Data Analysis for Biologists
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Singular Value Decomposition

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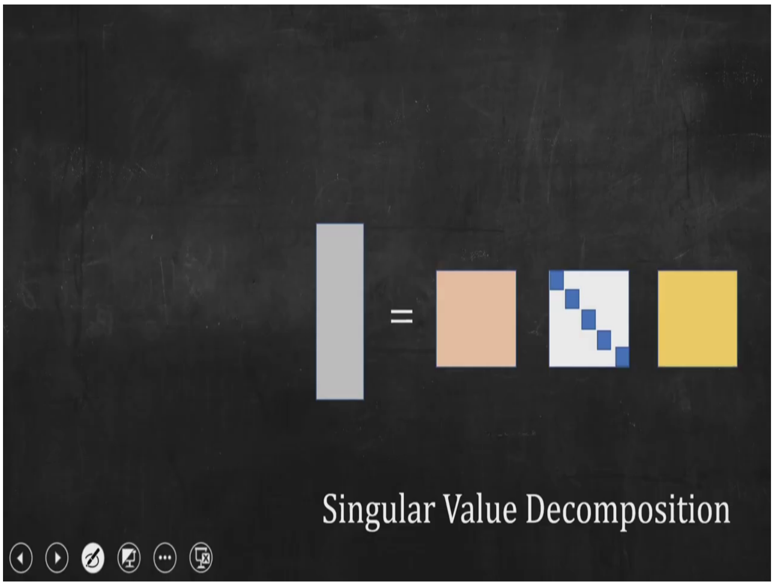

Welcome back everyone. In this video, we will discuss about singular value decomposition. It is a very powerful technique of linear algebra which is widely used in data analysis and machine learning. But I will start discussion with a different decomposition that is called Eigen decomposition. And we have studied Eigen value in one of the lecture in this series of linear algebra talk in this data analysis course.

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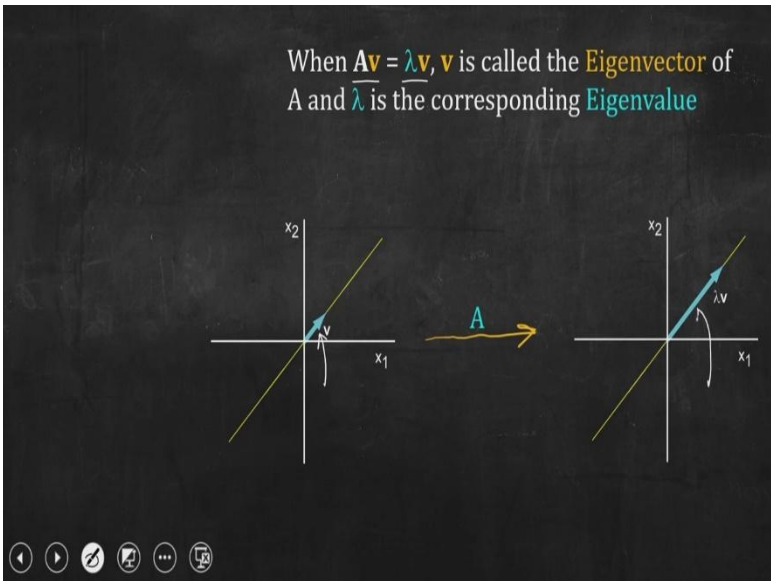

So, let me start with what is the Eigen value and what is the Eigen vector. So, if you remember if I have a square matrix, remember Eigen value Eigen vectors are defined only for square matrix. So, if I have a square matrix A and then its eigenvectors is called v and the Eigen value corresponding eigen values lambda is such that, if I make A into v, the matrix into the vector Eigen vector that will be equal to lambda into v that is our definition of eigenvalue and eigenvector.

$$Av = \lambda v$$

What is happening? Just to remind you, what we have discussed in another lecture on Eigen values and Eigen vectors this is my Eigen vector v. If I multiply this with A I am doing a linear transformation A into v and then I get a new vector lambda v. So, the direction of the vector does not change only it remains on the same span either it gets squeezed elongated or it get flipped on the same line, that is what is called eigenvalue and eigenvector.

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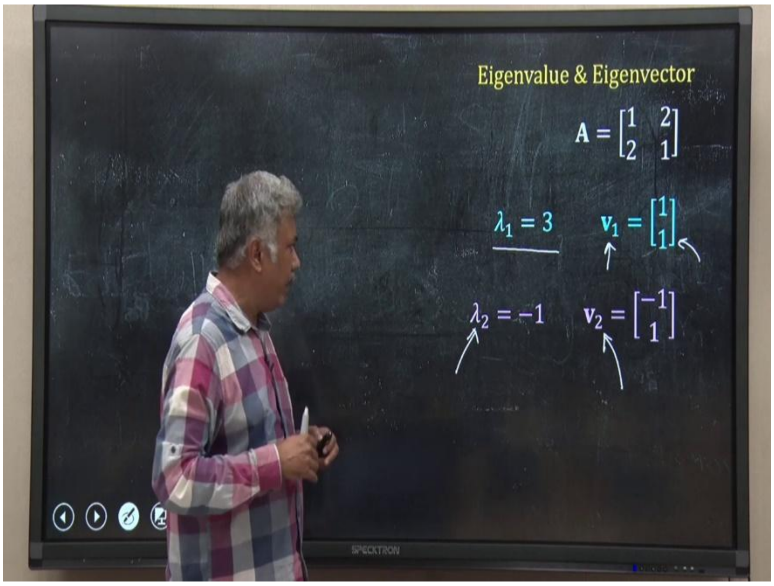

$$A = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$v1 = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$v2 = \begin{vmatrix} -1 \\ 1 \end{vmatrix}$$

$$\lambda_1 = 3, \; \lambda_2 = -1$$

So, now, let us take an example we have looked into that example earlier also. So, I have a matrix A 1 2 2 1 its actually a symmetric matrix if you remember. So, it has two eigenvalues and two eigenvector, eigenvalue one lambda 1 is 3 and the corresponding eigenvector is v1, this is 1 1 and the second eigenvalue is lambda 2 and the corresponding eigenvector is v2 minus 1 1. I have color coded them because I mix them together, create a new matrix and vectors, so it will be easy to understand.

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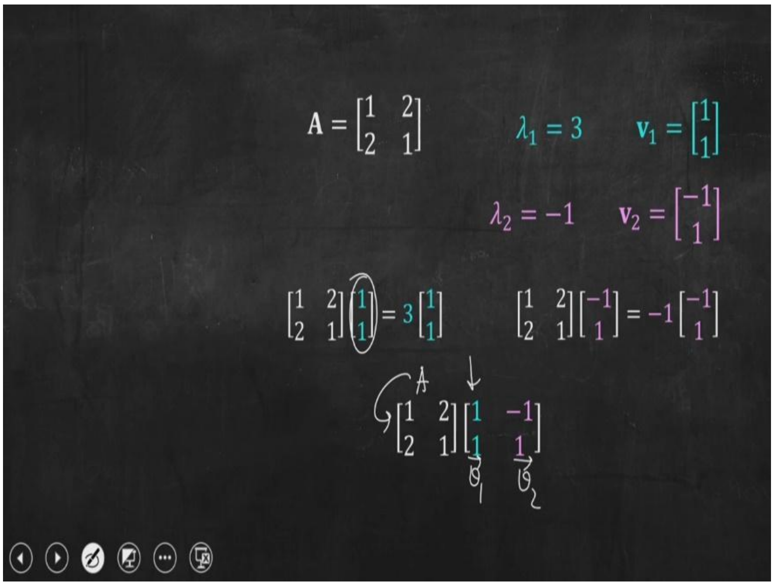

A =

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

v1 =

$$\begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

v2 =

$$\begin{vmatrix} -1 \\ 1 \end{vmatrix}$$

$\lambda_1 = 3, \ \lambda_2 = -1$

Av1 =

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

=

$$\begin{vmatrix} 3 \end{vmatrix}$$

$$\begin{vmatrix} 3 \end{vmatrix}$$

=

3 x

$$\begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

Av2 =

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} -1 \\ 1 \end{vmatrix}$$

=

-1 x

$$\begin{vmatrix} -1 \\ 1 \end{vmatrix}$$

So, now, let me rearrange this term and write it another way. So, what I have done here I have taken matrix A and I have multiplied with the first Eigen vector v1. So, I get that equal to lambda 1 and v1 that is my definition. Similarly, here I have A into v2 vector Eigen vector second one and the second eigenvalue into v2 the second eigenvector corresponding Eigen vector.

