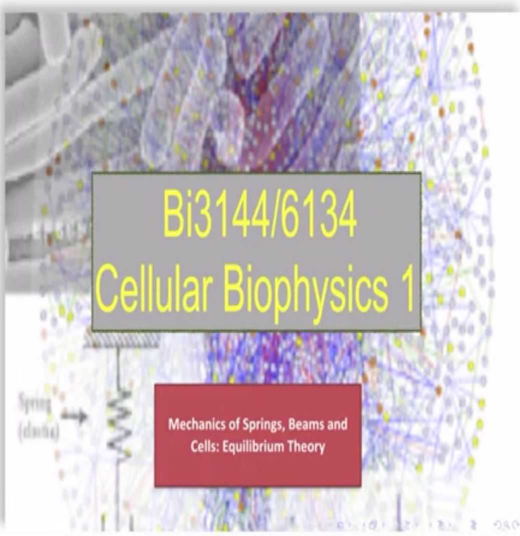

Hi welcome back this is bi3144/6134 and I am going to continue from where I left off last week. As I also announced in the class, we are going to have a live session on Wednesday rather than today on Monday.

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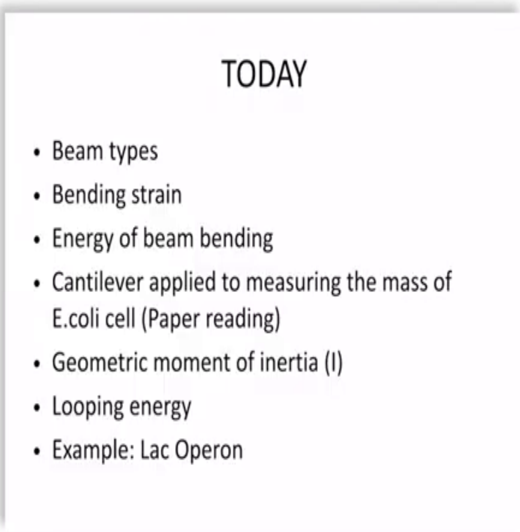

So, we are talking about beam mechanics and I had introduced beam mechanics to you last time and also ended with some discussion of prior to that with energies and energy skills and that is what we are going to discuss on Friday along with some sums for the spring energy skills but we are going to now go to more geometrically defined objects and discuss what are beams, bending strains, energy beam bending.

Usage to understand how you can measure the mechanics of cellular properties, the simplest being the mass of an *E. coli* cell, it seems very simple but you will see that the paper is a little more difficult, it is not that easy as we would hope. And then I discuss some aspects of the energy of beam bending in the parameters called geometric moment of inertia, looping energy and an example of all of this is the Lac Operon. Yes, exactly the Lac Operon has mechanics involved.

So, now one of the things I just want to point out is that this today's series of lectures is going to be in parts. So, I am going to record one part and then record the second part and keep uploading them as they go along, which means you have to look for part 1, part 2, part 3.

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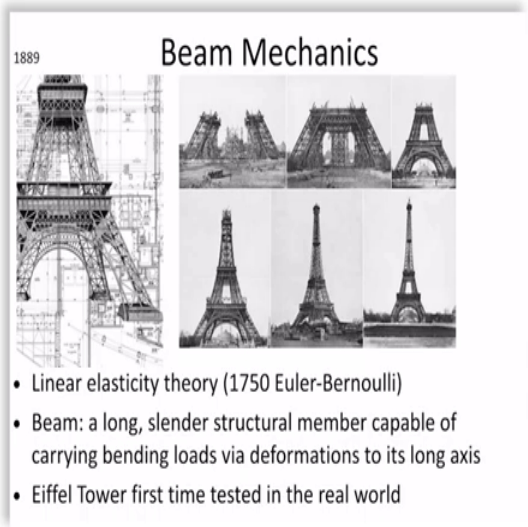

So, as I had mentioned last time beam mechanics arose out of linear elasticity theory from Euler and Bernoulli due to these 2 mathematicians synthesis. A beam is then defined as a long string slender structural member capable of getting bending loads via deformations to its long axis and the Eiffel Tower, the 2 Eiffel as it is called in Paris was the first time that it was tested in the real world.

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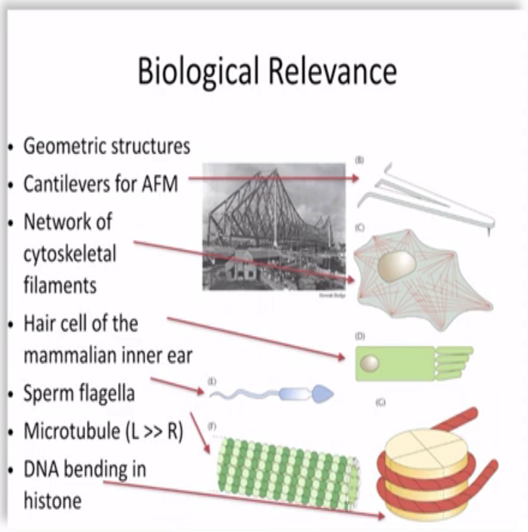

The biological relevance is to cantilevers as I mentioned in between, networks cytoskeletal filaments as we will be discussing a little later. The hair cell of mammalian inner ear, fertilization or reproduction and how you and I are made, sperms, eggs, birds, bees, intracellular mechanics through cytoskeletal dynamics and DNA bending involved in epigenetics and chromosome packing and therefore of some broad interest to biology.

