Data Analysis for biologists
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Lecture 11
Eigenvalue and Eigenvector

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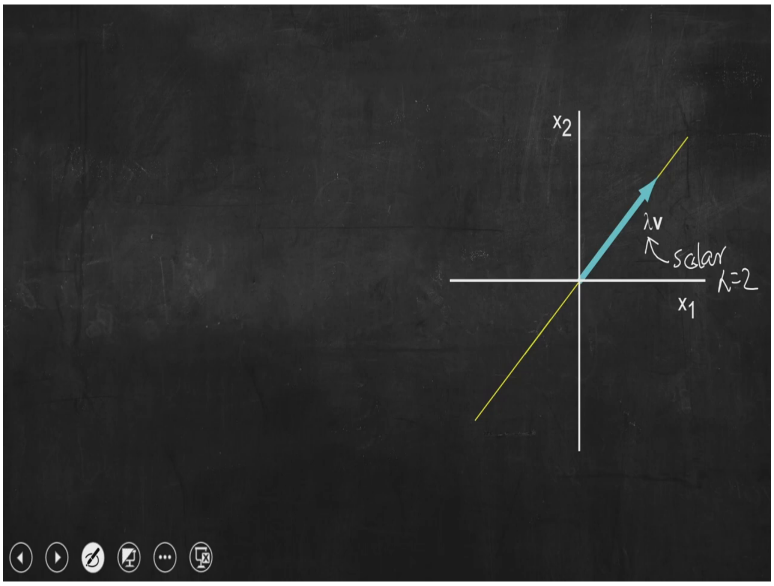