So, I have two equations. So, I will now take these first equation and the second equation, and I will club them together in a single unit, I will put them together in a single unit using matrix operations. So, what I will do? Let us start what I have done this is my A you can see this is my A and I have taken the first Eigen vector and put it here. So, this is my v1 and side by side arrange the v2 the second eigenvector, so, I have created a new matrix using v1 and v2 and I am multiplying A with that new matrix.

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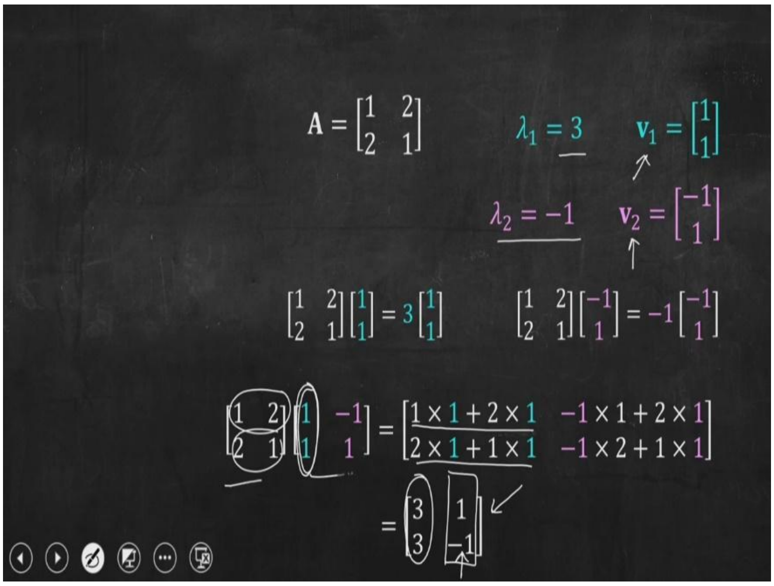

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \qquad \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$=$$

$$\begin{vmatrix} 1 \times 1 + 2 \times 1 & -1 \times 1 + 2 \times 1 \\ 2 \times 1 + 1 \times 1 & -1 \times 2 + 1 \times 1 \end{vmatrix}$$

$$=$$

$$\begin{vmatrix} 3 & 0 \\ 3 & -1 \end{vmatrix}$$

So, what do I get? I will do use the rule of multiplication of two matrices. So, I get this one you can try yourself. So, first you will take these column and the first row, you will get this one then take the first column and the second row, so you will get this one. In this way if you do the multiplication yourself, you will find I am getting a matrix 3 1 3 minus 1.

Now, look carefully to this matrix, this is 3 3, this vector, this column vector is 3 3. So, if I multiply my v1 vector first Eigen vector, with its Eigen value 3, I will get this vector. Similarly, look into the second of the column vector of this matrix that have got by multiplication 1 minus 1, if I multiply my Eigen vector v2 by this lambda 2 Eigen value, I should get this one, try yourself you will get that one. That means, in this new matrix that I have got by multiplying this matrix and this matrix is actually hiding those eigenvalues and eigenvectors. So, let me expand it.

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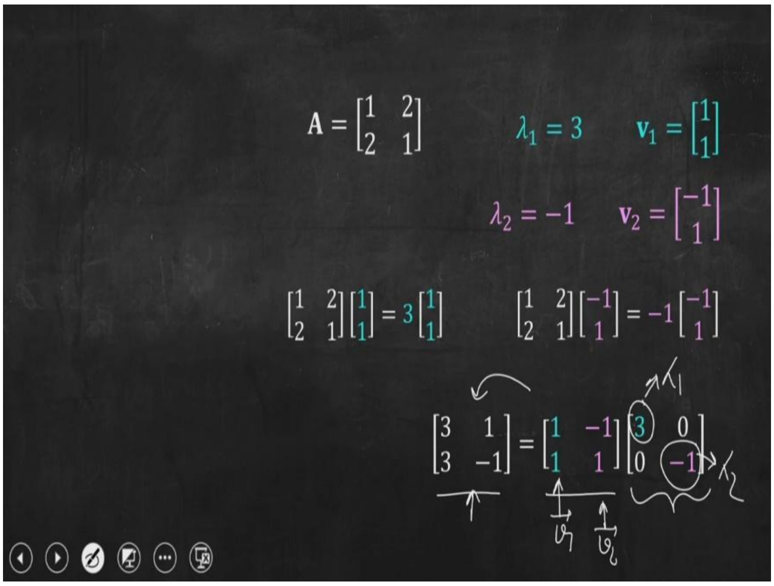

$$\begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \quad \begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix}$$

So, I can write this product that I have got just now is equal to again that same matrix this is v1 and this is v2. Now, what I have done here in the next matrix is a diagonal matrix only there are nonzero element in the diagonal of the matrix all other values are zero. And what are the diagonal element? The first diagonal element is this one is lambda 1 the first eigen value and this one is the second eigenvalue. Now, you do the multiplication using the rule of multiplication of matrices you will find you will get this matrix.

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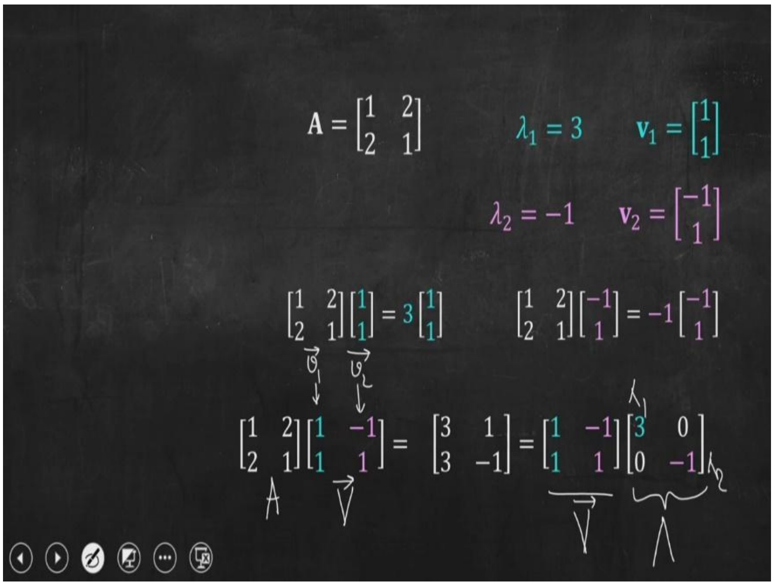

So, now, I will equate both the thing, what I have done from the multiplication that I have done in the previous slide and the multiplication in this slide. So, what I will get I have A I multiplied that with a vector V, I will write that as a capital V because that has two column vectors the first column vector is v1 the Eigen vector the first Eigen vector of A and it has a second column vector v2, which is the second eigenvector of A that will be equal to the same V matrix, capital V remember not small v and this one I will call lambda capital lambda because it is hiding my it is a diagonal matrix it is a diagonal matrix with lambda 1 here and lambda 2 here in the diagonal.

So, what I have got? I started with a single matrix I know its Eigen vectors and Eigen values. So, now, I have represented this Eigenvalue Eigenvector relationship in a different way, I have written A into V capital V is equal to capital V into lambda capital lambda capital lambda is a diagonal vector with the eigenvalues along the diagonal.

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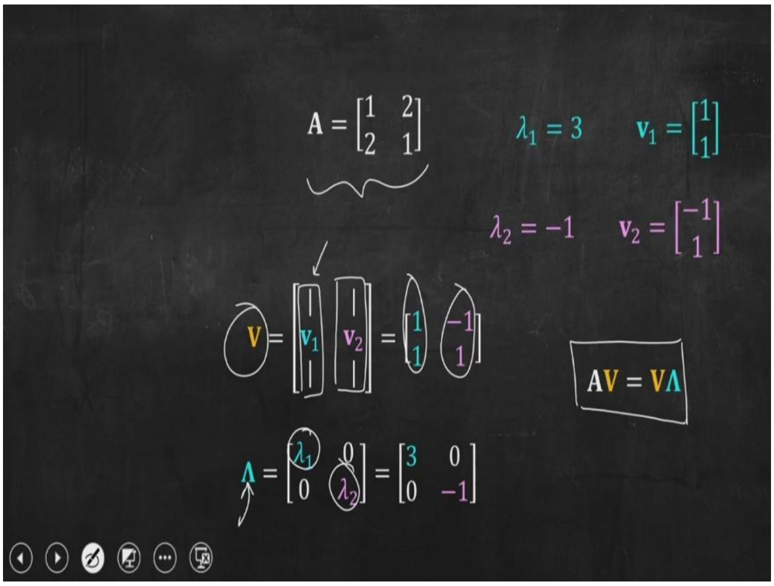

$$V = v_1 \, v_2 =$$

$$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$AV = VA$$

$$A =$$

$$\begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix}$$

$$=$$

$$\begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix}$$

So, if I just recapitulate what we have done till now, I started with A, I have club the eigenvectors to create a new matrix V, capital V and remember usually in the subsequent slide I will use this symbol, this V is these represent a column and this is the first eigenvector and this represent the second eigenvector. And capital lambda that is a diagonal matrix having two Eigen values along the diagonal and that gives me a relation that A into V is equal to V into lambda.

