Data Analysis for Biologists
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Lecture 09
Matrix and Matrix Operations

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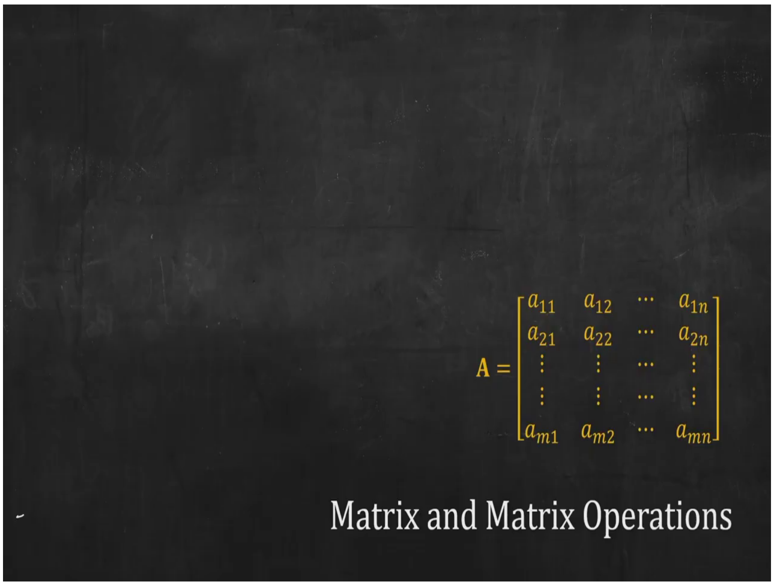

Hello, welcome back. In the last lecture, we learned about vectors. And we have seen how we can represent the data in vectorial form, and then do some vector operations, we learned about some vector operations.

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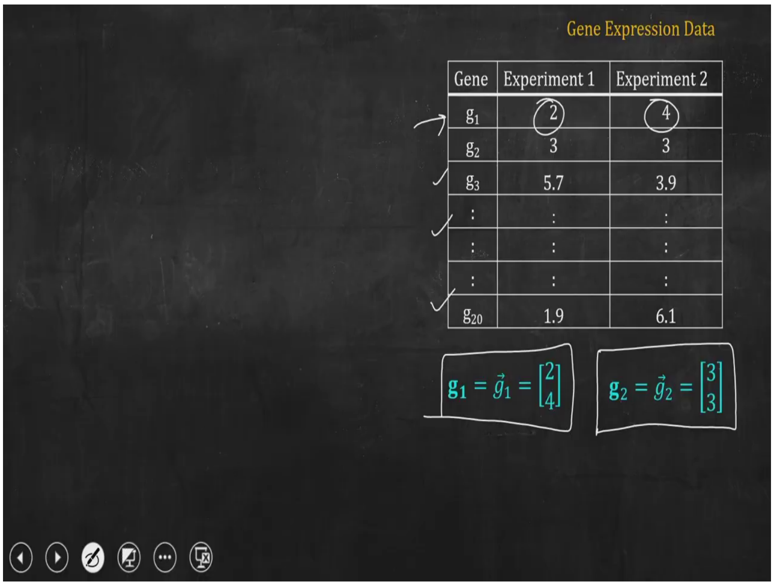

$$g_1 = \vec{g_1} = [2 \ \ 4]$$

$$g_2 = \vec{g_2} = [3 \;\; 3]$$

Today, we will learn about matrices. So, let us start with the same data set that we discussed in the last lecture, we have a gene expression data, we have 2 experimental conditions, in that we have suppose 20 different genes expression we have measured. So, this table shows that data. Now, in the last lecture, what we did, we took g1, and its value of expression for change 2 and 4, and represented that as a vector g1.

Similarly, we do the same thing for g2, its expression level fold change where 3 and 3, so we stack these two values together to create a vector is the least, least of values, and then we represented that by arrow in the space we did all the analysis. So, we did the same thing for all these genes. So, we have 20 genes, so you have 20 vectors. Now, suppose I have these 20 vectors corresponding to these 20 genes.

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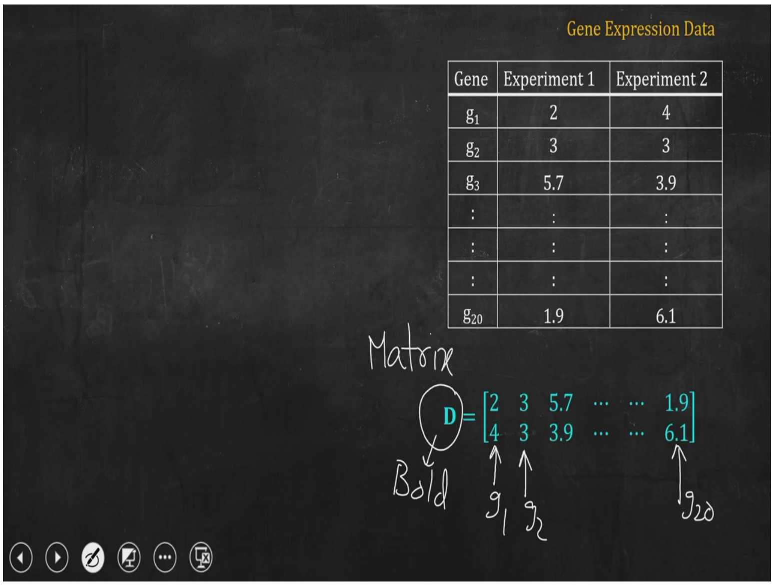

$$D = \begin{vmatrix} 2 & 3 & 5.7 & \ldots & \ldots & 1.9 \\ 4 & 3 & 3.9 & \ldots & \ldots & 6.1 \end{vmatrix}$$

Now, what if I arranged them as a set just like this. So, what I have done? This is the vector for g1, this is the vector for g2 and so on this is the you can easily recognize, this is the vector for g20. So, what I have done, I have arranged them side by side and I call these a matrix and I have given a name D for that and you know D I have written it as in a bold, because I represent is not just a variable d or something, it is a matrix. So, you can say that a matrix is nothing but a set of vectors.

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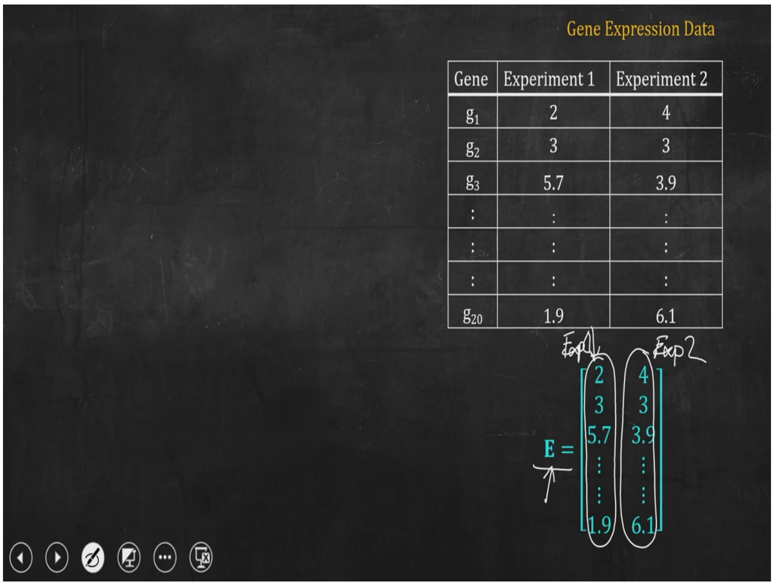

$$E = \begin{vmatrix} 2 & 4 \\ 3 & 3 \\ 5.7 & 3.9 \\ : & : \\ : & : \\ 1.9 & 6.1 \end{vmatrix}$$

Now, I can create a matrix from the same data set in a other way also, in this case, what I have done? I have made one vector here you can consider this as one single column vector, where all the fold change for the gene expression for all the genes from experiment 1 are stacked. And the next one is another column vector, which is for experiment 2, and I name this E, this is also a matrix. So, in a way a matrix is nothing you can imagine it as a set of vectors or you can also consider these as a matrix as a array of numbers. So, it is just a array of number and a table is also an array of number it is a matrix is also an array of numbers.