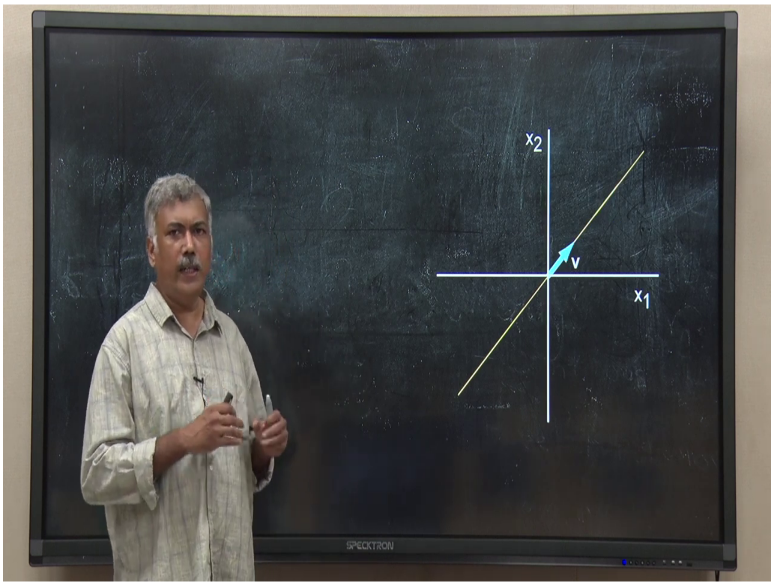

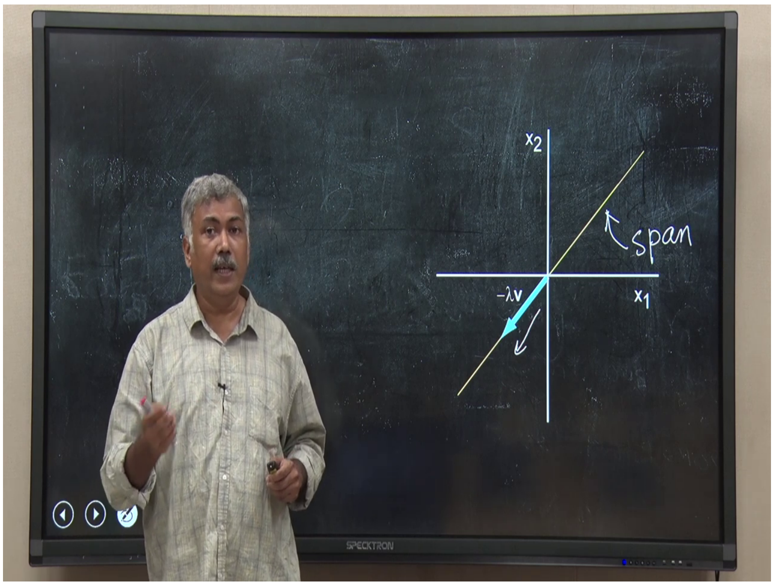

Welcome back. In this lecture, we will discuss about Eigenvalues and Eigenvectors. Suppose I have a vector v, as I have shown here in the diagram. And now, I want to stretch it by a particular scalar value, I want to make it longer. Or suppose I want to reduce its size, for example, suppose I want to stretch it by a scalar amount lambda.

So, lambda is a scalar, for example, it can be a 2, suppose lambda equal 2, that means I want to stretch this original vector v and make it double in size. And sometimes, suppose I take a negative value minus 2 or something so that the vectors direction change to opposite direction. So, in both the cases, in both the cases, what I want to do, I want to either stretch it, squeeze it or flip it in the opposite direction, but always along the same lines, on the direct that the line that follows the direction of the original vector, this yellow line.

So, this is called the span of the original vector, the span of the vector. So, I want to do this transformations, this stretching, flipping, squeezing all these things on this vector v, only on that span nowhere else, I do not want to rotate it in any direction or something like that. So, I want to do that. And I want to do that using matrix multiplication, we have learned about matrix, we have learned about how we can multiply a matrix and a vector.

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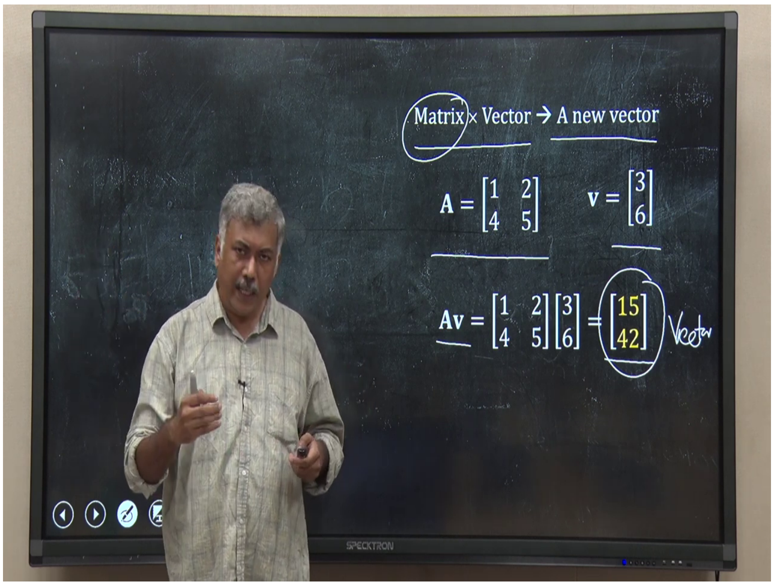

$$A = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$v = \begin{vmatrix} 3 \\ 6 \end{vmatrix}$$

$$Av = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \begin{vmatrix} 3 \\ 6 \end{vmatrix} = \begin{vmatrix} 15 \\ 42 \end{vmatrix}$$

So, if you remember if I multiply a matrix with a vector, it spits out another new vector, because you can imagine matrix as something like a function which takes a vector and it spits another vector doing the transformation on the vector. So, for example, take thid one, A is 1 2 4 5, A matrix and V is a column vector 3 6. So, I multiply A by V, and what do I get? I get 15 and 42. So, this is another new vector.