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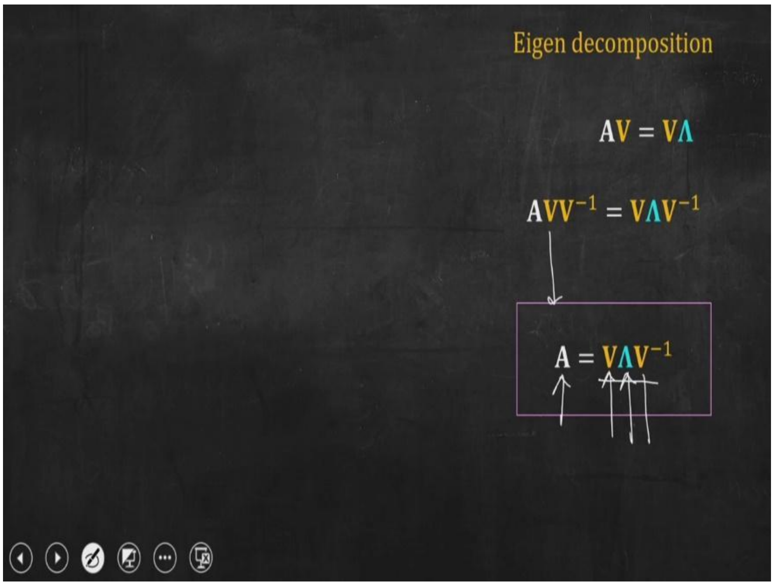

$$AV = VA$$

$$AV V^{-1} = VAV^{-1}$$

$$A = VAV^{-1}$$

Now, comes the part of Eigen decomposition. What is the meaning of decomposition? You want to break down something into parts, components. So, that is what we want to do in Eigen decomposition. I have proved the relationship A into V is equal to V into lambda, I hope you remember what is V, capital V and capital lambda are.

Now, I want to separate out A from this relationship, to separate out A the simplest one would be if I can take on the side the V on the side, but I cannot do the division by a matrix but I can do inversion. So, what I will do? I will multiply both sides by V inverse. So, if I multiply, I will get A into V into V inverse equal to V into lambda into V inverse. Now, what is this? This is the identity matrix.

If I multiply a matrix with its inverse, I should get an identity matrix by definition. An identity matrix is equivalent to 1 and if I multiply A matrix with the identity matrix, I will get back the same matrix. So, I will get A equal to V into lambda into V inverse. So, what I have done? I have broken down, I have decomposed the matrix A in three parts, and what are those three parts? Let us look into it.

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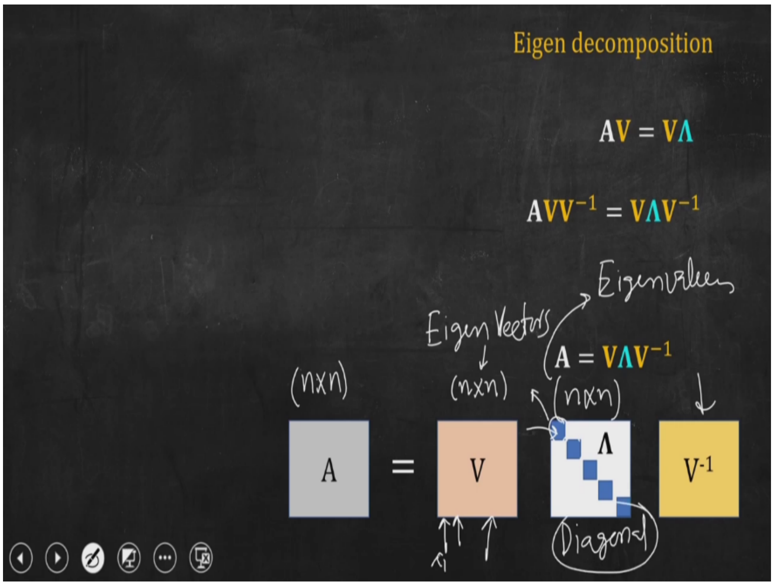

So, this is my square matrix, n into n, just to remind you again because we are using Eigen value Eigen vectors who you have to deal with square matrix. So, I have a square matrix A given to me I have broken it down into three components, the first is also n into n where I have all the Eigen vectors. I have the eigenvectors stacked side by side and then I have n into n diagonal matrix, that is why I have used this type of color in the diagonal, I have a diagonal matrix, where I have all the Eigen values, corresponding eigen values, Eigen values in the diagonal and this is the inverse of the first matrix V that is also a square matrix.

So, one square matrix is now divided or decomposed into three square matrices. And remember one thing we do usually in data analysis what we do, we arrange these Eigen values in order that means, the first eigen value at the topmost corner will have the highest value, then the next one then the next one so, that means the Eigen vectors are also arranged corresponding.

So, this is the highest Eigen value's Eigen vector in the first column something like that, we do that because in data analysis sometimes what we do in particular some dimension reduction techniques what we do we only consider those Eigen vectors and Eigen values, which are more important that means their values are higher, Eigen values are higher.

So, we may discard Eigen values and their corresponding Eigen vectors when the Eigen values are very low. So, if you arrange them in order then it become easier to implement. So, that is why you arrange them in that particular order. So, just briefly let me repeat what we have done. We started with a square matrix A and we have decomposed it by Eigen decomposition into three parts, V into lambda into V inverse.

And this method is very useful you may think why you are doing all this juggling of matrices vectors and Eigen values Eigen vectors, because this is very useful technique. Because under the hood many algorithms for example, you have to square this matrix or cube this matrix, it is very difficult not a trivial job to do, but if you use Eigen decomposition and then try to calculate the cube or square or higher order of A.

It becomes very easy, it becomes easy because you have to remember we have a diagonal matrix here and the mathematics of diagonal matrix is very easy, because it had only nonzero element in the diagonal, all other elements are zero. So, most of the multiplication addition all these things give rise to zero and is a square matrix in this case, so, its transpose is also a diagonal matrix.

So, as the math of diagonal matrix is very easy, as we decompose the matrix A in this fashion, the further analysis of A also becomes very easy. So, far so good, I have decomposed a square matrix A, but the problem is in most of the data analysis that we will deal in real life particularly in biology also, most of the data will not give you a square matrix, it will give usually rectangular matrix, just to remind you.

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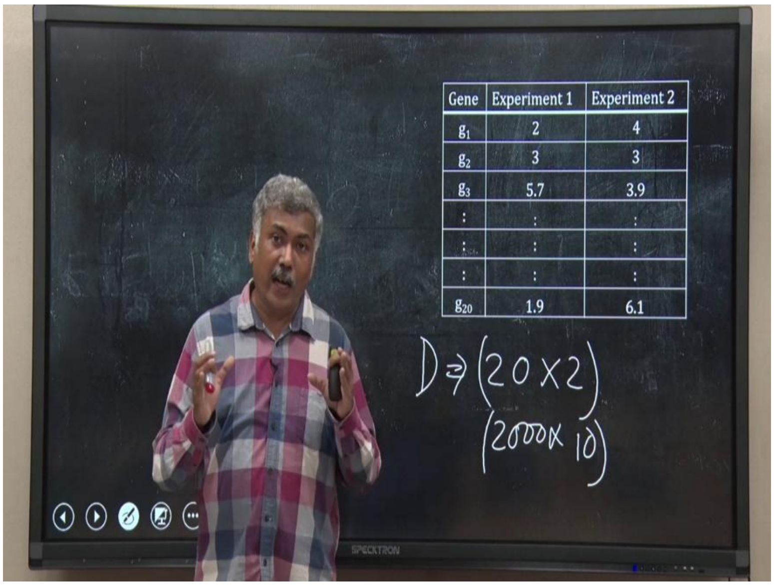

Gene	Experiment1	Experiment2

g1	2	4
g2	3	3
g3	5.7	3.9
:	:	:
:	:	:
:	:	:
g20	1.9	6.1

$D \Longrightarrow (20 \times 3)$

(2000×10)

Let us see one data set that we have handled earlier to understand matrix and vectors. So, this is a gene expression experiment data the hypothetical one. So, I have done two experiments and I have 20 genes. So, if I represent this as a matrix my data D matrix will have the dimension of 20 rows into 2 columns. So, it is no way square, its rectangular, it is skinny tall matrix and in reality.