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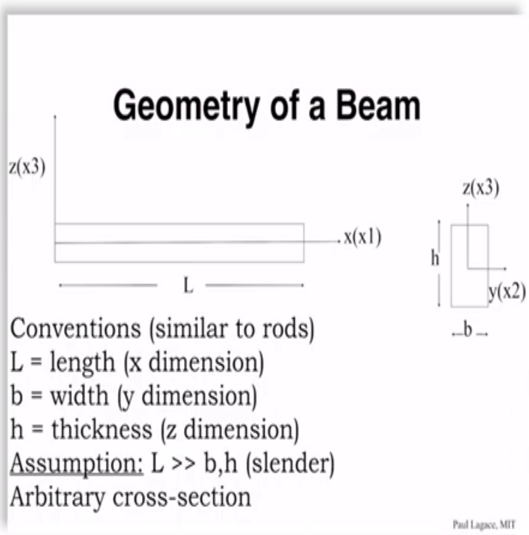

So, let us go to it, what is a beam? So, we say that in a similar convention to rods we define the length of the beam as the longest axis, in this case we call it the x dimension as you see in the diagram. Now, you can argue that orthogonal to the length there will be two dimensions this is what you see in the right hand side image, they have a breadth and a width.

You will see, I am sorry breadth and height or thickness. You will see very quickly that it does matter what this is. Now, in a simple case we could imagine a square cross section in which case h is equal to b in which case none of this makes sense. My point of shape makes no sense.

However, a basic assumption in slender beam theory is that length is much much longer than breadth and or width and height or breadth and thickness. It is a very simple question for you do microtubules and actin qualify for beam treatment as well as DNA, actin microtubule DNA. And if so is there a lower limit of length at which it does not apply to them. So, that L is much much greater than b, h.

For this you need to go back and refer to your notes about the typical heights, lengths and widths of these three macromolecular structures that we have discussed at length. It will hopefully also point out to you that there is a relevance to what we have done earlier and what we are doing now and what we will do near the end.

I want to emphasize that sometimes some of you may get a bit lost and I cannot really help it with this online mode it is and most of you unable to show up in live sessions I cannot address everyone's questions. But if you have any and if you are lost you need to talk to me. I am glad that some of you are brave enough to put in some very strong comments in your feedback, talk to me if you want me to solve them because I am not here to find out who wrote it, I am here to try and solve it. And I am going to make some efforts on my part but if I do not address your question, you need to bring it up, good.

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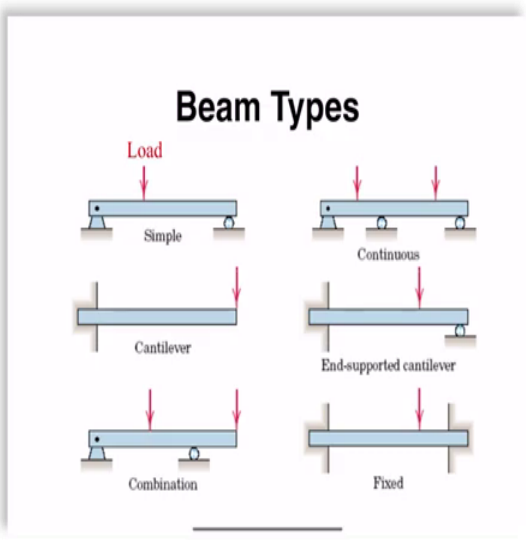

So, beams can be classified into these broad categories, this by the way is taken from engineering textbooks and from MIT online coursework for Mechanical Engineers. As a simple beam where a load is placed in the middle, the pivot point is to the left and the rolling joint is on the right. A continuous beam with multiple such flexible pivot points and multiple loads in between.

A cantilever where one end is fixed it is immobile there is no flexibility there and therefore all the flexure you will get is from the mechanics of the beam itself. The load is then applied to the trip. End-supported cantilever is a modification of the cantilever. A combination cantilever is one where the end support is now moved a bit away from the end inwards and a fixed cantilever or a fixed beam is where both ends are completely mobile, so the only flexing that happens in the middle.

Think of yourself holding a ruler and pressing it, yeah, you see this bending that comes out of it. Now, I know I cannot show you this because my video is not turned on, feel free to experiment a little bit on your own to gain an intuitive sense to this. Now, we are not going to talk about all these we are probably going to focus largely on cantilever like motion and there may be some examples of fixed beams and I hope you will see where the relevant parts show up.