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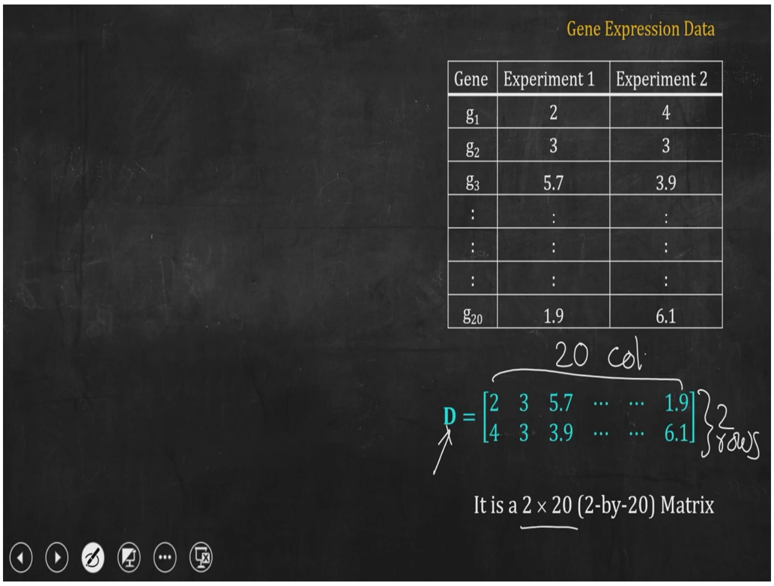

D =

$$\begin{vmatrix} 2 & 3 & 5.7 & \dots & \dots & 1.9 \\ 4 & 3 & 3.9 & \dots & \dots & 6.1 \end{vmatrix}$$

Now, let us go back to the D matrix. So, if you see how many rows we have? We have 2 rows and we have how many columns? 20 columns. So, these matrix D is called a 2 by 20 matrix.

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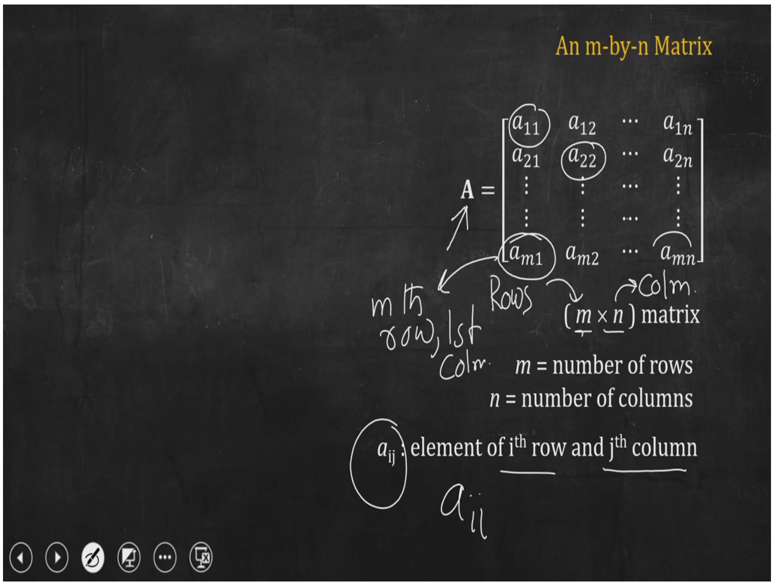

A =

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \end{vmatrix}$$

$$\left| \begin{matrix} a_{m1} & a_{m2} & \ldots & a_{mn} \end{matrix} \right|$$

Similarly, imagine a something bigger something like m by n matrix, that means in that case, I have m number of rows. And n number of columns, note that we always write the row first and then we give the multiplication sign and the column numbers and is this product of m into n is the total number of elements, total number of numbers in these matrix that is the size of the matrix.

So, I have an m by n matrix, which A here and try to follow the naming convention that we have used it always does not maintain these naming conventions, but it is always good that you follow these specific naming conventions. So, for example, this is am1 that means this is the m th row. So, this is am1 is the m th row and the first column. So, m is the row number, n is the column number and aij is the element of i th row and j th column. So, what will be aii? aii will be the diagonal element, for example, a11, a22 for example, somewhere we will have a am somewhere here. So, those are the diagonal element.

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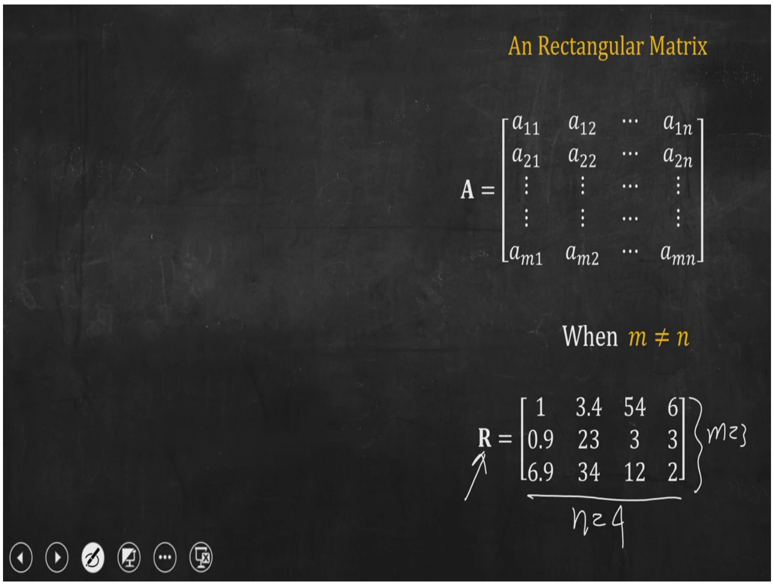

R =

1	3.4	54	6
0.9	23	3	3
6.9	34	12	2

Now, depending upon the value or m and n, we can have different types of matrix. For example, we can have a rectangular matrix, where m and n are not equal, in the example given here is R. R has you can count it has 3 rows m equal to 3, whereas, the number of column n is equal to 4. So, I got a rectangular matrix and in data analysis in biological will regularly meet this type of rectangular matrices, these matrices can come from suppose that type of gene expression experiment.

Suppose, you have 4 experimental conditions, and you have measured the change in gene expression of 20 or 200 genes. So, 1 dimension or the number of rows or number of column will be a the number of experiments which is lesser than the number of genes. So, I will always get a rectangular matrix either a vertical one or a horizontal one. But sometimes, we will get also square matrix, although rectangular matrix we meet regularly in data analysis, we actually like to work on square matrix.