If you do a microarray or RNA seq analysis in that case maybe your experiments will be a 10 at max because these experiments are very costly to do. So, you may have 10 experiments and you may have 2000 genes. So, you can imagine these are tall skinny matrices, rectangular you can have the opposite one also in some experimental cases some particular data set, but whatever it is, in most real life cases, particularly in biology that data analysis that you will do, you will not have square matrix, you will always have some sort of rectangular matrix.

Now, can't we have a technique of decomposition, where I will take a rectangular matrix and decompose it and decompose it in such a way that I will have those advantages like I will get a diagonal matrix as diagonal matrix math is easy so that I can do lots of other calculation very easily. So, I want to achieve that just like Eigen decomposition.

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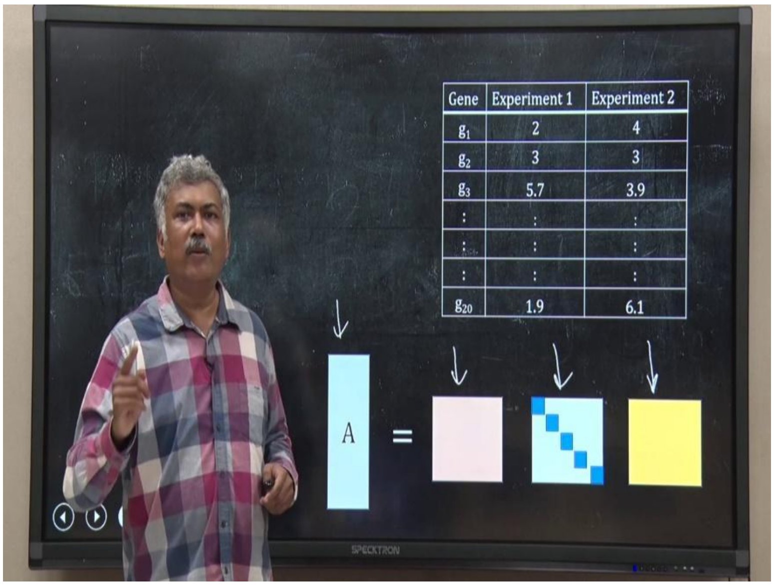

So, that is my wish that I have a rectangular matrix and I want to decompose in a three part, a square matrix, a diagonal matrix and another square matrix that is all. Remember, if I can bring down my data or something to square matrix, math is much well known, well establish very easy lots of tools are there, if I can create a diagonal matrix also lots of tools are there and math is easy. So, I want to decompose my rectangular matrix in this way and that is where singular value decomposition SVD comes, let us learn that.

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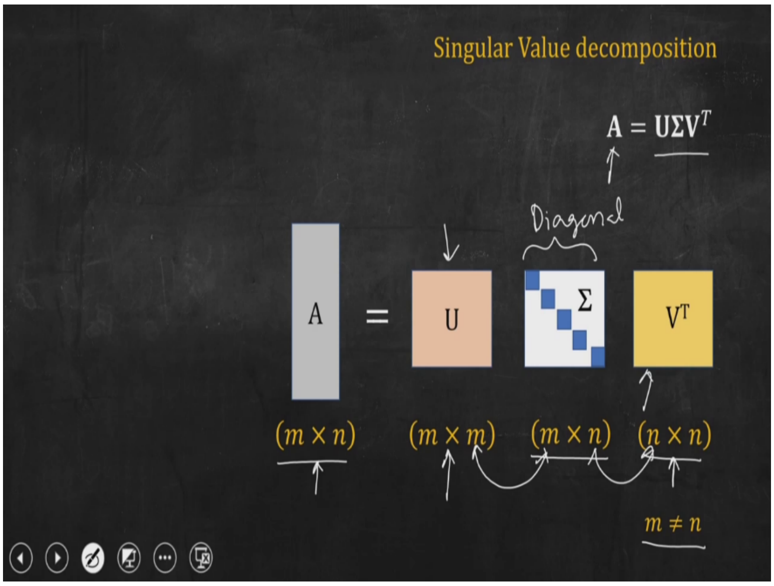

$$(m \times n) = (m \times m)(m \times n)(n \times n)$$

$$m \neq n$$

So, what is SVD? SVD will fulfill my wish. The wish is, I have a square matrix A of dimension m into n, where m is not equal to n in general, for this particular drawing, I have shown m is bigger than n, number of rows is more than number of columns. And I will break it into three components, U will be a square matrix of m into m, V transpose the last component will be n into n square matrix and I will have a diagonal matrix here in the middle whose dimension will be m into n, just to try to use the rule of multiplication this is m into n this is m into m, so m in m matching whereas, these n and n are matching, so, eventually I will get a matrix by multiplying these three matrices m into n matrix.

So, the rule of multiplication is satisfied. So, this is what I want and as I want it just be a wish will not fulfill my wish. So, what I do I have to put some more constraint in it and let me discuss what will be the form of U and V and this sigma, which is the diagonal matrix.

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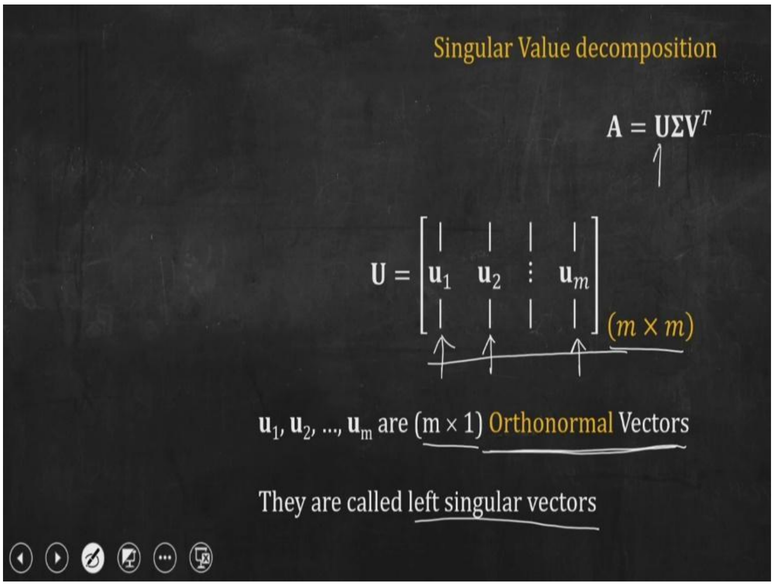

U = [u1 u2 : um] (mXm)

So, the U, I want the U should be a special type of square matrix. It is obviously m into m but it is a square matrix of orthonormal vectors. Now, what is orthonormal? If you remember the lecture of vectors, we started the linear algebra discussion with vectors. In that case, I have discussed that two vectors are orthogonal or perpendicular in laymen terms, if that dot product is zero.

Now, if both the vectors are unit vector, means you have normalized with respect to their individual length. So, if I have two-unit vector and they are orthogonal that means, they are, dot product between them is 0 then we call them orthonormal., orthogonal normalized vector, orthogonal unit vectors. So, that is called orthonormal vectors. So, what I am assuming here is that this U is a square matrix having orthonormal vectors. So, all these vectors are unit vectors and they are orthogonal to each other.

So, U1, U2, Um and I represent them by these notation, this is one vector this is another vector, I have m number of vector and their length is also m, they have m rows. So, I have a m into one size vector m number of m into 1 size vector. And these vectors let me tell it at the very beginning these vectors will be called left singular vectors, because they are on the left side of this equation, they will be called left singular vectors and they are orthonormal.

