Transport Phenomena in Biological Systems
Prof. G. K. Suraishkumar
Department of Biotechnology
Indian Institute of Technology, Madras

Module - 1
Lecture - 5
Useful Derivatives

Welcome back. Let us continue with material balances. It is a very useful principle. We are going to show where all we can use that; at least, we are going to highlight where all we can use that. This is an introductory course. So, this session or this lecture is on material balances, the useful form of that in fluid systems.

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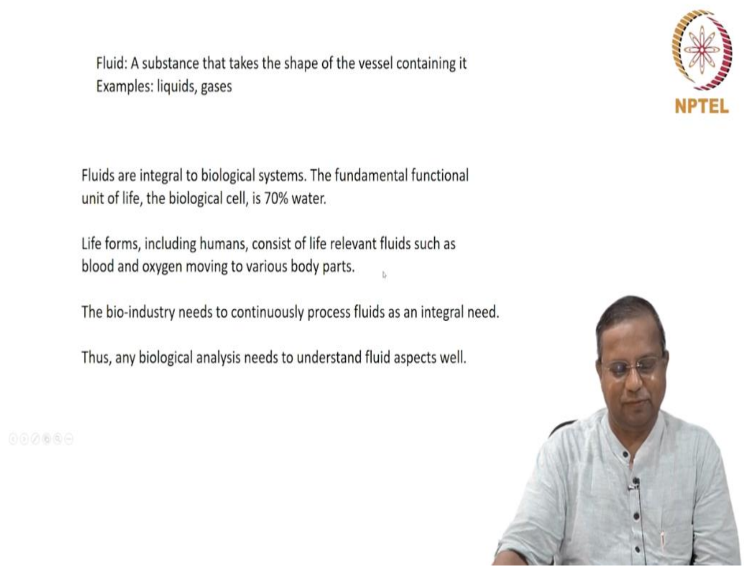

We all know what a fluid is, from high school. Fluid is a substance that takes the shape of the vessel containing it. And, examples are liquids, gases and so on. And fluids are integral to biological systems. The cell which is the fundamental functional unit of life is about 70 percent water, which is a fluid. So, they have it; 70 percent of something; the fundamental unit of life itself is a fluid.

Not just that, life forms including humans consist of life relevant fluids such as blood, oxygen, lymph and so on so forth. Moving to various body parts. We are human beings made up of various cells, various organ systems and so on so forth. There are fluids in constant flow in our body; both liquids and gases, which make life possible, human life possible.

Not just that, the bio industry needs to continuously process fluids as an essential need, as an integral need. Things, fluids need to move; liquids and gases need to move from one place to another. The bio reaction itself happens usually in a liquid. And then, you take out the products of interests and process them to get of a pure form, concentrated form and so on so forth.

So, all this requires processing fluids in a continuous fashion, usually; continuous or batch fashion, that does not matter. And therefore, any analysis of biological systems needs to understand the fluid aspects very well. The fluid aspects are very different. It is, you cannot very easily use whatever we know so far directly. They are complex things. Some of those are still not understood properly, in our frameworks.

And therefore, we need to look at fluids. This is probably the first time you are looking at fluids, unless you have done a course on fluid mechanics already. It is a very interesting thing to look at. And let us start with material balances in fluid systems.

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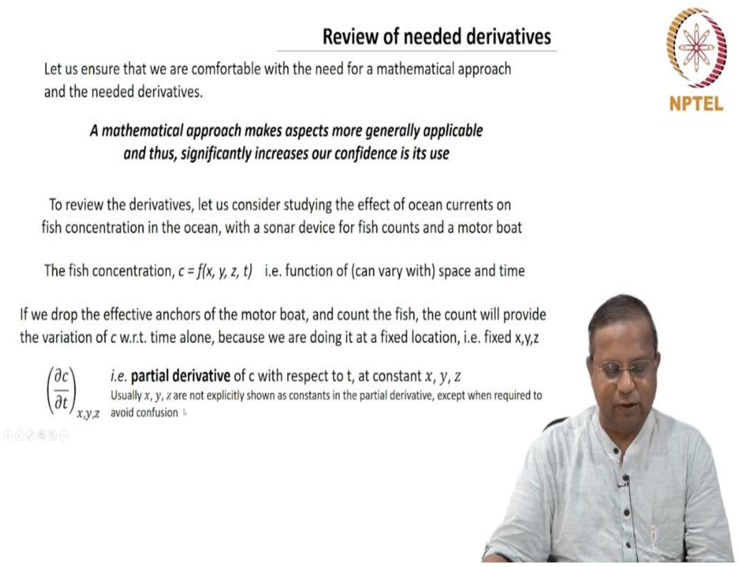