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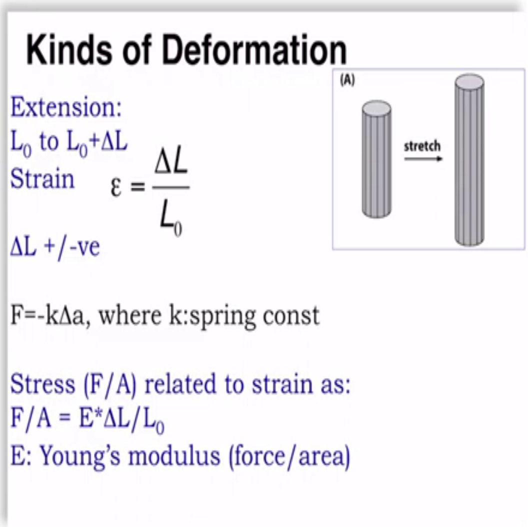

So, the kinds of deformations we have talked about so far with springs have been extensions and compressions, stretch elongation and reduction in length, which means that our length changes from L_o to $L_o + \Delta L$, the strain is defined by,

$$\epsilon = \frac{\Delta L}{L_o}$$

ΔL upon L_o and ΔL can be positive or negative which is what I call the extension in the case of positive and compression in the case of negative ΔL.

Since we know that

$$F = -k\Delta a$$

where k is the spring constant and then stress is related to strain as F/A is equal to E times ΔL upon ΔL_o that is stress is related by the Young's modulus to the string proportionately, this Young's modulus has units force per unit area similar to stress simply because $\frac{\Delta L}{L_o}$ is dimensionless and unit less.

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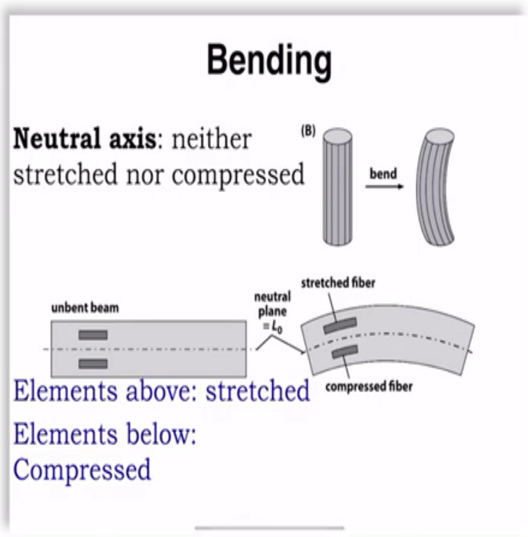

Bending can have modifications but they are neither stretching nor compression by themselves. Instead what we see is that if we look along the length of the beam at a cross-section through the beam then we will find that we can define a so-called neutral plane or neutral axis, above that neutral axis elements of the beam are being stretched, below it they are being compressed.

What about the neutral axis, well, as the term defines it and you can see probably and also intute I hope at and along the neutral axis deformation is neutral, in other words there is no change. So, this is quite useful, because it tells us that effectively there is a gradient of the nature of deformation across the axis that is orthogonal to the axis along which the bending is happening and that we can get some kind of a pattern that comes out of this or maybe you may say a law that may come out of it and I am going to discuss that in my next video.

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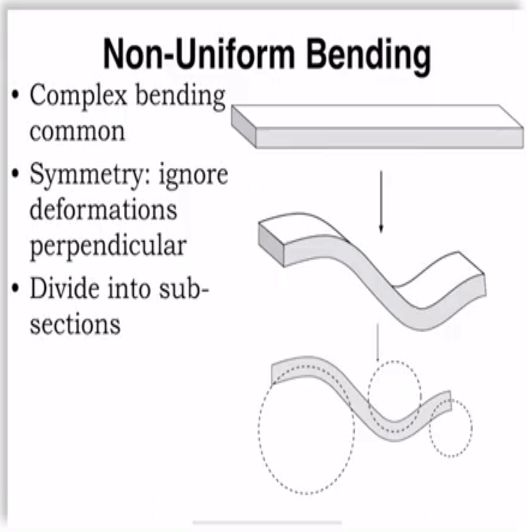

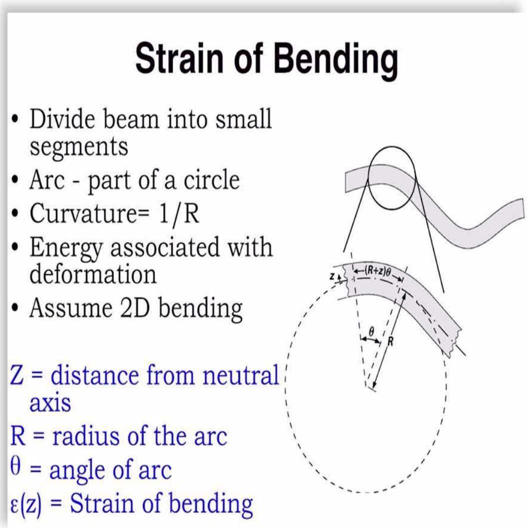

There is just one minor thing that it is obvious to most of you that if you take a cable or a very flexible structure or anything that you can try to bend there may be more than just one curvatures, you may have complex curvatures as it is called or complex bending, this is quite common in fact, we have to deal with it.