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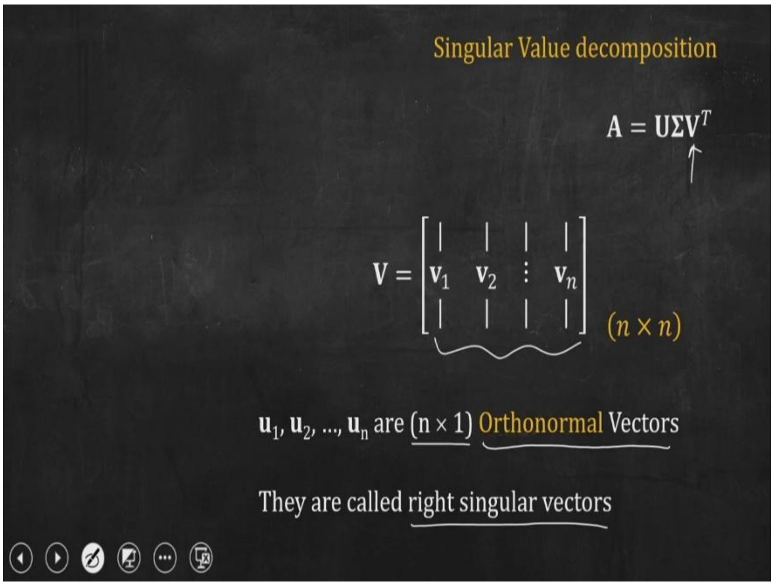

V = [v1 v2 : vn] (n X n)

V is also similar although we are writing V transpose I am defining V here, V will be n into n matrix where it is made up of vectors n number of vectors each of them has n into 1 size, n rows one column vector and they are also orthonormal that means, these vectors are unit vector and they are orthogonal to each other. So, this is the condition same as that for the U matrix and the vectors here will be called right singular vector, because V is on the right side of sigma that is why we will call it right singular vectors, not so difficult to remember.

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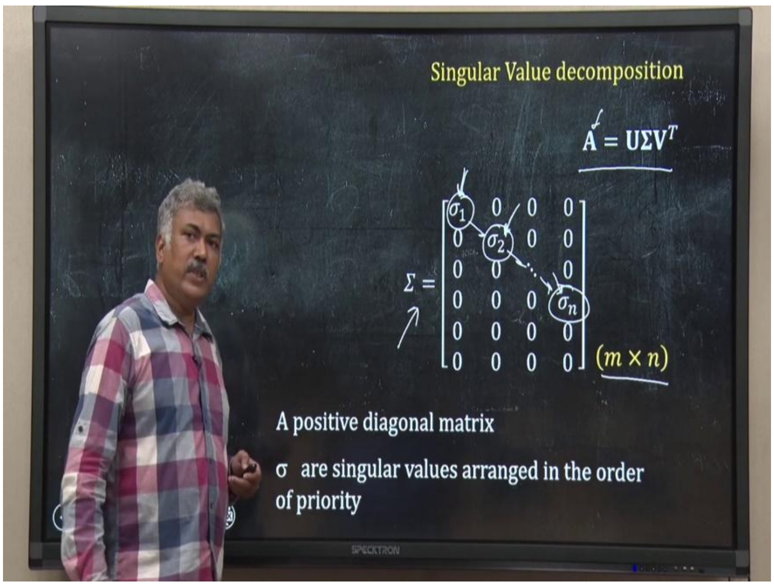

Σ =

$$\begin{vmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & : & 0 \\ 0 & 0 & 0 & \sigma_n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

(m X n)

Now come the middle part which is a diagonal matrix. So, in sigma capital sigma is a rectangular matrix because I have to match the dimension in multiplication. So, m into n in is a diagonal matrix. So, its diagonal has some term which are nonzero I will call them sigma 1, sigma 2, up to sigma n, these are small sigma. And as it is a rectangular matrix, obviously, I will have to put, use the diagonal, on the main diagonal of that rectangular matrix.

So, I have these values along this main diagonal and rest of the values are zero. One important thing in this case we have to remember we want that this is a positive matrix that means, all these sigma values that I have written, are positive values So, this is one constraint we are putting and also what we do just like in case of arrangement of Eigen values in Eigen decomposition, we usually in data analysis what we do, we arrange this term sigma's in order of priority that means, I will put the highest positive value first that will be sigma 1 and next the second value will be sigma 2 and this will be the lowest nonzero value like that, we arrange it in this way.

So, I have defined U, V and sigma and I want them in such a way that I will, if I combine them in this particular form I will get A. So, in other words, if I decompose A I should get U sigma and V transpose. So, now comes the question that is my wish and I have defined these properties there will be orthonormal, there will be diagonal matrix, but how should I get those U, V and sigma?

That is not very difficult, this is a straight forward linear algebra method I will show part of it and I will skip some steps if you are interested you can do it yourself also, maybe here and there, you have may have to consult a linear algebra text somewhere to know some rules, but otherwise it is not so difficult. But for this course, you do not need to go into detail of the proofs either.

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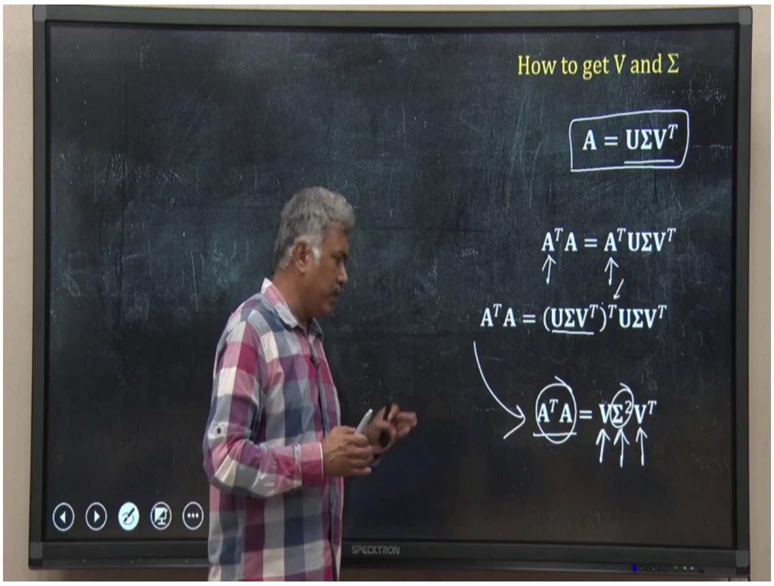

$$A = U\Sigma V^T$$

$$A^T A = A^T U\Sigma V^T$$

$$A^T A = \left(U\Sigma V^T\right)^T U\Sigma V^T$$

$$A^T A = V\Sigma V^T$$

So, let us start and see the logic and try to understand the logic. So, I want to calculate V and sigma. So, I have these relationships, that is what I want. So, multiply both sides of the equation with A transpose. So, I multiplied this side also with A transpose. Now, what is A? A by my definition is U sigma V transpose and I transpose the whole thing and now, in this step I have skipped lots of steps if you want you can check it out.

So, after multiple steps of using linear algebra rules, you will end up into this relation A transpose A is equal to V into sigma square V transpose. I will discuss what is sigma square and what is A transpose meaning are. So, I will get A transpose A, so, you are multiplying the transpose of A matrix with A that is on the left hand side, on the right hand side what you have got? You have got V into sigma squared and V transpose.

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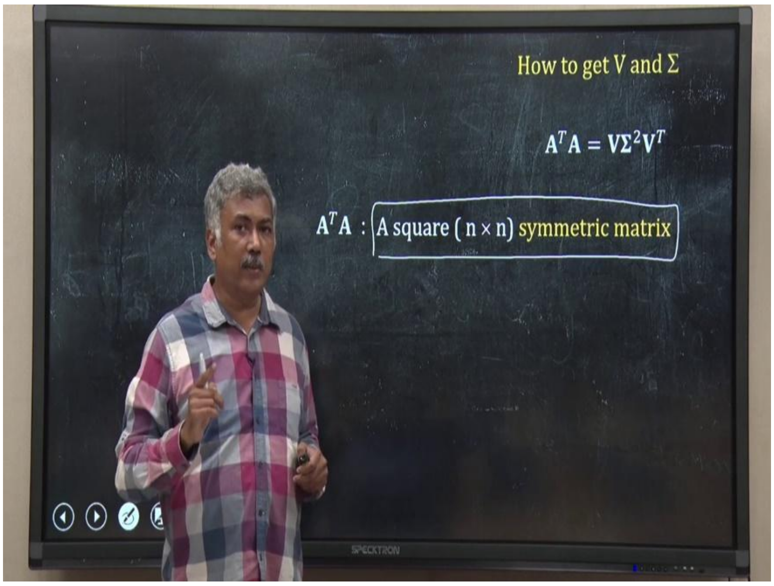