To do this, we will be employing a few derivatives. Okay. All done math; you know what; you have all done calculus; you know what derivatives are, the meaning of derivatives, the way to write them and so on so forth. There are various kinds of derivatives. There are a few kinds that we use normally. You would have seen 2 of these. Maybe you have not seen all of the 3 as yet.

Or maybe you have, in a physics course or something like that. So, let us see what those are. This is a review of the needed derivatives. This is essentially to make sure that all of us are comfortable with the meaning and use of these derivatives. To do this; and not just that, before I get into the actual derivatives, let me repeat, one of the aspects, one of the points that I have made already.

A mathematical approach makes aspects much more generally applicable; and thus, significantly increases our confidence when we use it. That is the reason for converting many things onto a mathematical framework. We can use it with confidence. And one of the common questions as I had mentioned earlier is, why so much math? Okay. This is the reason for so much math.

It gives us so much confidence to blindly apply it. We do not have to worry about: Is this really going to be applicable here? Will I get; Can I rely on it? Can I use the results that I get here, confidently, to know what is going on, to design things, to operate things and so on so forth? All these questions go away when we use this framework. And that is the reason why a mathematical framework pretty much is used in most of these subjects.

And especially this one is highly mathematical. To review these derivatives, let us consider this scenario. We are in a boat, a motorboat that is equipped with powerful motors, as well as with a powerful anchor, a strong anchor. And we want to measure the fish concentration, let us say, in the ocean. And what we have to measure the fish concentration is a sonar device which pings with every fish it encounters and gives you a count.

So, we have a sonar device for fish counts. And we have a motorboat. And the motorboat has strong motors, as well as strong engines, as well as a strong anchor. Which means, if the anchor is dropped, it will stay at that place, even in the ocean. Because, ocean has a lot of currents. So, if you do not drop the anchor and if you stop the engines, you can still move with the current.

That is the situation here. Okay. Let us do step by step. The fish concentration is a function of the position. These are position coordinates: x, y, z and then time t. Okay. What do we normally mean by a function is that, it can vary; the fish concentration can vary with space, x, y, z coordinates, as well as time. If we drop the effective anchors of the motorboat, it is going to stand still at that place and count the fish at that place.

The count is going to provide the variation of concentration of fish, with respect to time alone, because the position is fixed. On other words, the x, y, z coordinates are fixed. And therefore, we are going to have the variation with time alone, at that particular point. And that is represented as you know, by the partial derivative of the concentration of fish with time. We normally say, $\frac{\partial C}{\partial t}$. This, as you know is the way we represent derivatives. It is not the ratio here. It is $\left(\frac{\partial C}{\partial t}\right)_{x, y, z}$. That constant is indicated here as a subscript after the bracket. And many a times, if it is clear enough, we do not even mention this x, y, z.

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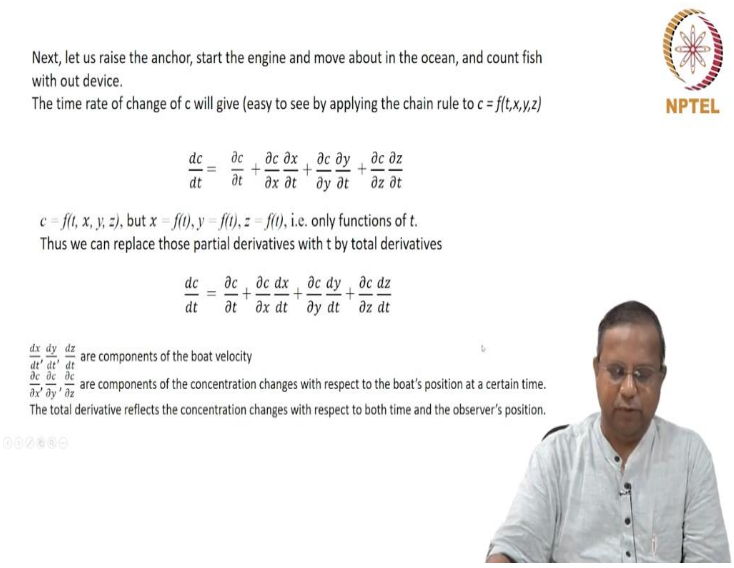