So, for beams at least, for beam mechanics the trick is to ignore deformations perpendicular to the axis and divide it into subsection so that you get many curvatures and then find a way to sum them over the beam. So, the strategy we are going to take for deriving the strain of bending is dividing the beam into small segments, consider an arc as a part of a circle with curvature 1 by R and calculate the energy associated with the deformation. In order to do that we need to define a few terms Z is the distance from the neutral axis, R is the radius of the arc, θ is the angle of the arc and $\epsilon(z)$ is the strain of bending.

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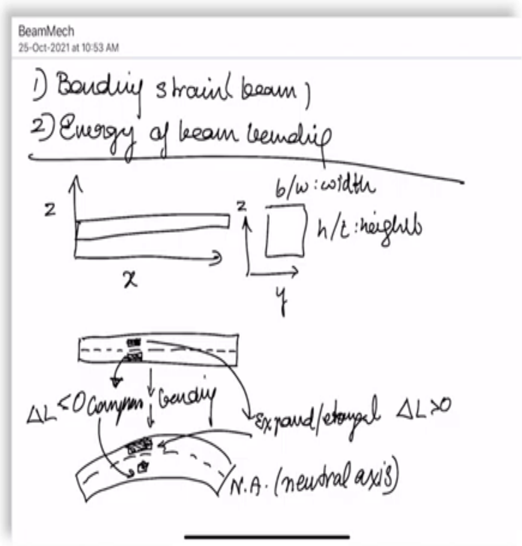

I want to talk to you about bending strain of a beam and the energy of beam bending. So, let us get to it. So, again just to remind ourselves the geometry of a beam involves this diagram I have shown you earlier along some axis where the beam is elongated and in cross-section we argue that it has a along the orthogonal axis of breadth or thickness and height I am sorry breadth or width and height or thickness, we will conventionally call this height and we will call this conventionally width.

You can look up the literature and find that this is typically the nomenclature used. So, in the case of deformations as we said earlier, beam bending, no not the shape I was looking for, along some axis results in a lack of deformation that access we defined then as the neutral axis not applicable but neutral axis and indeed those components below the neutral axis will undergo compression while those above it will undergo expansion.

So, when the beam bends this will be compressed as seen here and this will expand or elongate. And by this I am referring to the segments, yes, sections taken for illustrative purposes but this of course applies all through. What do we mean by expander elongate? That means just like our spring mechanics we talked about ΔL is greater than 0 and ΔL is that less than 0, negative or positive. In the context of defining the bending strain and energies we need to define a few terms.

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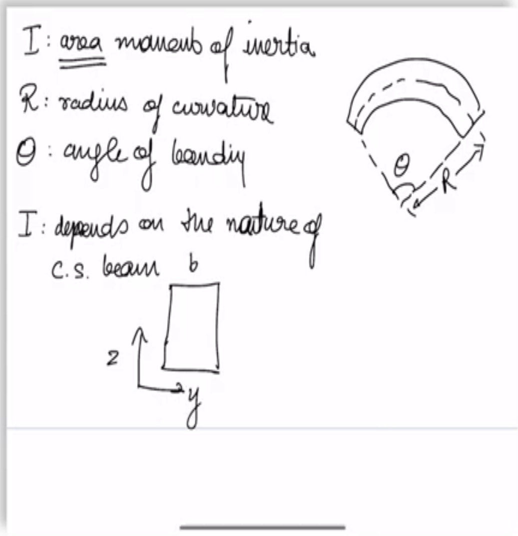

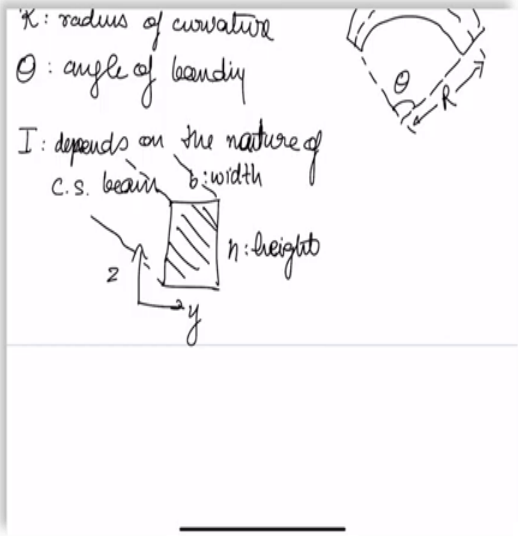

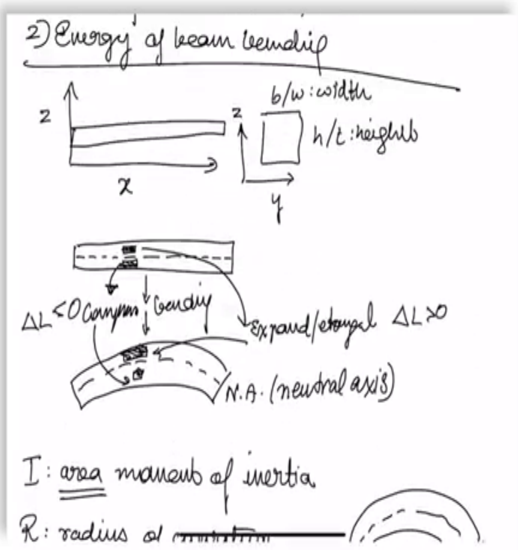

Those are I, the area moment of inertia, I urge you to be careful about this because of this business this is not the conventional moment of inertia that they used, R is the radius of curvature, θ is the angle of bending and we can see from classical geometry that sorry my beam is not looking very beam like let us try to draw it again, curved beam, bent beam, this is what I mean graphically as θ and this is what I mean by R, sorry up to here.