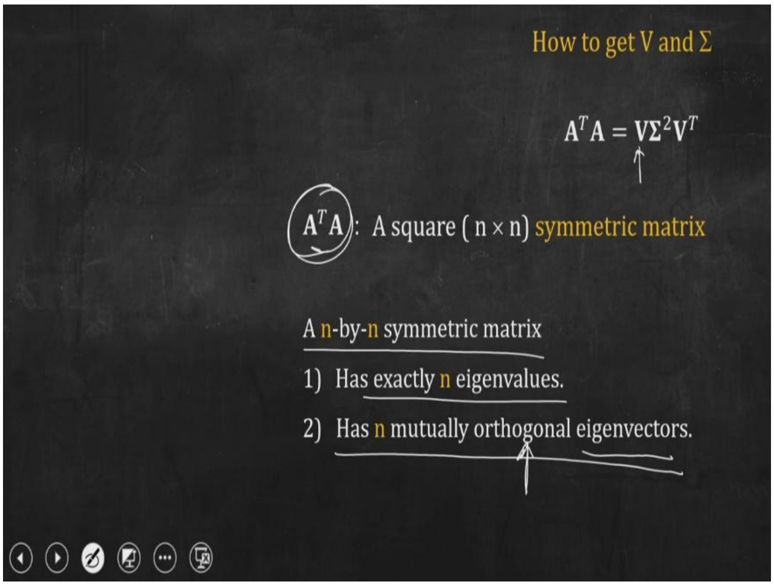

$$A^TA = V\Sigma^2V^T$$

So, now look into A transpose A and its properties. Just like magic of linear algebra A transpose A whatever matrix you take A you will have a very interesting property. A transpose A will be a square symmetric matrix, it will be a square matrix, because you started you may have started with the rectangular matrix, but when you multiply this transpose you will eventually end up with a square matrix. So, we have a square symmetric matrix.

Now, if you remember from our Eigenvalue Eigenvector discussion, symmetric matrix has a very interesting property in terms of eigenvalue and eigenvector. What is that? just to remind you, if I have n by n symmetric matrix like this one, then we have discussed in the eigenvalue lecture is that it will have exactly n eigenvalues and it will have n mutually orthogonal eigenvectors. So, I will get n mutually orthogonal Eigen vector, pay attention to this, it should be orthogonal.

Remember, I want the vectors in this capital V should be orthonormal. So, one requirement is that the vectors has to be orthogonal, normalization you can always do of any vector by dividing it by length. So, the first priority is obviously it need to be orthogonal and we are moving towards that, that A transpose A will have eigenvectors and those Eigen vectors will be orthogonal.

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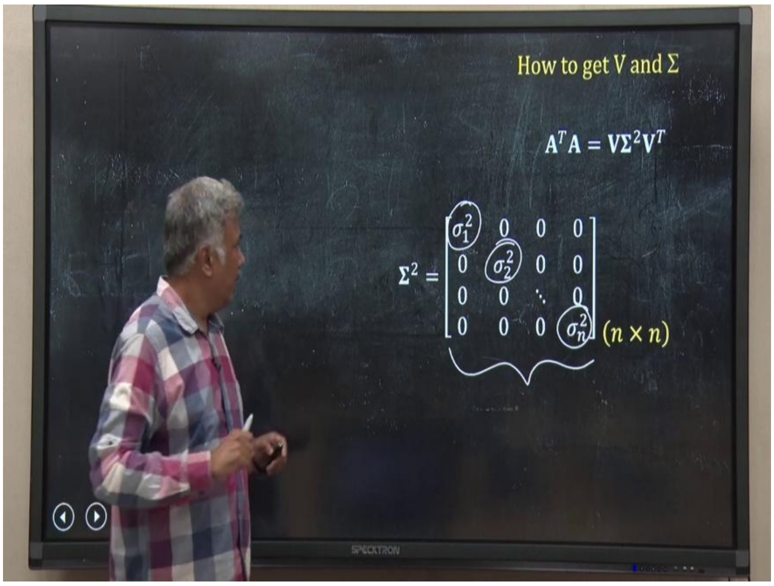

$$A^T A = V\Sigma^2 V^T$$

$$\Sigma^2 =$$

$$\begin{vmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 \\ 0 & 0 & : & 0 \\ 0 & 0 & 0 & \sigma_n^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

(n X n)

Let us move into sigma square, sigma square here in this calculation, if you do the derivation, you will find the sigma square you will land up of this form it will be n by n diagonal matrix with diagonal terms sigma 1 square, sigma 2 squared and sigma n square. So, it will be n by n square matrix diagonal with the Sigma terms along the diagonal in the square form.

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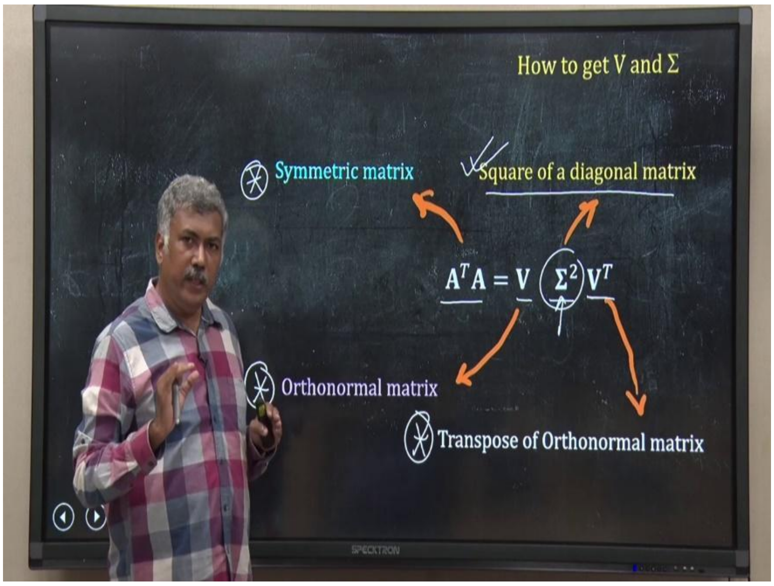

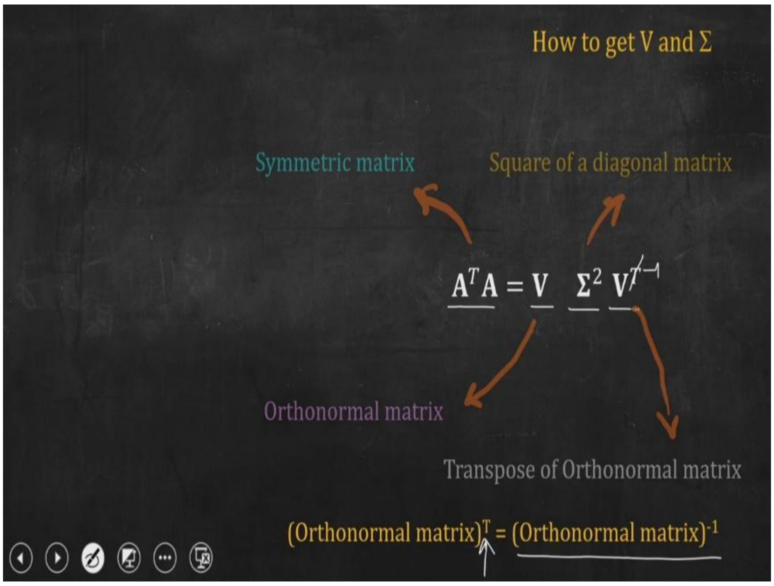

$$A^T A = V \Sigma^2 V^T$$

$$A^T A = V \Sigma^2 V^{-1}$$

$$(Orthogonal\ Matrix)^T = (Orthogonal\ Matrix)^{-1}$$

So, let me rewrite the whole scheme. What do I have? I have A transpose A that is equal to V into sigma square into V transpose. A transpose A is a symmetric square matrix, V is a orthonormal matrix that you want, we defined it that way. And I am calculating V I am trying to find out V. What is V transpose? V transpose is transpose of that orthonormal matrix and this sigma square is nothing but a square of a diagonal matrix.

Remember, if I give you a diagonal matrix suppose 2 by 2, it has a and b along the diagonal and zero other parts, if you square it, it is very simple the diagonal terms will get multiplied. So, I can say sigma is nothing but a diagonal matrix, so sigma square is also the square of the terms of the diagonal matrix. So, sigma square is a square of a diagonal matrix. Now, there is a very interesting property in linear algebra and that will save us in this derivation, it is known or rather it can be shown that for an orthonormal matrix that means, a matrix is made up of vectors, column vectors which are orthonormal.

If I have a orthonormal matrix its transpose is same as its inverse. So, the transpose of an orthonormal matrix is equal to the inverse of that matrix. So, what I can do here I can replace this transpose by inverse. So, I can write A transpose A equal to V into sigma square into V inverse, I can write that, so that is what I have written.