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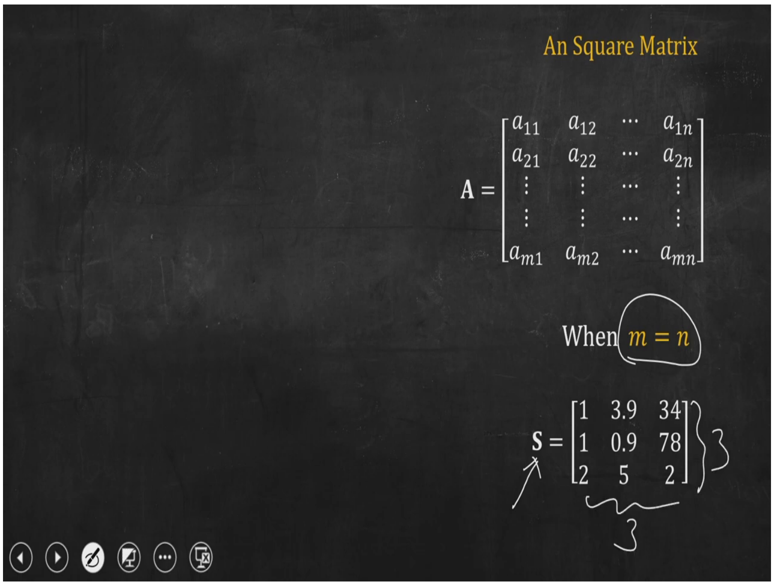

$$S = \begin{vmatrix} 1 & 3.9 & 34 \\ 1 & 0.9 & 78 \\ 2 & 5 & 2 \end{vmatrix}$$

What is the square matrix? In square matrix, both the row number and column number are same. Now, example is suppose this S matrix, you see, I have 3 rows here, and I have 3 columns. So, the square matrix, and analysis a mathematical analysis a square matrix is very rich, and we have lots of tools to analyze square matrix and square matrix has certain unique properties. Those properties are very useful for our data analysis.

So, although you will meet rectangular matrices very regularly in our raw data, we do some transformation or analyses. And eventually, will end up most of the time with some square matrices and do lots of beautiful mathematical analyses… on that matrix and try to get a meaning out of the data.

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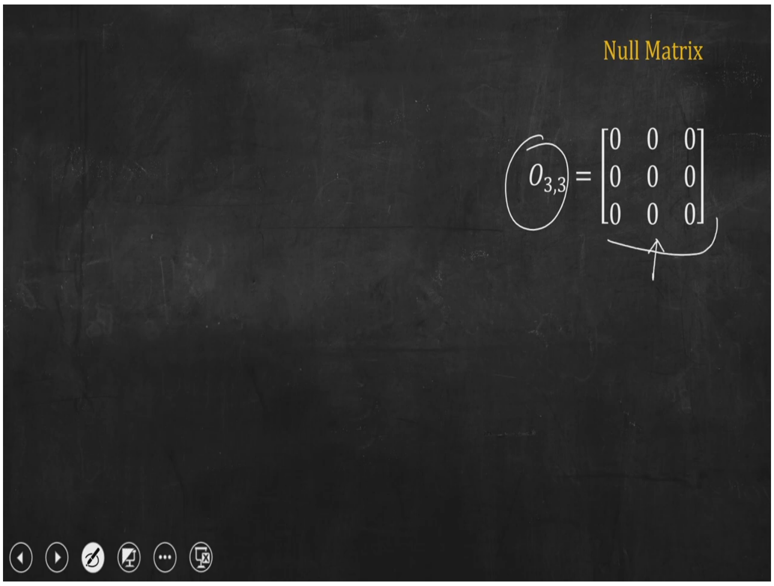

$$O_{3,3} =$$

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

So, let us look into a few other types of matrix, just like your scalar value can be 0 a number can have 0 value. So, my matrix can be also a 0 matrix or a null matrix what will happen in case of a 0 or null matrix? O usually a capital O is used to represent a null matrix, all the elements if you see here are 0.

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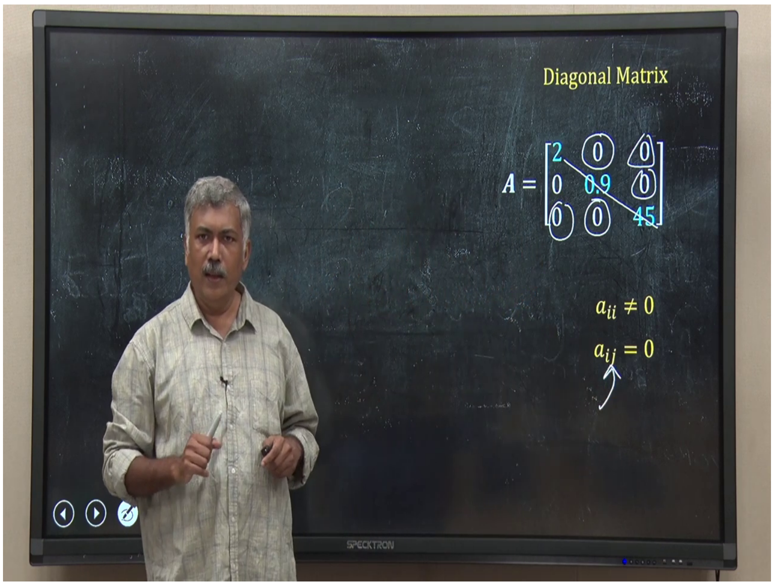

A =

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 45 \end{vmatrix}$$

$$a_{ii} \neq 0,\ a_{ij} = 0$$

Similarly, I can have a diagonal matrix I have shown squared diagonal matrix here, you will see that all the values are along the diagonal only, of diagonal element this one, this one, this one, this one and this one are all 0. So, in a diagonal matrix aii, aii represent the diagonal element. aii is has numerical values, non-zero values, where are all other thing aij, all aij are off diagonal values are all 0.

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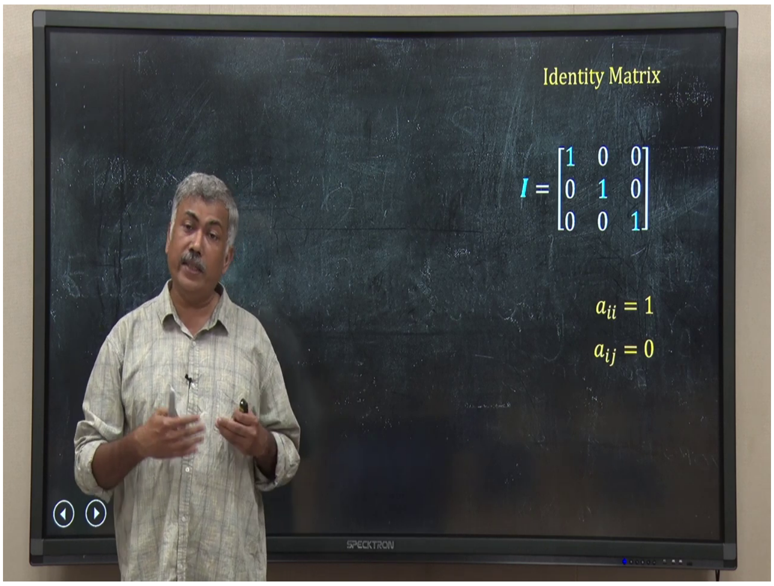

I =

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$a_{ii} = 1,\ a_{ij} = 0$$

There is an interesting diagonal matrix we call identity matrix, what it is a diagonal matrix, but the diagonal elements are 1. And I will show you it is just like number one, a scalar value of one that we use in normal arithmetic identity matrix has that type of utility.

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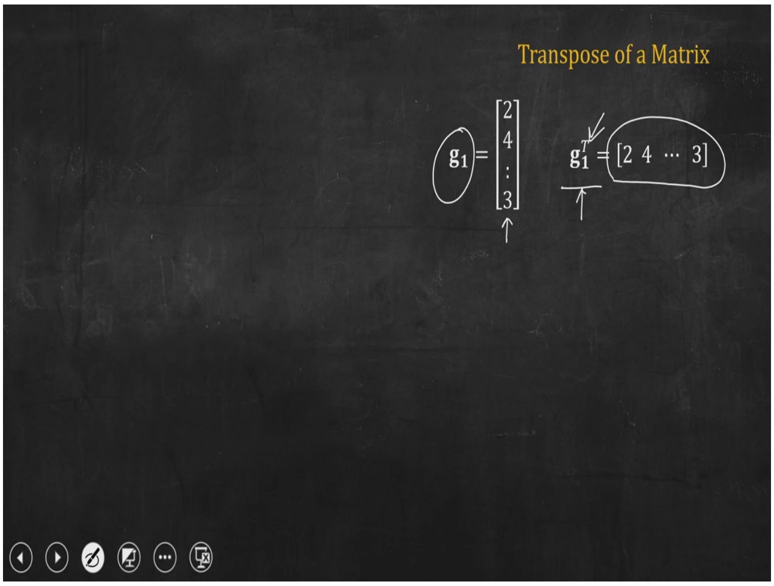

$$g_1 = \begin{vmatrix} 2 \\ 4 \\ : \\ 3 \end{vmatrix}$$

$$g_1^T = [2 \;\; 4 \;\; \cdots \;\; 3]$$

Now, in the last lecture, we discuss about transpose of A vector, is not it? What we did? You gave me a vector suppose a column vector, when I transpose it, I simply flip it make it row vector. So g1 I am given you 2, 4, 3 is a column vector. If I take g1 transpose T in the superscript, I get a row vector just flipped it. Can't I do the similar thing in matrices? Yes, for matrix also I can do that.