So, indeed I itself depends on the nature of the cross section of the beam. In other words, what we had drawn earlier as b and h which we defined as width and height, which is effectively along the cross section of the beam. You know this that if you have intuitively if you have a ruler that is what we call a foot ruler and maybe I can turn on my camera and show you what I am trying to do. Then there is one direction in which it is much harder to bend than in another. So, let us see what I can do with my camera, all right I think you see it.

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So, if I have this, if I have my ruler which I try to bend in this direction versus in this direction you understand the difference. Now, if I call this my height yes cross sectional height and if I call this the thickness it is a reasonable assumption to make. Then if I am bending when the h is in the is perpendicular to the direction of bending versus h being out of plane in another plane to the bending plane.

So, I am bending in this plane but h is in this direction, then I am going to have an easier time bending it as opposed to this I do not know if you see this so if I am trying to bend this way it is not easy. And this is what is meant by the direction mattering and this exactly is what the moment of inertia expresses. Now, we wanted to go a bit further with the beam bending energy and we will return to the moment of inertia in a bit.

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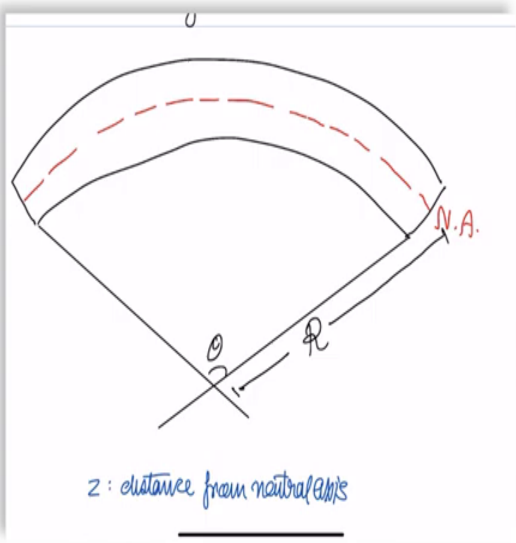

So, for that we need to go back to our drawings crude as they are, state that the so let us make it a less steep end, that is easier, this is my neutral axis. And now I can be above or below this and indeed I can say that the distance from the neutral axis Z is going to define what is going to happen to me as I bend to the beam as I bend.

Perhaps I need to move this text a bit down below because I needed to also draw something else which I forgot. And this is my θ with this R the radius of curvature and the angle of curvatures as we define them earlier.

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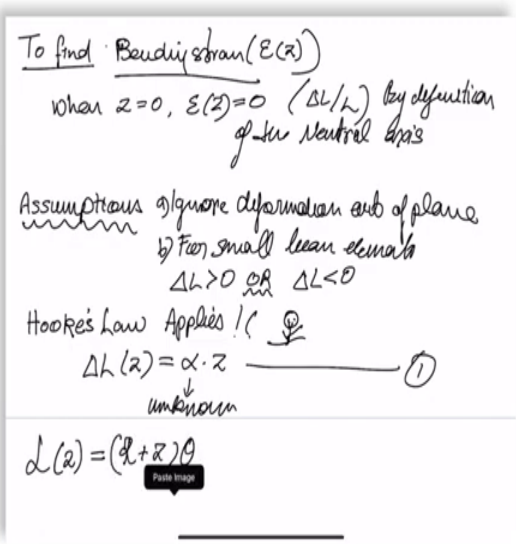