So, I can convert one vector to another by multiplying it with a matrix. So, we will use this technique so, that we can actually stretch, flip, squeeze a given vector, but along its span only, in no other direction. So, take an example take the same diagram.

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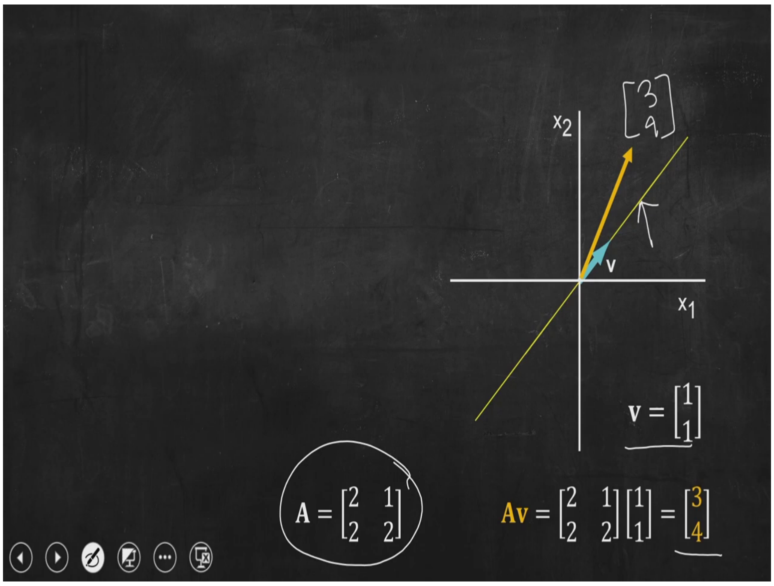
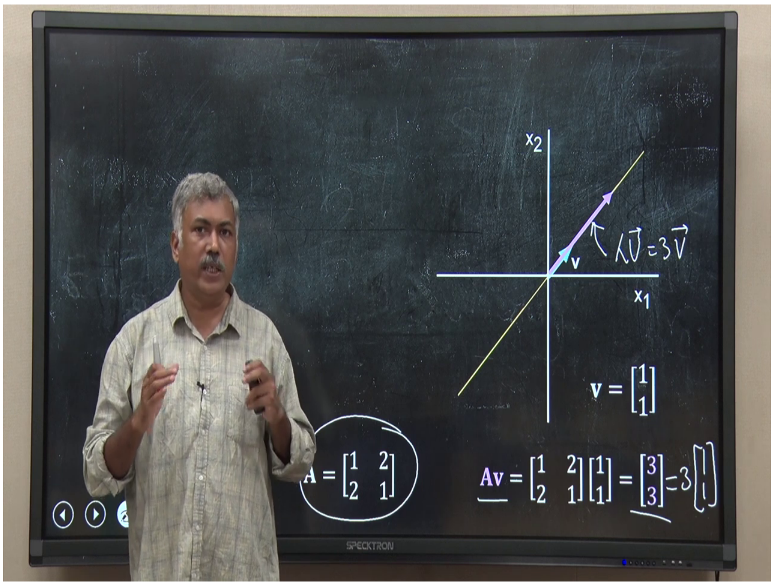


$$A = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix}$$

$$v = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$Av = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 3 \\ 4 \end{vmatrix}$$



$$A = \begin{vmatrix}1 & 2\\2 & 1\end{vmatrix}$$

$$v = \begin{vmatrix}1\\1\end{vmatrix}$$

$$Av = \begin{vmatrix}1 & 2\\2 & 1\end{vmatrix}\begin{vmatrix}1\\1\end{vmatrix}$$

$$= \begin{vmatrix}3\\3\end{vmatrix}$$

$$= 3 \times \begin{vmatrix}1\\1\end{vmatrix}$$

So, the vector given here V is 1 1, this is 1 1 vector. So, now, I want to extend it by a scalar amount of three. So, I want to make it 3 by 3. So, lambda here in this case is three. So, I want the vector to become thrice in length, but remember on the span, on this yellow line, so, how can I do it as I said I want to do it by matrix multiplication that means, I have to get some matrix which can take this vector 1 and 1 and spilt out 3 3 how can I get that.