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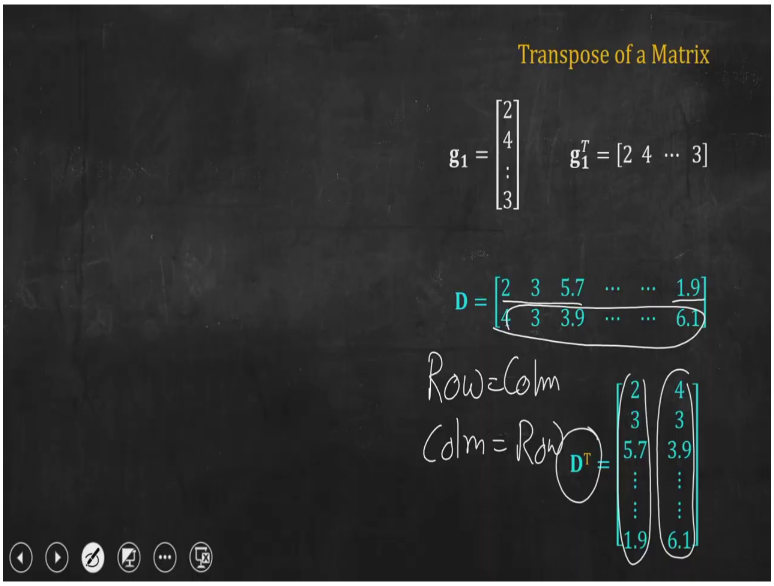

D =

2	3	5.7	...	...	1.9
4	3	3.9	...	...	6.1

D^T =

2	4
3	3
5.7	3.9
:	:
:	:
1.9	6.1

So, D is a rectangular matrix suppose? So, I have 2 rows and suppose 20 columns, something like that. Now, if I flip it, flip it in the sense that the rows now, each row now becomes a column. And inversely, a column becomes a row. See what I have done here, 2, 3, 5.7 up to 1.9 is the first row, I have flipped it and made it a column. The second row is 4, 3, up to 6.1. And I have flipped it to get 4, 3, 3.9 and 6.1 a column vector. So, I flipped it. So, rows become column, column becomes rows. So this is transposition of a matrix. So, that is why I have written D transpose T, T is the superscript.

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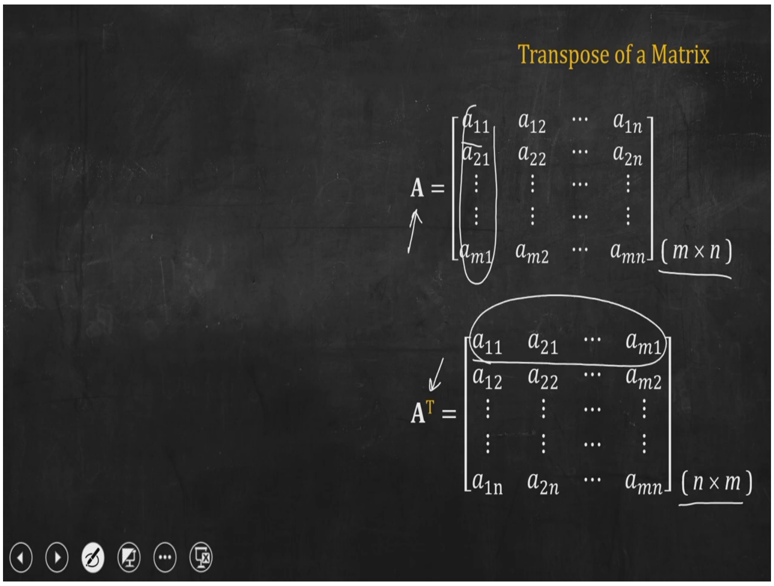

$$A = (m \times n)$$

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix}$$

$$A^T$$

$$= (n \times m)$$

$$\begin{vmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{vmatrix}$$

So, if I have an m by n matrix, m number of rows n number of columns, if I make a transpose of that, I will get a matrix which will have n number of rows and m number of columns. So, that is what I have shown here. A is a m by n matrix I have flipped I have converted rows into columns. So, I get the transpose of A, A transpose which is nothing but now n into m matrix, n number of rows m number of columns.

So, a11, a21 this one you can see has become this row it is easy. Now, what we have learned? We have learned till now, just like vectors we can actually stack vector, arrange vectors, as I take as a set of vectors and create a matrix and matrix can be rectangular, it can be square, and just like a vector, I can also do transposition of a matrix. So, cannot I do other operations like addition and multiplication that we learned for vectors? Yes, we can do that. So, let us learn those one by one.

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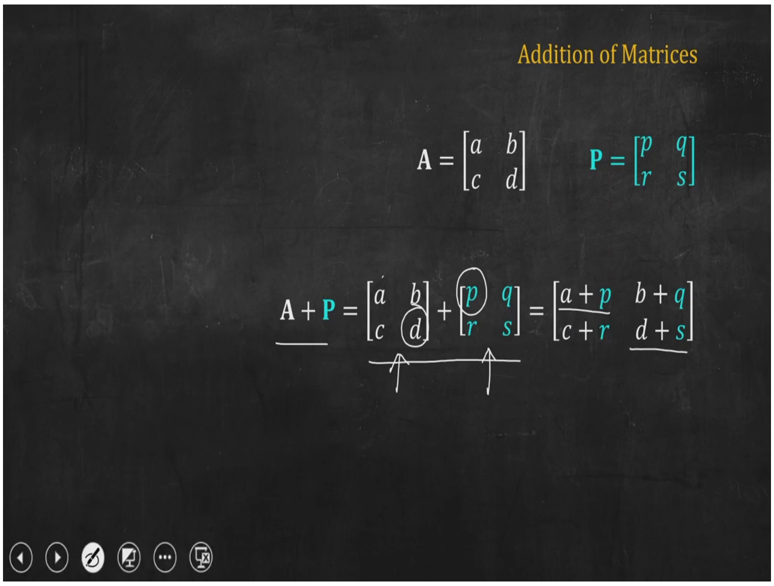

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$P = \begin{vmatrix} p & q \\ r & s \end{vmatrix}$$

$$A + P = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} p & q \\ r & s \end{vmatrix} = \begin{vmatrix} a+p & b+q \\ c+r & d+s \end{vmatrix}$$

First the addition of matrices, two matrices you have given me, I want to add them. So, A and P are given. So, A plus P will be equal to the I have written it explicitly. So, a, b, c, d, e is the

A matrix and p, q, r, s is my P matrix. So, what I will do? I will take the first row first column element A, and I will sum that with first row first column element of the second matrix P. So, I will get a plus p.

Similarly, if I take the second column second rows D, then the corresponding second matrix will be s, so I sum them together and I get d plus s. So, in this way element by element matching their column row position, we sum them, add them. So, we get the at the product… result of the addition of these 2 matrices.