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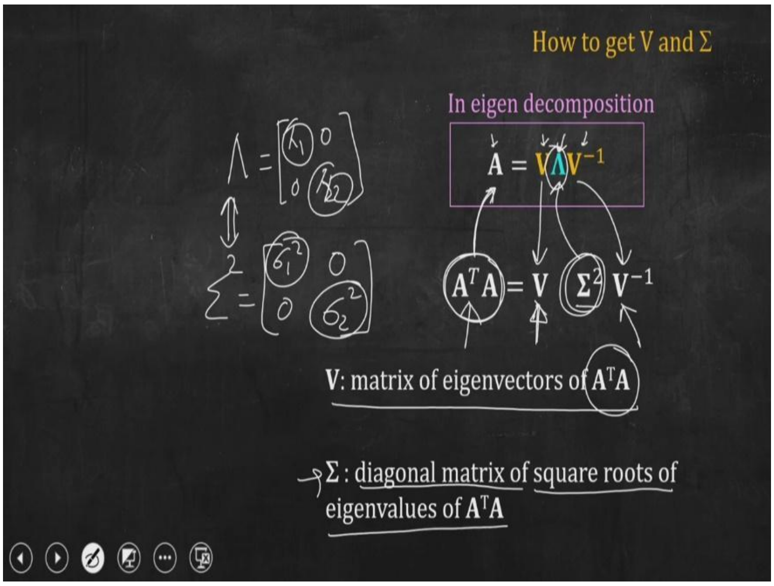

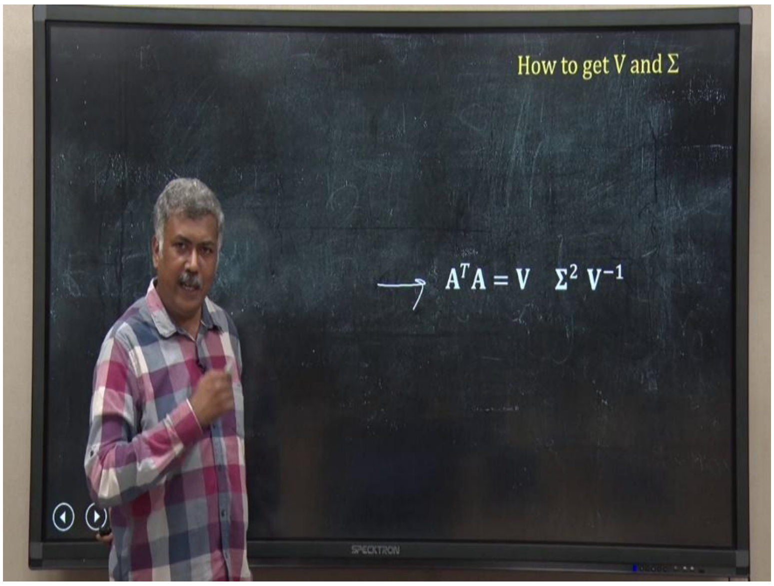

$$A^T A = V \Sigma^2 V^{-1}$$

Pay attention to this relationship do you find something hidden in this some relationship that we have discussed a few minutes back hidden in this particular structure? If you pay attention, you will find this has a similarity with Eigen decomposition. Let me show that let me write Eigen decomposition form. So, in Eigen decomposition what I have done A is equal to Eigen vector matrix into my lambda matrix the eigenvalue matrix and the inverse of the Eigen vector matrix that is what Eigen decomposition.

Now, imagine A transpose A is same as A. A transpose A is a square matrix. So, it can have Eigenvalue Eigenvector, A is a rectangular matrix I cannot have Eigenvalue Eigenvector for that, but A transpose A is a square matrix, is a symmetric matrix, it can have Eigenvalues Eigenvectors. So, this is A suppose then I can map them, so, this V is this V, this V inverse is these V inverse and this sigma square is nothing but this lambda, one to one correspondence is there.

So, that means, if I consider this as a single matrix, obviously this is a single matrix then on the right-hand side what do I have? I have a matrix of Eigenvectors of A transpose A and remember A transpose A is a symmetric matrix. So, its Eigenvectors are orthogonal and that is what we want. So, this is A matrix, square matrix of vectors which are orthonormal. Similarly, this will be just the inverse of that one and this sigma square is a diagonal matrix and this is also a diagonal matrix.

So, in case of lambda what I have? I have suppose if I take two by two lambda 2 sorry lambda 2 0 0 and for sigma square if I have sigma 1 square 0 0 sigma 2 square then if these two things are equal then lambda is nothing but sigma 1 square, lambda 2 is nothing but sigma 2 square. Now, what is lambda 1 and lambda 2? Lambda 1 and lambda 2 are now the eigenvalue of this A transpose A. So, sigma is a diagonal matrix of square roots of eigenvalues of A transpose A that is what I have got and calculated an Eigenvalue Eigenvector we know.

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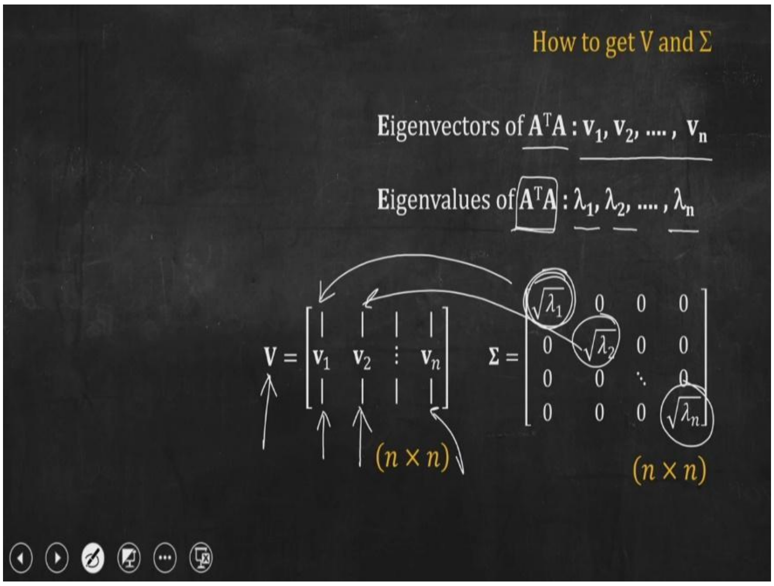

V = [v1 v2 : vn] (n X n)

$$\Sigma =$$

$$\begin{vmatrix} \sqrt{\lambda_1} & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & 0 \\ 0 & 0 & : & 0 \\ 0 & 0 & 0 & \sqrt{\lambda_n} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

(n X n)

So, if you give me a rectangular matrix general matrix A, then what I can do is that as I am doing Eigen decomposition, I have to calculate V and sigma. So, I will calculate the eigenvectors of A transpose A and suppose their V1 V2 up to Vn and the corresponding Eigenvalues of A transpose A, lambda 1, lambda 2 and lambda n then the v matrix, square matrix for my SVD will be V1 the first Eigen vector, second eigenvector, third and so, the n-th eigenvector.

Whereas, the sigma will have the square root of the eigenvalues, lambda one’s root lambda two’s root upto the square root of lambda n the n-th Eigenvalue of A transpose A. As I said by convention what we do, we take the largest value here, second largest value here and then the other in order of priority. Similarly, this one will be corresponding the first vector will be corresponding to lambda 1, second one will be corresponding to the second eigenvalue, this way we arranged them.

So, I have got V and sigma i can use actually the similar technique to get U, I will not go into the derivation of how we get U, just I will show in one line.

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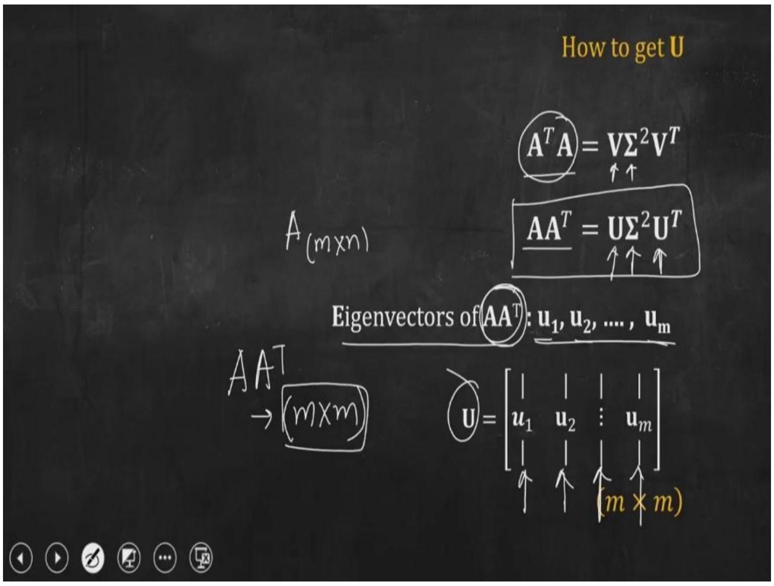

$$A^TA = V\Sigma^2V^T$$

$$AA^T = U\Sigma^2U^T$$

$$Eigenvectors\ of\ AA^T: u_1, u_2, \cdots, u_m$$

U = [u1 u2 : um] (mXm)

So, I have got V by taking A transpose A and that gives me V and sigma square. And if I do A into A transpose just the opposite one, A into A transpose I will get U sigma square U transpose I will get this relationship. So, what is U now? U is the eigenvectors, a matrix of eigenvectors of A, A transpose. So, they are U 1, U 2 up to U m, remember, if I have A is m into n matrix, then A into A transpose will be of m into m dimension.