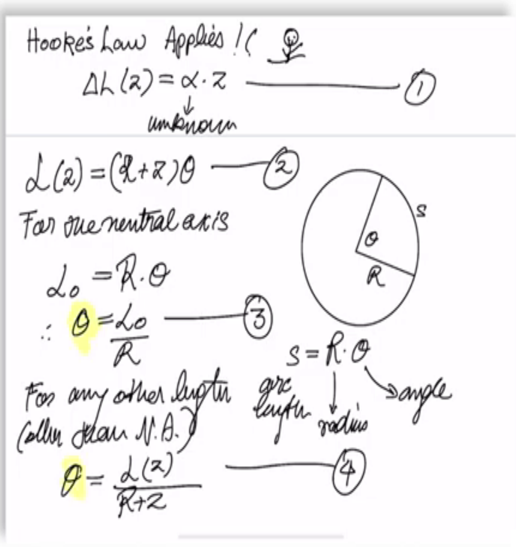

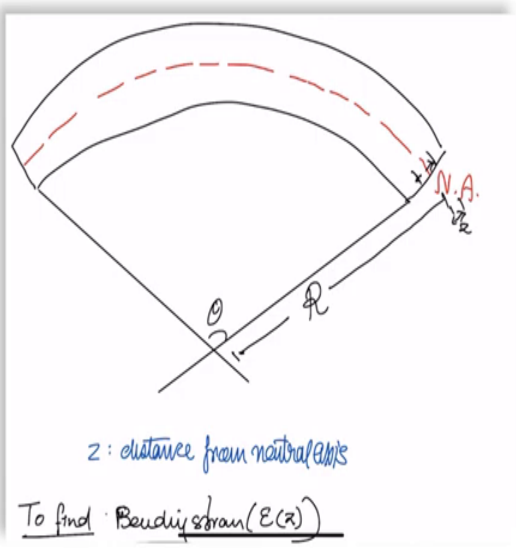

So, our aim is to find an expression for bending strain, that is $\epsilon(z)$, because Z remember is going to decide there is nothing changes in other words when Z is equal to 0, epsilon Z is equal to 0 because it is $\frac{\Delta L}{L}$ and there is no deformation by definition of the neutral axis, yeah, you agree.

So, we need to make some assumptions to derive the general expression for bending strain and a few of those are the following, so let us take them. a, ignore deformations out of plane for small elements of the beam, small beam elements bending is likely to result in positive or negative ΔL deformation or, meaning to say because it is above or below.

Remember, when we say that we can now go back to our previous section and say Yippy! Hooke's Law applies, yeah! All right jumping for joy. Let us say that

$$\Delta L(Z) = \alpha Z \ (1)$$

$\Delta L(Z)$Z that is to say with reference to Z is equal to αZ.

Then,

$$L(z) = (R + Z)\theta(2)$$

this is from geometry, because remember if we go, let us say we go here Z is in this direction or Z is in this direction, let us assume we go up so R plus some amount Z gives us the radius and times theta is the angle and this is from the simple principle that for a circle with θ is the angle and R as the radius the relationship between the arc S and the radius is R times θ, this is the arc length, this is the radius and this is the angle of the arc corresponding to the arc defined certain value.

$$\theta = \frac{L_o}{R}(3)$$

So, for the neutral axis what is the relation between this arc length and the angle, it is L_o resting length is equal to R times θ. So, in that sense θ is equal to $\frac{L_o}{R}$.

For any other length other than the neutral axis what is the relationship between theta and L and R. Well, we can say that θ is equal to L as a function of Z because changes as a function of Z upon R + Z, this is an equation,

$$\theta = \frac{L(z)}{R+Z}(4)$$

equating 3 and 4, this is θ, this is θ, this is θ so we can hopefully equate these.

$$L(Z) = (R + Z)\frac{L_o}{R}(5)$$

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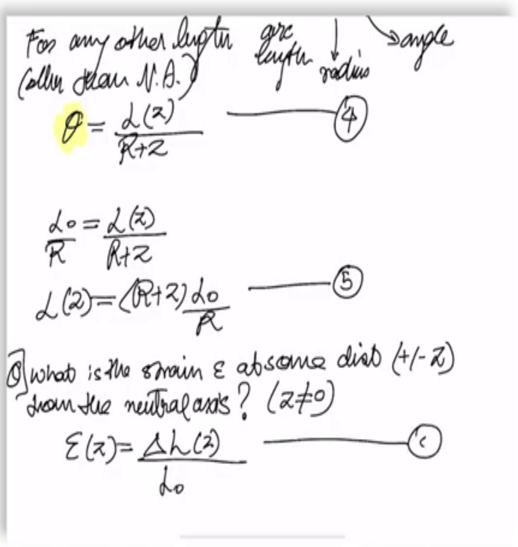

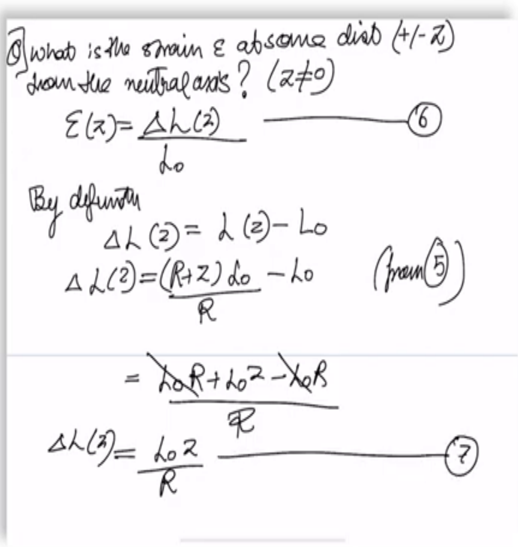

We get $\frac{L_o}{R}$ is equal to $L\frac{(Z)}{(R+Z)}$ because the angle is the same remember it is obvious. This means we can simplify and we get L(Z) is equal to $(R + Z)\frac{L_o}{R}$. So, we of course are led by this expression for L(Z) to the question what is the strain which is what we originally set out to answer by the way is the strain ϵ at some distance ±,Z from the neutral axis.

$$\epsilon(Z) = \frac{\Delta L(Z)}{L_o}(6)$$

Assuming Z is not equal to 0 meaning other than at the neutral axis itself. So, then $\epsilon(Z)$ by definition must be d$\Delta L(Z)$eformation at a position Z upon resting length. This is our equation 6. By definition we can say that $\Delta L(Z)$ is equal to $L(z) - L_o$ that is the meaning of change in L.