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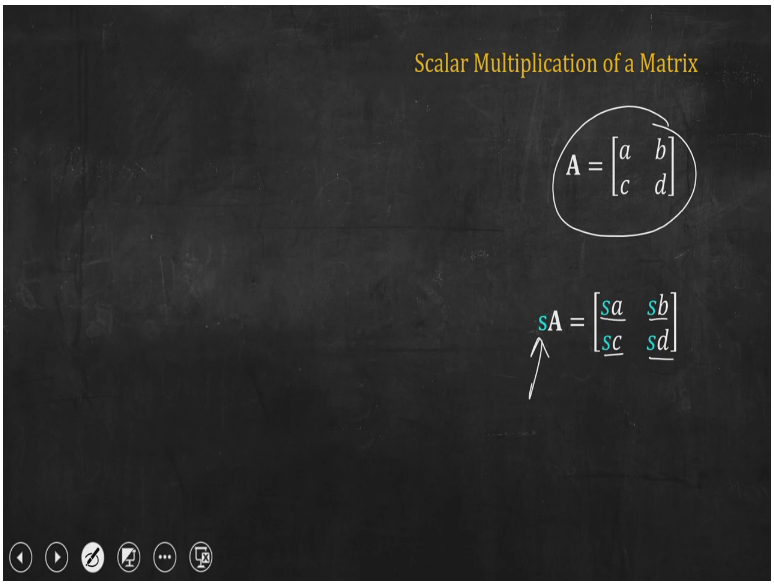

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$sA = \begin{vmatrix} sa & sb \\ sc & sd \end{vmatrix}$$

Now, in the lecture of vectors, we have learned about scalar multiplication, I had a vector I multiplied by a scalar value, normal simple number. So, here also I can multiply a matrix by a simple number a scalar, so suppose the scalar is s, and I am multiplying A, which is a, b, c, d, a matrix a square matrix. So, what will happen in this case? Just like the vector, all the terms, all the elements in this matrix will now get multiplied by their scalar smallest. So, what I will get? I will get sa, sb, sc, sd as simple as that.

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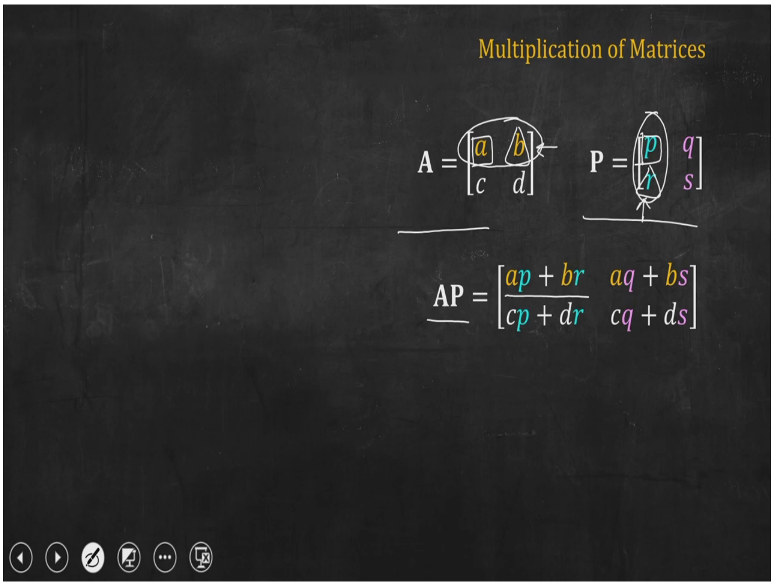

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$P = \begin{vmatrix} p & q \\ r & s \end{vmatrix}$$

$$AP = \begin{vmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{vmatrix}$$

Now, comes the tricky part, I want to multiply one matrix with another matrix. So, I want to do multiplication of matrices. So, this one is bit tricky, although not very difficult, just try to follow me and you will get the logic what we are doing here, two matrices are given to me p and q. What I have done, I have color-coded the rows and columns. So, the rows of A are with yellow color and white color, whether the column of P are blue and pink.

So, what I will do? I will take first, the row first row of matrix A and the first column of the P. What I am doing, I am multiplying A by P, A by P. So, I will take the first row of A and the first column of P matrix, then what I will do? I will multiply A by this P, the first element of these two columns and rows and then add that with b into r. So, that is what I have done here a into p plus b into r.

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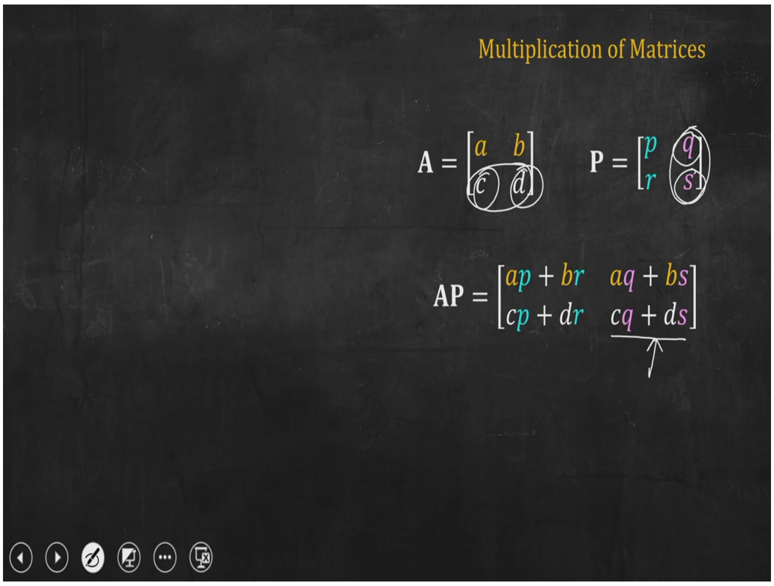

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$P = \begin{vmatrix} p & q \\ r & s \end{vmatrix}$$

$$AP = \begin{vmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{vmatrix}$$

Now, try it again for another column and row. So, suppose c and d. And I want to multiply with q and s. So, what I will do in this case? c into q, s into d and they will go and multiply and they will go here. Let us try another one.

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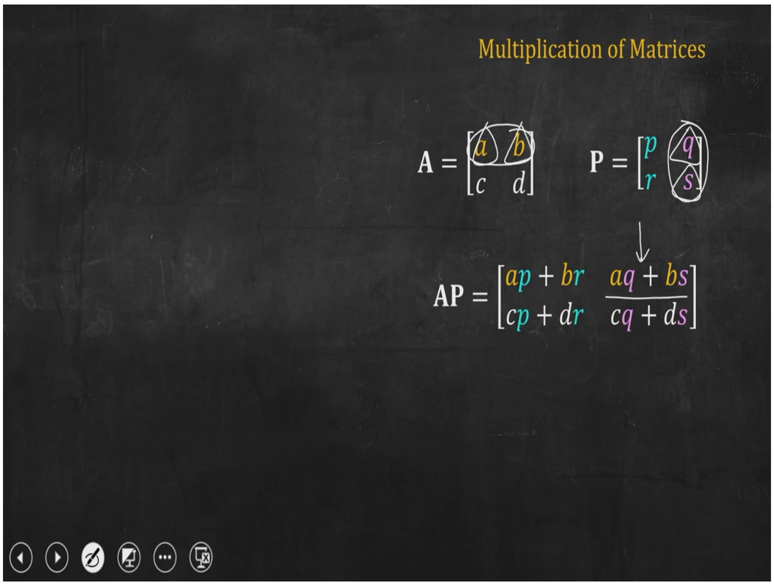

A =

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

P =

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix}$$

AP =

$$\begin{vmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{vmatrix}$$

I want to get this element, this value, how I will get it? I will take this row and this column. So, a into b, sorry q and b into s. That will give me aq plus bs. So, in this way, we multiply rows and columns, the rows of the first matrix with the column of the second matrix, and then we get the results and we arrange them in a matrix format.