So, that means, it should it will be again a symmetric matrix. So, it should have m Eigen values and m Eigenvectors each of these Eigenvectors will be orthogonal to each other. So, I have U 1, U 2, U n all these are orthogonal eigenvectors of A A transpose and I have stacked them together side by side to get n by n matrix which is my capital U matrix in SVD.

The Eigen values if you look into it, you can try yourself or look into the textbook you will find the eigenvalues of A A transpose will be same for nonzero values for the eigenvalues of A transpose A. So, for these two things, the Eigen value, nonzero Eigen values are same only there will be some Eigen values extra in one case depending upon the value of m and n which is zero, because the dimension has to match to because this is m into m in the other case it was n into n.

So, nonzero terms will be same zero terms will be extra zeros or less zero will be there depending upon the relation between m and n. So, what I have got? I have got V, I have got sigma I have got U. So, now, I go back and represent the whole SVD in one slide.

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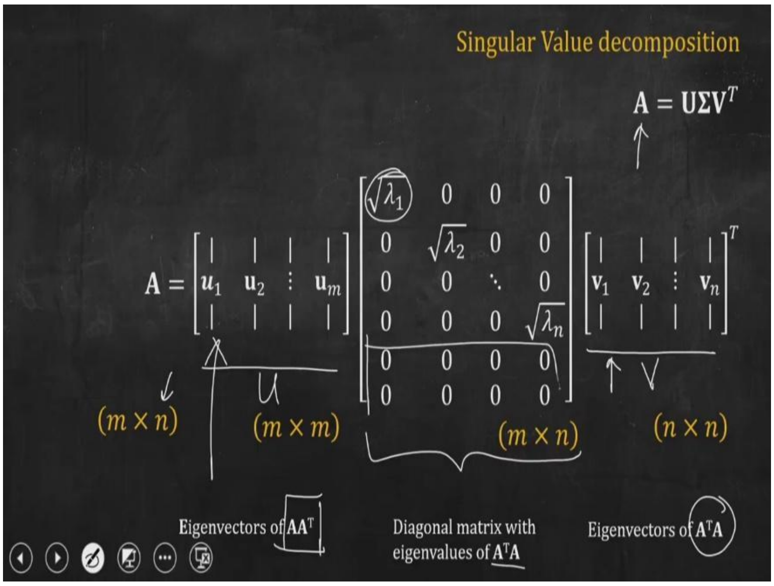

$$A = [u1 \;\; u2 \;\; : \;\; um] \;\; \mathbf{X}$$

$$\begin{vmatrix} \sqrt{\lambda_1} & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & 0 \\ 0 & 0 & : & 0 \\ 0 & 0 & 0 & \sqrt{\lambda_n} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$\mathbf{X} \;\; [v1 \;\; v2 \;\; : \;\; vn]$$

What I am doing? I have a m into n generalized rectangular matrix A and I am decomposing it into three component, U matrix it is square, A sigma matrix and a V transpose which is also square matrix. So, what do I have? A is m by n, I have this U matrix which is m into m and the elements here are the Eigen vectors of A A transpose. Whereas, this one is my V matrix having m into n size and these vectors are nothing but Eigen vectors of A transpose A.

In the middle I have a m into n matrix where I have the eigenvalue, square root of those eigenvalues of A transpose A arranged on the main diagonal. So, lambda 1, lambda 2, lambda n are the eigenvalues of A transpose A and we have taken the square root of that and we have arranged them along the main diagonal. So, this is what we call singular value decomposition.

There is something called economic size or economic SVD, what they do? See we have lots of zeros here and if you take a large dimension, suppose 2000 into 10 or something like that, then m is suppose 2000 and this is 10, so, you can imagine there will be lots of rows or lots of column whatever you do, we have lots of zeros.

So, it does not make sense to keep those zeros. So, that is why many times we chop those zero columns or zero rows and that is called economy size or economy SVD. So, that is all for this singular value decomposition lecture. Let me jot down what we have learned in this lecture.

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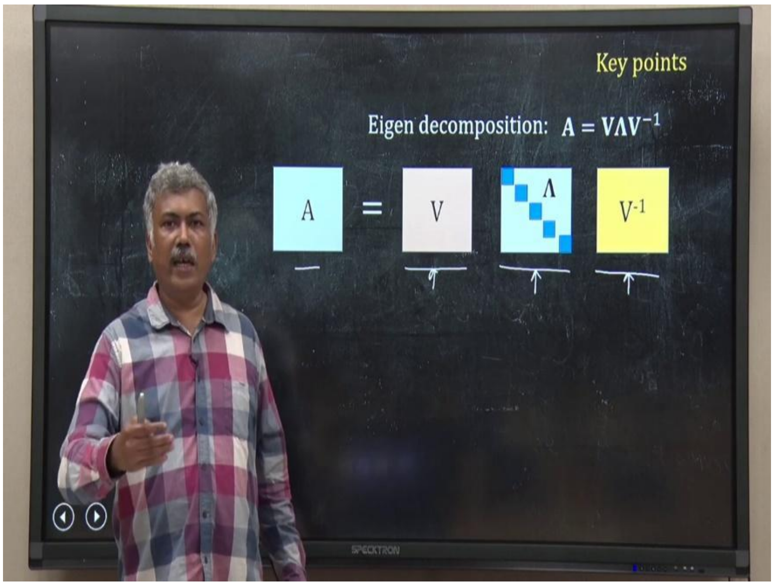

$$A = VAV^{-1}$$

In this lecture the first thing we have learned is called Eigen decomposition and this is remember related to a square matrix A in this what we do we decompose the matrix A into eigenvector matrix, into the diagonal matrix based on the Eigenvalues and the inverse of V. Based on this idea, we moved forward from square to rectangular generalized matrices and we have learned singular value decomposition, and what is that?

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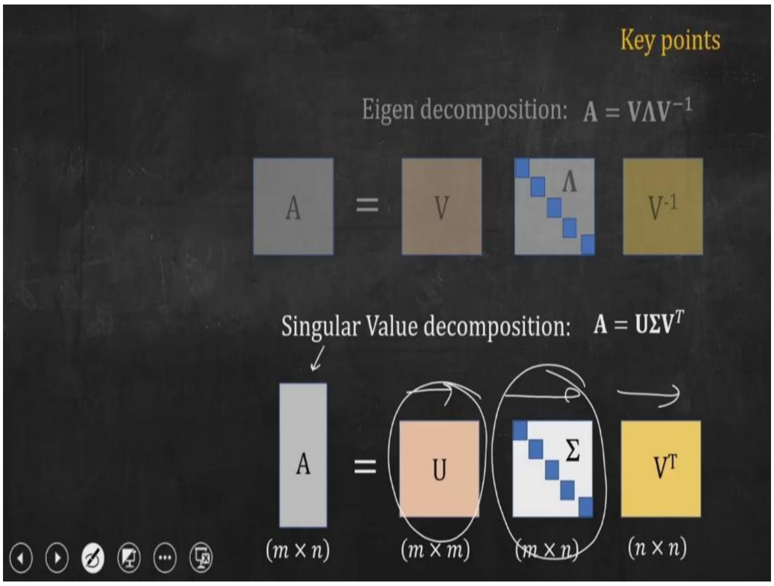

$$A = U\Sigma V^T$$

Singular value decomposition, I am taking a rectangular matrix and I am dividing into three square matrices, in the middle, I have a diagonal matrix, where in the diagonal elements are called singular values, they are nothing but the eigenvalues of A transpose A matrix, nonzero Eigenvalues and I have a square matrix here, which are also Eigen vectors of A A transpose matrix and they are orthonormal.

Similarly, you have a V transpose here where V is also a matrix measure of Eigenvectors. So, in this way I have decomposed rectangular matrix into A. That is all for this lecture. See you in the next lecture. Till then Happy Learning.