Next, what we are going to do is, we are going to raise the anchor, start the engines and move about in the ocean. And while moving about, we are going to count the fish. We are going to use our sonar device and count the fish. In this case, the rate of change of fish concentration, what will it give? We will see when we go through this. If we apply chain rule; you recall chain rule, function of a function, right.

So, $\frac{dc}{dt}$, the total derivative of c with respect to t is the partial of c with respect to t plus the partial of c with respect to x. But x is a function of t. Since we are moving about, the position varies with time. So, $\frac{dc}{dt} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial C}{\partial y}.\frac{\partial y}{\partial t} + \frac{\partial C}{\partial z}.\frac{\partial z}{\partial t}$. . This is the chain rule that is coming in here . We know that concentration itself is a function of time, as well as space x, y, z coordinates.

But x is just a function of time alone. It is not a function of space. y is a function of time alone, z is a function of time alone, by the same argument. They are only functions of time x, y, z. And therefore, we do not need to write this as a partial. You know, partial means, there is a function of many variables. In this case, it is a function of only one variable. And therefore, you can replace the partials with the total derivative. If you do that, $\frac{\partial C}{\partial t}$ is still partial. We get $\frac{dC}{dt} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial x}\frac{dx}{dt} + \frac{\partial C}{\partial y}\frac{dy}{dt} + \frac{\partial C}{\partial z}\frac{dz}{dt}$. Now see this. $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are nothing but the components of the boat velocity. And therefore, let me also say this. $\frac{\partial C}{\partial x}, \frac{\partial C}{\partial y}, \frac{\partial C}{\partial z}$ are components of the concentration changes with respect to the boat's position at a certain time.

And therefore, the total derivative reflects the concentration changes with respect to both time as well as the observer's position. With respect to time are these $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ and with respect to the observer's position are $\frac{\partial C}{\partial x}, \frac{\partial C}{\partial y}, \frac{\partial C}{\partial z}$. And, that is what the total derivative gives you. We know what the partial derivative gave at; when we dropped the anchor at that particular point. The total derivative gives you these two. So, look at this as some sort of a review. The next one you may or may not know.

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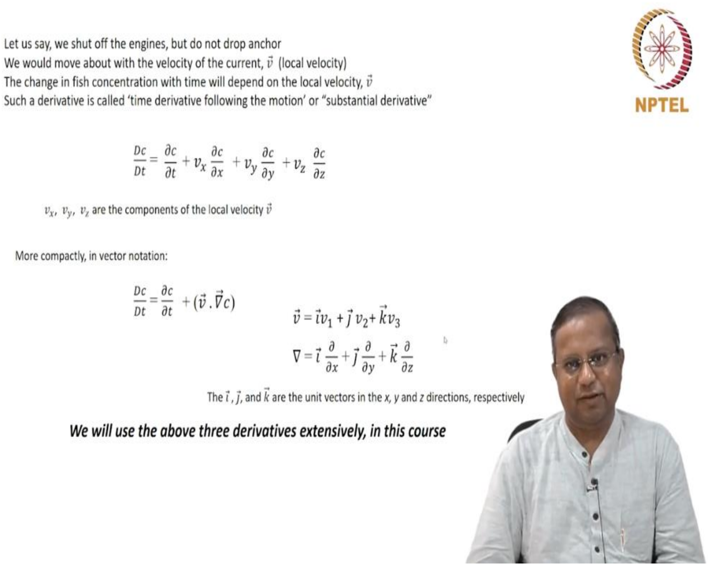

To understand that, let us say that we shut off the engines, but we are not going to drop the anchor. If you do not drop the anchor, there are currents in the ocean; and the boat is going to move according to the current. Therefore, we would move about with the velocity of the current. v is called the local velocity here. The change in fish concentration with time will depend, of course, on the local velocity.