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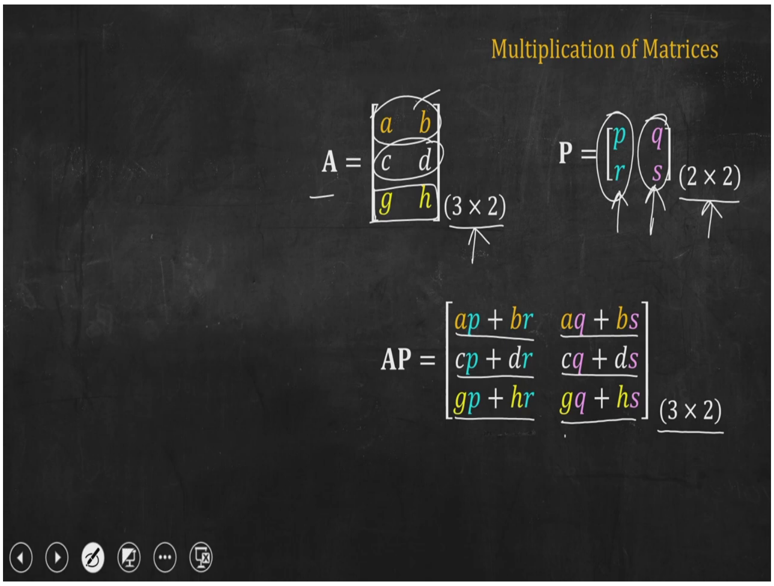

$$A = (3 \times 2)$$

$$\begin{vmatrix} a & b \\ c & d \\ g & h \end{vmatrix}$$

$$P = (2 \times 2)$$

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix}$$

$$AP = (3 \times 2)$$

$$\begin{vmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \\ gp + hr & gq + hs \end{vmatrix}$$

Now, let us try something where the matrix multiplication will get affected by the dimension like the rows and columns number of the matrix. So, what I have here as example? I have A, which is 3 by 2 and P, which is 2 by 2, so one is rectangular one is a square. So, let me try how to do try to do the multiplication. So, I will take the first row, and I will multiply with the first column. So, I will get ap plus br simple, I will take the first row and the second column, so I will get aq plus bs. In this way, I will take the second row and multiply first with the first column, and I will get cp plus dr.

And then with the second column, I will get cq plus ds and then I multiply take the last group of matrix A, and I do the similar operation, so I get gp plus hr, because I am multiplying with the first column, then I multiply with the second column. So, I get gq plus hs. So, the end result is I have multiplied at 3 by 2 and 2 by 2 by 2 by 2 and I get a 3 by 2 matrix. Now, I will reverse the multiplication here in this example, I have multiply A by P.

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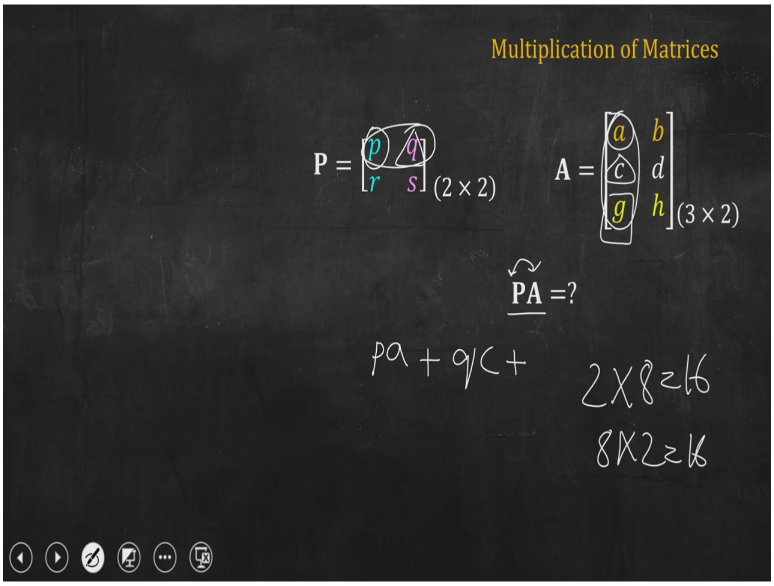

$$P = \begin{vmatrix} p & q \\ r & s \end{vmatrix} (2 \times 2)$$

$$A = \begin{vmatrix} a & b \\ c & d \\ g & h \end{vmatrix} (3 \times 2)$$

$$PA = ?$$

Now, I want to multiply P by A. So, I want to calculate this let us try. So, take the first row and take the first column. So, what will happen, I will have p into a this one and this one, plus q into c, very good, we are moving fine plus. Now, I am supposed to take this one g and multiply with something in the first row, but we have nothing in the first row anymore. So, that means it does not make any sense.

So, now that means in this case, I have just reversed the order of P and A and now, I cannot do my matrix multiplication. In normal multiplication, you can always write 2 into 8 is equal to 16. And similarly, you can say 8 into 2 equal to 16 you can do both away, but in matrix multiplication, you cannot do that. The row number color number should have a particular matching, what is that?

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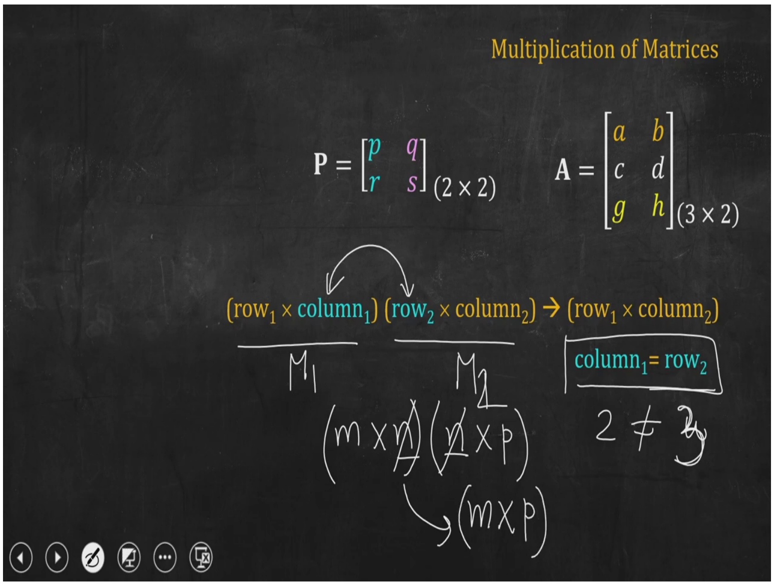

$$\left(row_1 \times column_1\right)\left(row_2 \times column_2\right) \to \left(row_1 \times column_2\right)$$

$$column_1 = row_2$$

Suppose this is for the first matrix M1, this is the second matrix, so you are multiplying of M1 by M2, M1 has the row 1 and column 1 and the row 1 column number, row 2 and the column 2 are the for the second matrix. So, the most important requirement here is that these column number of the first matrix and row number of the second matrix must be equal, that means, column 1 must be equal to row 2.

In the previous example, just when I was trying to multiply P by A, this condition was not met, because the number of column in that case was 2 and the number of row in the second matrix was 3. So, they are not equal so, I cannot multiply them. So, this is what you have to keep in mind.

So, in general, it is very easy if you think this way, I have a m into n matrix and I want to multiply with that with n into p matrix, this n and this n are same, the number of columns of the first matrix is equal to number of rows of the second matrix, the result tend to be this n and this n will vanish and I will get a matrix of m into p. Now, due to this property, as I said earlier, the scalar thing when you take normal arithmetic 8 into 2 and 2 into 8 are same both will give you 16.

