Transport Phenomena in Biological Systems

Prof. G. K. Suraishkumar
Department of Biotechnology Bhupat and Jyoti Mehta School of Biosciences Building
Indian Institute of Technology - Madras

Lecture – 27
Application of Equation of Motion to Flow Over an Inclined Plane

Welcome. In this class, we look at the application of the momentum balance equation that we derived okay. What I am going to do is I am going to show you or I am going to apply the equation of motion to the same situation that we considered earlier. The case of a thin film of liquid falling over an inclined plane, a Bostwick viscometer case, and you will be able to see how simple it is to arrive at the final relationships of use when we use this equation compared to the shell balances.

Shell balances took a long time and so and so on. Now, we spent enough time and effort to set these equations of motion that we can directly use in appropriate situations okay, so we might as well use that, you just need to have it by your side, you could refer to that, I will not expect you to remember any of those equations, do not worry about it. Normally, those are very long equations, very complex terms there, different subscripts and so on and so forth.

I will not expect you to remember it. You could just take those equations there and directly start using them okay, that is the reason I asked you to make a copy of those equations and keep it aside. So, some applications of the equation of motion, first to the same situation the steady state falling film case.

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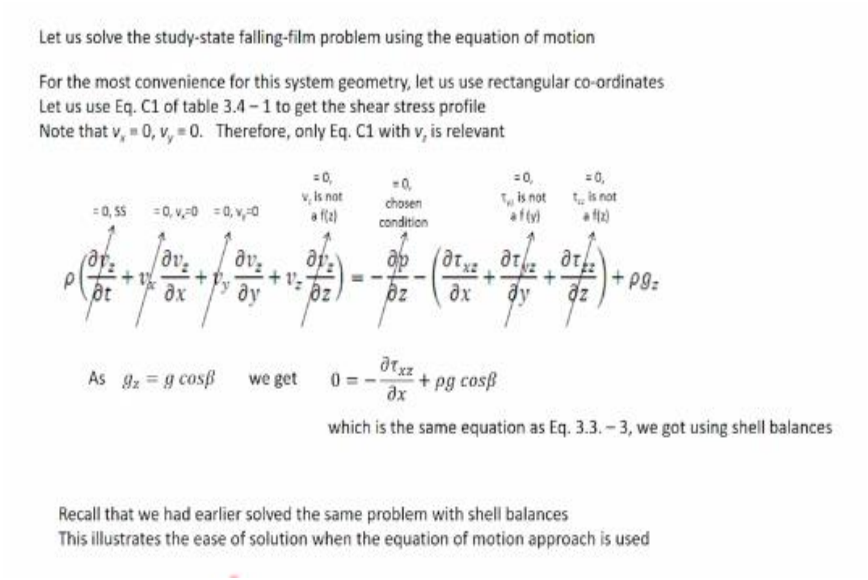

So here is the rectangular Cartesian coordinate geometry, right, that is quite easy to see. So, let us use equation C1 of table 3.4 – 1 to get the shear stress profile okay. What was the equation 3.4 – 1, I think I need to open up that equation again okay. So, this is the rectangular Cartesian coordinate system for the equation of motion. So, this is the x direction, y direction, z direction.

z direction

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}-\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z \qquad \text{(C1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z \qquad \text{(C2)}$$

I think we are looking at the z direction, here we had written it for z direction. This is a general equation. The second ones are for Newtonian fluid constant with ρ and μ. Here, this is a general equation, let us consider this equation okay. So this if we take and apply it here, it is quite easy to see that there is no bulk velocity in the x direction, there is no bulk velocity in the y direction of the film. This is the z direction, the bulk velocities only in the z direction. Therefore, only the v_z is relevant and the equation that contains v_z is C1 and that is the reason why we went to C1. When we take the C1 and cancel the terms, I will show you one by one, then we get something like this.

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0,\ SS}+\underbrace{v_x\frac{\partial v_z}{\partial x}}_{=0,\ v_x=0}+\underbrace{v_y\frac{\partial v_z}{\partial y}}_{=0,\ v_y=0}+v_z\underbrace{\frac{\partial v_z}{\partial z}}_{=0,\ v_z \text{ is not a } f(z)}\right)=-\underbrace{\frac{\partial p}{\partial z}}_{=0,\text{ chosen condition}}-\left(\frac{\partial \tau_{xz}}{\partial x}+\underbrace{\frac{\partial \tau_{yz}}{\partial y}}_{=0,\ \tau_{yz} \text{ is not a } f(y)}+\underbrace{\frac{\partial \tau_{zz}}{\partial z}}_{=0,\ \tau_{zz} \text{ is not a } f(z)}\right)+\rho g_z \quad (3.4.1\text{-}1)$$

This is a steady state analysis as we said, therefore there is no variation of time, the first term goes to 0. There is no v_x right, there is no velocity in this direction therefore that is 0. There is no v_y, so that goes to 0. There is certainly a v_z. However, we have taken the case of well-developed flow, which means the v_z is not varying with z. Therefore v_z is not a function of z and therefore this derivative goes to 0 and we have already chosen that the pressure does not vary with the length and therefore $\frac{\partial p}{\partial z}$ is 0 by chosen condition. This term remains.