So, $\Delta L(Z)$ by substituting now with the expression we have from equation 5 can be written as, $L(Z)$, $\Delta L(Z)$ is equal to $\left((R + Z)\frac{L_o}{R}\right) - L_o$. Simplifying we end up

$$\Delta L(Z) = \frac{L_o R + L_o Z - L_o R}{R} = \frac{L_o Z}{R} \quad (7)$$

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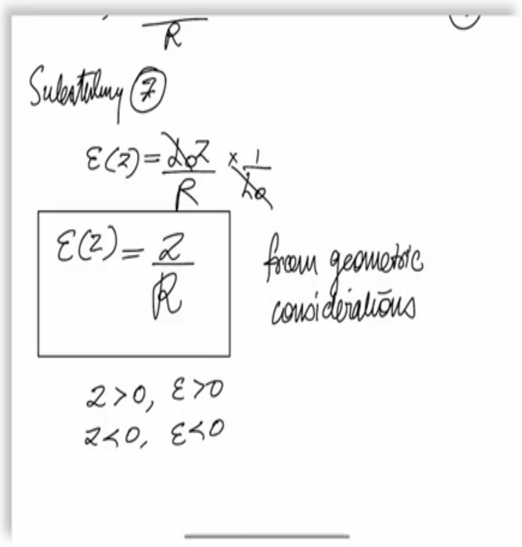

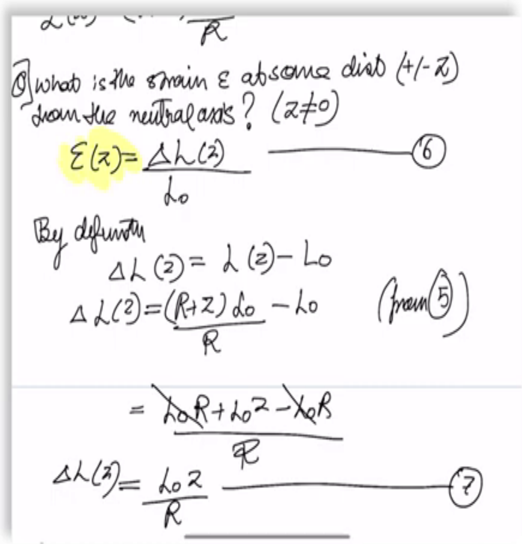

And now substituting 7 that is this deformation in our original equation for how we are going to derive ϵ that is the strain we get $\epsilon(Z)$is equal to $\frac{L_o Z}{R}$ time $\frac{1}{L_o}$ which is nothing but Z upon R.

$$\epsilon(Z) = \frac{Z}{R}$$

And this came from purely geometric consideration. Meaning to say we could arrive at this expression purely from geometry, when Z is great is, sorry, the equation also allows us to see what we could have probably intuited which is that when Z is greater than 0 positive, ϵ is also greater than 0 and when said is less than 0 ϵ is also less than 0. And to now my second point about energy of beam bending.

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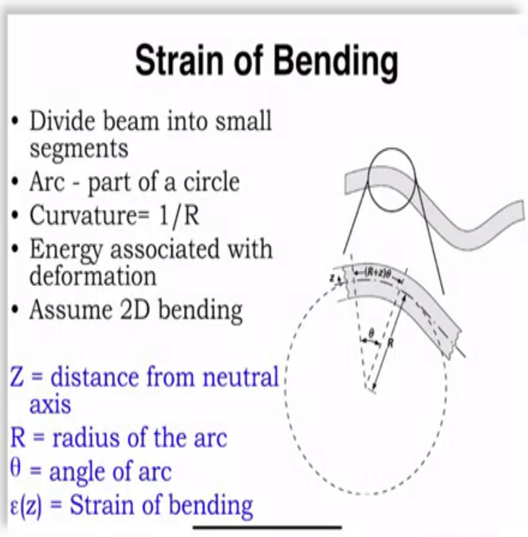

So, as we saw earlier the strain of beam bending can be derived from curvature considerations and geometric assumptions about how we can understand the role of height from the neutral axis to arrive at an expression for strain.