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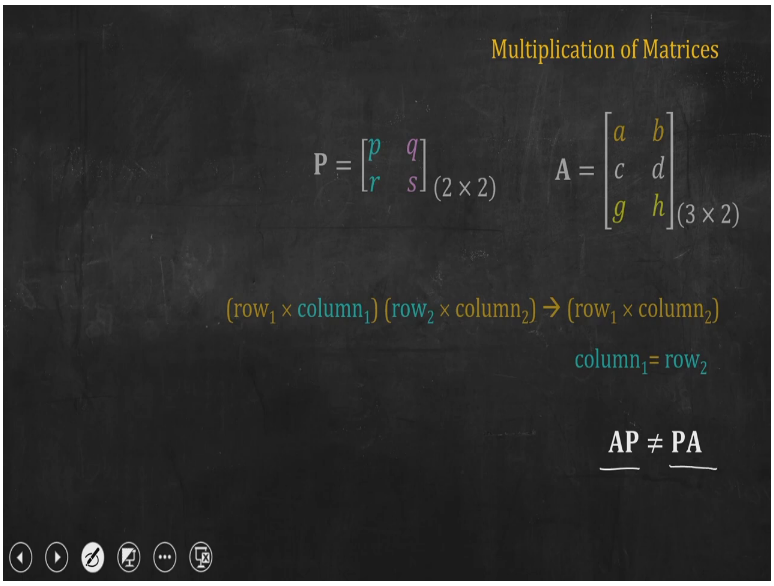

$$AP \neq PA$$

But multiplication of matrices does not have that property in general A into P is not equal to P into A. This property we have to keep in mind when we are doing matrix manipulation and lots of multiplication in our code or somewhere in data analysis. Now, let us do some multiplication with some unique type of matrices.

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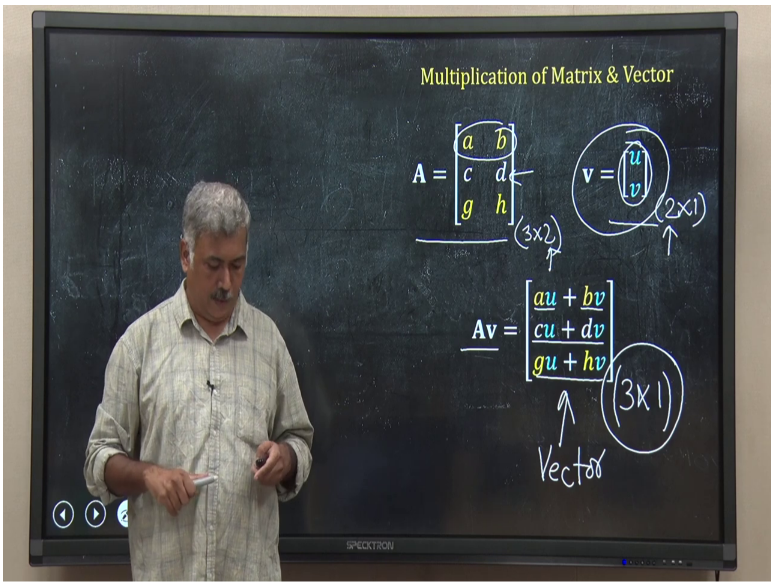

$$A = (3 \times 2)$$

$$\begin{vmatrix} a & b \\ c & d \\ g & h \end{vmatrix}$$

$$v = (2 \times 1)$$

$$\begin{vmatrix} u \\ v \end{vmatrix}$$

$$Av = (3 \times 1)$$

$$\begin{vmatrix} au + bv \\ cu + dv \\ gu + hv \end{vmatrix}$$

For example, I want to multiply a matrix with a vector. So, I have a matrix A, and I have a vector, a vector has this one as a column vector. So, you can imagine this is vector is also a matrix. It has 2 rows, and just 1 column as simple as that. So, I can write these as 2 by 1. Whereas this one is, I have 3 rows and 2 columns. And you can see, this is 2, this is 2, so the number of columns of A is equal to number of rows or V, that means I can multiply A by v, so I can multiply A by v.

So, what will happen, I will take this row and take this column, so I will get au plus bv, then you take the next row, the column remains same, so I get cu plus dv. Similarly, for the last row of A I get gu plus hv. Now, pay attention to the number of rows and columns for this product, the product that we have got, here in this case, I have 3 rows and 1 column that means this is also a vector.

So, what we have got? We have taken a vector and use that to multiply A matrix and I got back another vector. So, you can imagine a matrix just like a function, this is another way of thinking of a matrix. You would consider imagine the matrix A in this case as a function, it takes a vector and spit out another vector. In more mathematical terms, what matrix A has done on matrix v is that, it has linearly transformed it has done a transformation where the vector v has now transformed into another vector.

And one important thing in this particular example, because we have different row and column numbers in these matrix and vector you can see I started with a vector which is 2 by 1, 2 rows. So, the input I have given is a vector with 2 rows that means a 2-dimensional vector, whereas the output I have got is 3 by 1. So, it is 3-dimensional vector. So, it has moved into a different space. So, you can imagine a matrix also in this way, rather than just imagining it as a you know set of vectors or area of numbers, you can imagine it as if it is a function who is do some sort of linear transformation, a transformation on a vector, a given vector.

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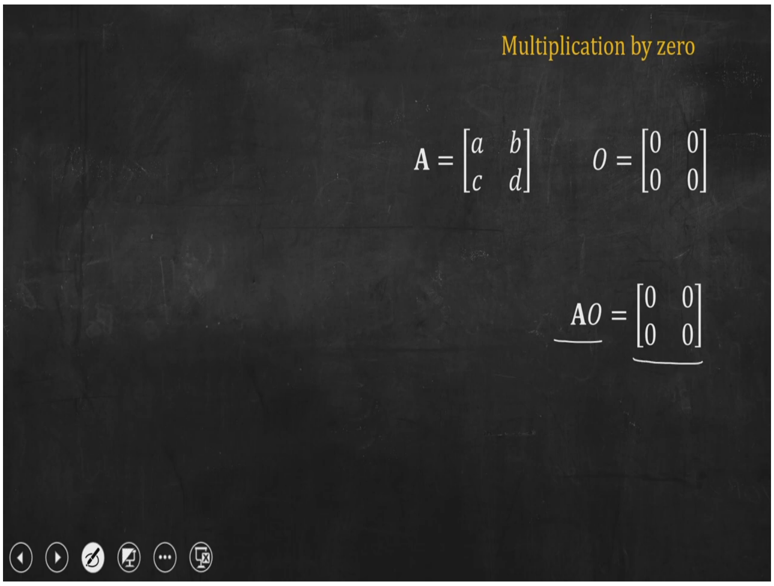

A =

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

O =

$$\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$$

AO =

$$\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$$

Now, another special type of multiplication, which you may encounter data analysis is multiplication by 0. So, if I take a scalar value and multiply it by 0 what do I get I get 0. So, similarly, here if I multiply A with a null matrix, I should get a 0 the only condition obviously, here is that you have to match the row number column number issue that is required for matrix multiplication.