τ_{yz} is not a function of y, therefore that goes to 0, τ_{zz} is not a function of z, therefore that term goes to 0, ρg_z remains. Why did not do I have the details here? See for getting an idea of why it is not a function of y, why τ_{yz} is not a function of y, you could resort to the direction of motion and the direction of action, simplistically speaking, we already seen that 2 different shear rates could contribute to a shear stress and so on and so forth. So to get an idea here, this is the shear stress that arises in the y direction because of the motion in the z direction okay. There is motion in this direction, there is no shear stress in this direction, so that is the reason why this is 0. It is not a function of y and there is no shear stress that is going to result in this direction because of the motion in this direction that is the reason why this normal stress is 0.

Since

$$g_z = g \cos\beta$$

we get

$$0 = -\frac{\partial \tau_{xz}}{\partial x} + \rho g \cos\beta$$

which is the same equation as Eq. 3.3-3.

There is no way variation in velocity and so on and so forth, so that is 0 okay. So what we get from this equation is again need to get back to this, g_z is nothing but gcosβ okay, g was in this direction, this was the β, so the component here is gcosβ, and if we substitute here we are going to get 0 on the left hand side. The equation we got now is exactly the same equation that we had earlier okay.

So just by using this equation, cancelling out the relevant terms, we could directly get this equation okay. We do not have to go through significant amount of effort, which we need to spend if we do shell balances.

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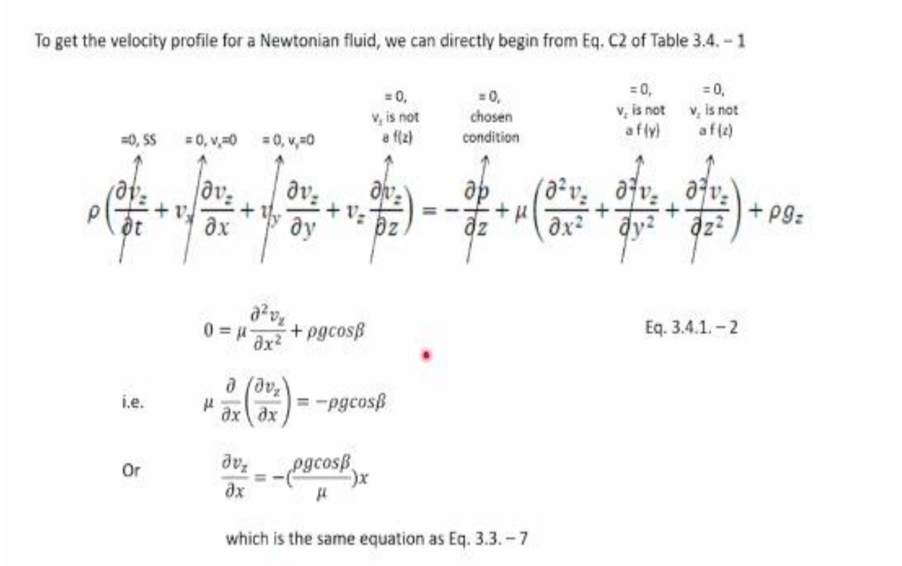

Now, let us try to get a velocity profile in this Newtonian fluid, it is water flowing as we said. Water is a Newtonian fluid and for this we can use the next equation C2 in table 3.4 – 1. I am going to write the equations, I am not going to go back and show you that equation. Please look at the table that you have made a copy of and find out that equation, yeah that would be an easier way to do. Initially, I need to make this connection and then once you have already made the connection you can directly refer to the equation. This would have been the equation C2 in table 3.4 – 1.

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0,\ SS}+\underbrace{v_x\frac{\partial v_z}{\partial x}}_{=0,\ v_x=0}+\underbrace{v_y\frac{\partial v_z}{\partial y}}_{=0,\ v_y=0}+\underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0,\ v_z \text{ is not a } f(z)}\right)=-\underbrace{\frac{\partial p}{\partial z}}_{=0,\text{ chosen condition}}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\underbrace{\frac{\partial^2 v_z}{\partial y^2}}_{=0,\ v_z \text{ is not a } f(y)}+\underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0,\ v_z \text{ is not a } f(z)}\right)+\rho g_z$$

When we cancel the terms, the first term is the processes at steady state therefore there is no variation with time, the first term goes to 0. You have already seen there is no v_x, there is no v_y, therefore these 2 terms go to 0. We also said that v_z is not a function of z, well-developed flow, and therefore that is 0. We also said that pressure is not a function of length, the fluid is very thin, we went through a long argument to establish this earlier, you can go back and look at that argument, $\mu \frac{\partial^2 v_z}{\partial x^2}$, this term of course remains, there is a variation of v_z with x. There is no variation of v_z with y. It is all the same at various ways and similarly there is no variation of a v_z with z, well-developed flow okay. So, these 2 terms go to 0, these are quite straightforward to see, ρg_z will be there. Now, the only terms that remain are written as,

$$0 = \mu \frac{\partial^2 v_z}{\partial x^2} + \rho g \cos\beta \qquad (3.4.1\text{-}2)$$

i.e.

$$\mu \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial x}\right) = -\rho g \cos\beta$$

or

$$\frac{\partial v_z}{\partial x} = -\left(\frac{\rho g \cos\beta}{\mu}\right)x$$

which is the same equation as Eq. 3.3-7.

So, by applying the equation of motion, we significantly simplify the effort that we need to put in to arrive at relevant insights. This is a reasonably short lecture. I think I have shown you one application, I think I let it sink in and when we come back in the next class, we will look at other applications, all of heavy relevance to whatever is being used in the body, in the industry, in various biological systems. See you then.