We are moving about. We are at different points and space. The way it is going to change the time and so on so forth. And therefore, the fish concentration is going to depend on the local velocity *v*. And such a derivative which is called the time derivative following the motion, is also called the substantial derivative. You may have seen this in your physics class, but this is what it means.

We are moving about and we are counting fish in this case. And therefore, we get it as a function of the motion of the fluid itself, as well as time. That is what we represent; the substantial derivative is what is represented as capital Ds. $\frac{DC}{Dt}$, which is the combination of partial derivative of c with respect to t and the velocities of the stream have come in the place of the velocities of the boat. $\frac{DC}{Dt}$ is written as follows.

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + v_x \frac{\partial C}{\partial x} + v_y \frac{\partial C}{\partial y} + v_z \frac{\partial C}{\partial z}$$

It is essentially the same as earlier. Only thing is that, the motion is now dependent on the motion of the boat with the current. v_x, v_y, v_z are the components of the local velocity. And we can try this more compactly, in the vector notation. You are all familiar with vectors from your math course. In short, we have $\vec{v}.\vec{\nabla}$c where $\vec{v} = \vec{\imath}\, v_x + \vec{\jmath}\, v_y + \vec{k}\, v_z$. $\vec{\imath}$ is unit vector in the x direction. $\vec{\jmath}$ is the unit vector in the y direction. $\vec{k}$ is the unit vector in the z direction. v_x, v_y, v_z are the components in these 3 respective directions. And ∇ is nothing but,

$$\nabla = \vec{\imath}\ \frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\ \frac{\partial}{\partial z}.$$

We have the dot product of $\vec{v}$ and $\vec{\nabla}c$ i.e. $\vec{v}.\vec{\nabla}$c. When you take the dot product, we do a term by term thing. Now, $\vec{v} = \vec{\imath}\, v_x + \vec{\jmath}\, v_y + \vec{k}\, v_z$ and $\vec{\nabla}c = \vec{\imath}\ \frac{\partial c}{\partial x} + \vec{\jmath}\frac{\partial c}{\partial y} + \vec{k}\ \frac{\partial c}{\partial z}$

$\vec{\imath}.\vec{\imath} = \vec{\imath}\,\vec{\imath} \cos 0 = 1$, (Angle is 0 and cos 0 is 1)

$\vec{\imath}.\vec{\jmath} = \vec{\imath}\,\vec{\jmath} \cos 90 = 0$ (Angle is 90 and cos 90 is 0).

$\vec{\imath}.\vec{k} = \vec{\imath}\,\vec{k} \cos 90 = 0$. (Angle is 90 and cos 90 is 0).

Thus, if you do a term by term thing, what remains are only these aspects: $v_x \frac{\partial C}{\partial x} + v_y \frac{\partial C}{\partial y} + v_z \frac{\partial C}{\partial z}$.

And therefore, this can be represented as $\vec{v}.\vec{\nabla}c = v_x \frac{\partial C}{\partial x} + v_y \frac{\partial C}{\partial y} + v_z \frac{\partial C}{\partial z}$

$$\boldsymbol{\frac{DC}{Dt} = \frac{\partial C}{\partial t} + (\vec{v}.\vec{\nabla}c) = \frac{\partial C}{\partial t} + v_x \frac{\partial C}{\partial x} + v_y \frac{\partial C}{\partial y} + v_z \frac{\partial C}{\partial z}}$$

Hope you got this. If not, you, why do not you write it? Do a term by term thing and see that everything else cancels out, except for this? This is the reason why we use a compact notation. Expanding it and writing it every time is rather cumbersome. So, we would prefer a compact notation. But a compact notation has this much meaning that goes into it, which I am sure you already recognize. If not, you need to internalize that now.

There is so much processing that is represented by a compact notation. We are going to use these 3 derivatives extensively in this course. I think we will stop here for this lecture. We have been at it for some time. It is good to process this first. And then, we will move on to the next thing, next lecture. See you in the next lecture then.