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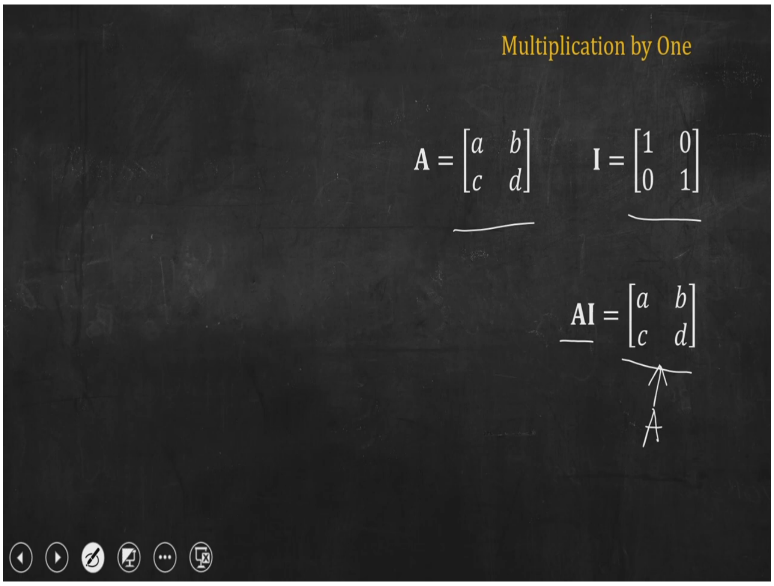

A =

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

I =

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

AI =

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Now, similarly, I can multiply just like a scalar value I can multiply with 1 I can multiply a matrix also something like one. So, in this case, I do not have 1 I will have identity matrix identity matrix is nothing where you have all the diagonal values are 1. So, you have to match the rows and column number. So, I have 2 by 2 here. So, I have here 2 by 2. And if I multiply A by I, I will get the same matrix A, you check both them are same. So, I am getting back the same metrics.

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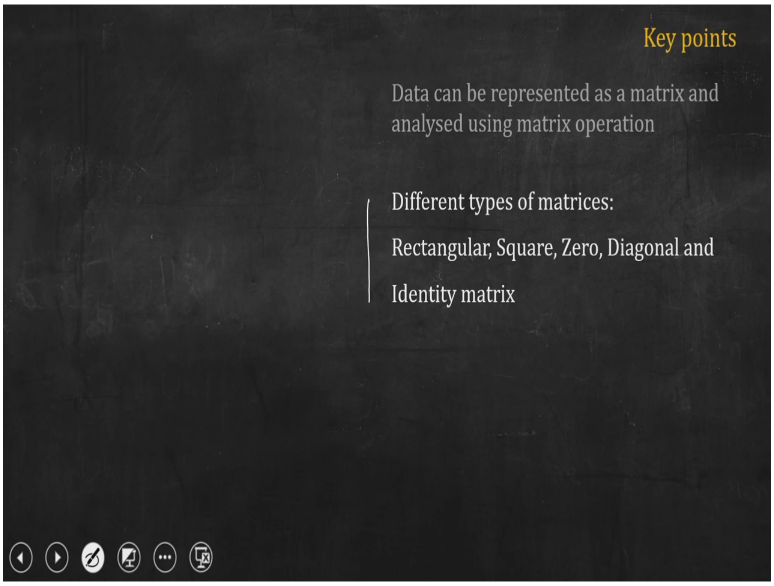

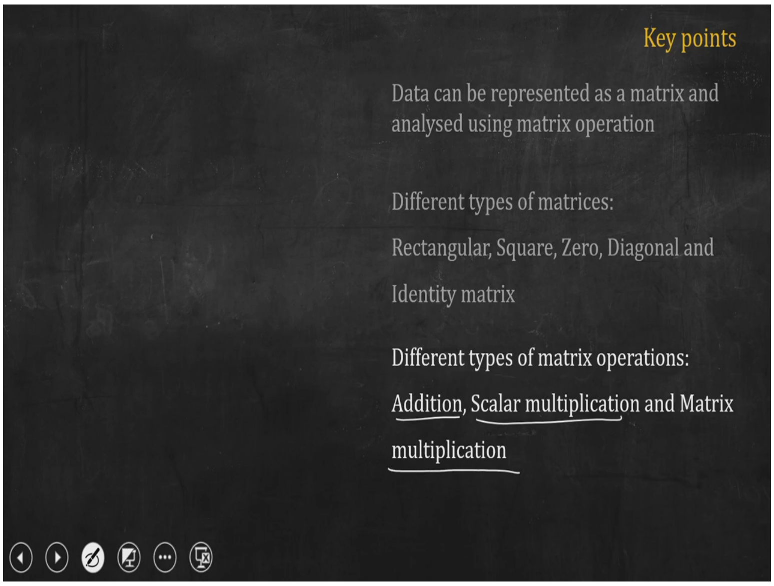

So, now I will jot down the key points that we have learned in this lecture. In this lecture, we have learned about how we can represent a data as a matrix. And just to remind you, you can think of a matrix as a set of vectors, you can think of it as a array of numbers or even as I discuss in later on, when I was discussing multiplication of a matrix with a vector, you can imagine a matrix as a function which transform a vector, it takes up a vector and spit out a new vector. So, in these all these 3 ways you can imagine a matrix.

Now, just like vector you have different operations on a vector, we can do all those arithmetic operations also on matrices. And we have learned also in this lecture, certain particular type of matrices for example, we have learned about the rectangular matrices square matrices 0 and diagonal matrix and identity matrix.

And then just like other operations, we have done we in case of vectors we have done learned about addition, scalar multiplication and matrix multiplication. That is all for this particular lecture on matrix and matrix operation before I leave I will leave you with a problem to think over.

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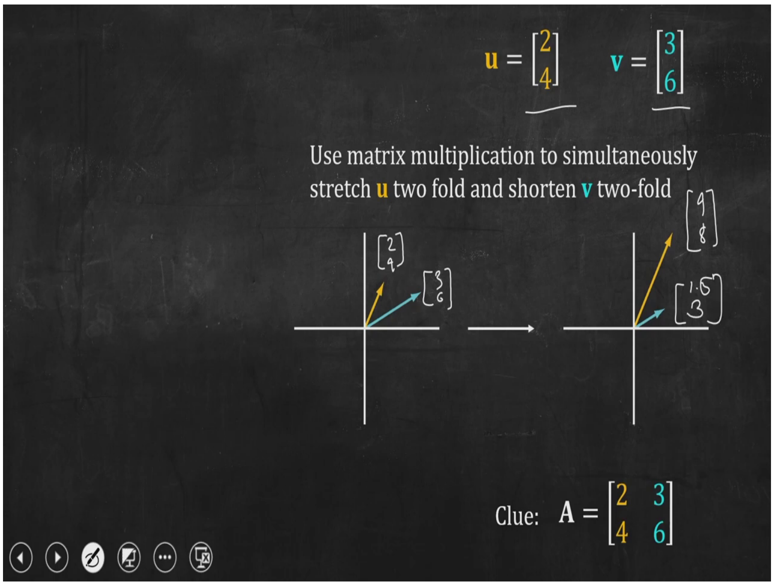

u =

$$\begin{vmatrix} 2 \\ 4 \end{vmatrix}$$

v =

$$\begin{vmatrix} 3 \\ 6 \end{vmatrix}$$

Clue: A =

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}$$

Suppose 2 vector are given to you, u and v. So, u is 2 by 4, 2 and 4 and 3, 6. So, I have shown it here, this is u 2, 4 and this is 3, 6 suppose, what I want? I want to convert this matrix so, sorry these vectors in such a way that these 2, 4 it becomes double. So, it becomes something 4, 8 whereas, the other one give becomes half. So, it becomes 1.5 and 3 suppose.

So, I want to transform them in this way, the one vector will get stretched it become double in size the other one will get squeezed and it will become half in size, but I want to do these transformations simultaneously not one by one I want to do the simultaneously and using some matrix manipulation technique.

So, this is a trick that we use many times in data analysis and then repeatedly doing some a particular operation, we can actually use matrix operation which are very easy to do in R and other programming and many other programming languages. We will do some matrix manipulation, and I will get the result. So, try to do that. I will leave you a clue, what you have to do? You have to consider a matrix where you will put your original vectors u and v are as their columns.

So, you create a matrix where u and v are stacked side by side and you have created a matrix. Now, think about something what can you do on this matrix? So, that simultaneously one vector will become extended stretched twice, whereas the other one will we can shorten by 2-fold. Think about it and join back in the next lecture. Till then Happy Learning.