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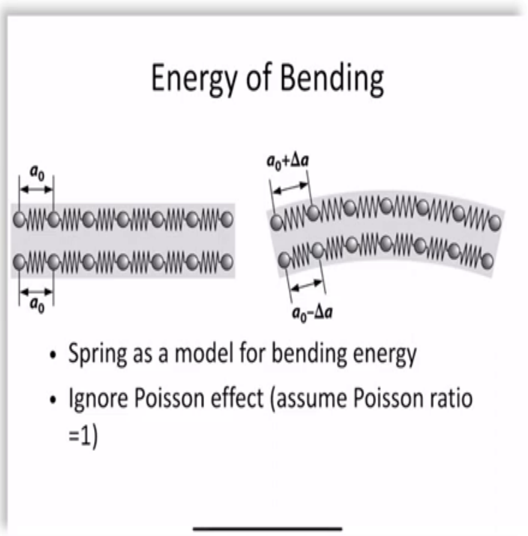

Now, for the energy of being bending we can of course go back to the microscopic view of a beam where you see a bunch of springs, a series of springs which are attached to the atoms or molecules that make up the material solid in this case with the resting length of a_o so when the beam is deformed the ones above the neutral axis undergo a deformation which is $+\ \Delta a$ and the one below it proportionately at the same distance undergo $-\ \Delta a$ deformation we ignore the Poisson's effect, meaning to say we assume that the Poisson's ratio is 1.

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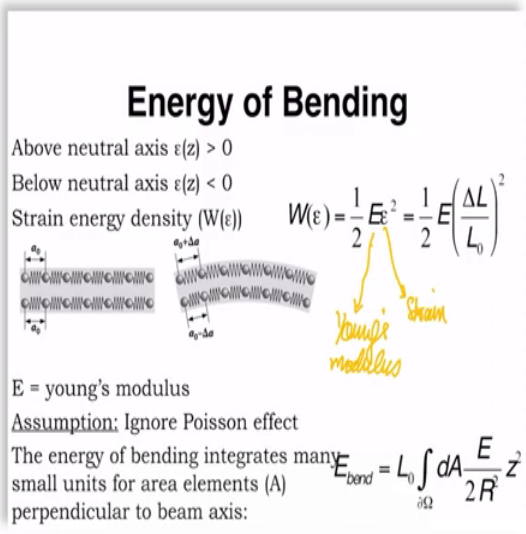

Then the energy of the bending is related somehow to this idea that above the neutral axis epsilon's at the strain energy, strain, sorry the strain is greater than is positive and the strain is negative below the neutral axis. So, we end up having to write something which introduces a strain energy density term which is W epsilon and that can be related

$$W(\epsilon) = \frac{1}{2}E\epsilon^2 = \frac{1}{2}E\left(\frac{\Delta L}{L_o}\right)^2$$

that is our strain and E is nothing but Young's modulus.

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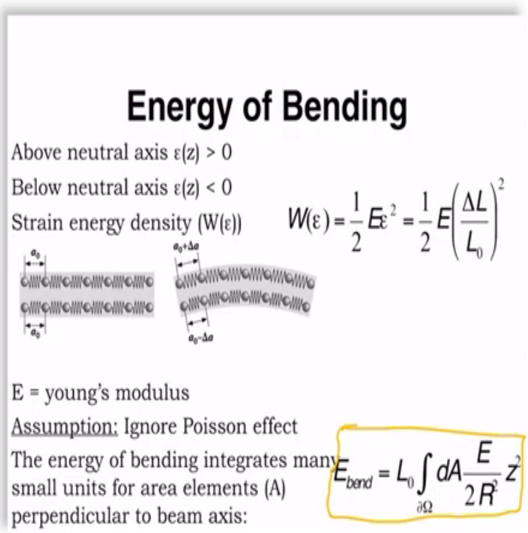

In such a case it is also clear that we can write the energy in terms of W that is the energy density times the volume and which is because of the energy density is in terms of per unit volume and so since $\epsilon(z)$ is the strain change with Z we need to integrate the strain in order to get at an expression for bending energy which is this expression here which is telling us that the energy bending

$$E_{bend} = L_o \int dA \frac{E}{2R^2} Z^2$$

is to an expression that is a resultant of the integration.

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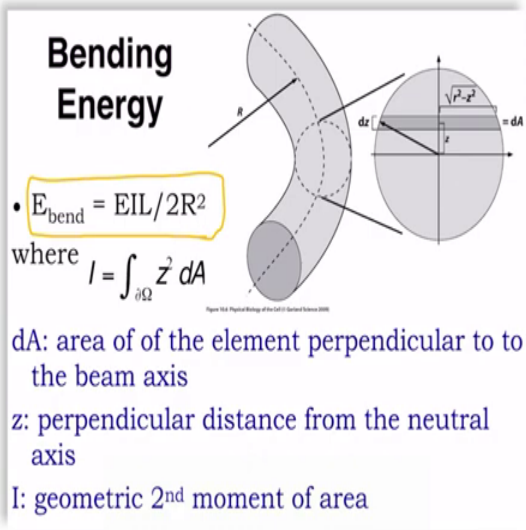

And you can find a full work solution in landau Lifshitz textbook on mechanics, the classic Physics textbook. But we are not going to derive it here and for those of you interested please refer to Landau Lifshitz but the working idea is that how does this $EIL/2R^2$ relate to what we are trying to discuss and that is what I am going to talk to you about next in terms of the meaning of I that is to say the geometric moment of inertia and the energy of looping, thank you.