Suppose I have a machine or I am imagining it and I try to get the matrix which can do that. So, one case that I have made suppose, is this one, the matrix A is 2 1 2 2. This is my matrix, this is my guess. So, I multiply A with V which is 1 1 and what I get, I get 3 4 a vector I got but its value is 3 4, so, this is that vector this yellow colored one, so, this is 3 4 okay it is longer in length it is longer than my original vector V.

But obviously, it is not in the span, I want it on the span. So, this matrix has not worked for me. And suppose I have done lots of these matrices, created lots of matrices, tried them, tested them, and eventually I landed up with a matrix A, whch is 1 2 2 1 and now multiply A with V and you see, you get 3 3, this is nothing but 3 into the original vector 1 1. So, what I have done, I have an original vector V, which I have multiplied with a given matrix A

1 2 2 1 And I now had got this pink vector. This is three times of original vector. So, this is lambda V or in a way, this is 3 into V because 3 is lambda. So, I have found the matrix that I was looking for. So, now let me generalize it.

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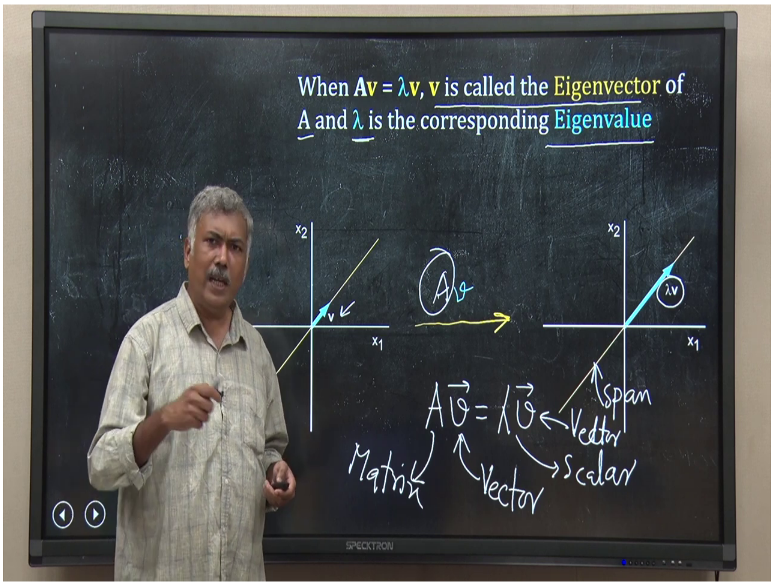

$$A\vec{v} = \lambda\vec{v}$$

I have a vector V and I want to transform it using a matrix A. So, that that vector now give rise to a new vector or the vector gets stretched squeezed or flipped or linearly transformed along the span of that vector, original vector along this yellow line. So, what I have done, I have multiplied a matrix A with a vector and I have got another vector which is lambda into V, because lambda is a scalar.

So, if I have this property, if I have this situation, where A into V, A is a matrix and V is a vector. If I multiply them, I get a scalar into the original vector. So, if you have this situation, if you have this property altogether, then what do we say, we say V is the Eigen vector of A and lambda the scalar is the corresponding Eigen value, this is how we define the Eigenvalue or Eigenvector of a particular matrix A.

So that's the definition. Now, how will you calculate eigenvalues and eigenvectors obviously, not by guessing game that I have shown? So, there are techniques to calculate Eigen value on Eigen vectors for a given matrix. I do not want you to go into details of those, those calculations are required, all the calculated eigenvalue and eigenvector for a square matrix is very easy.

It is school level thing you can use, technique you can use, do not require complicated thing, but in reality for our data analysis course, we will not calculate them manually, our computer program will calculate them. So you do not need to go there. You can also use online tools that are online calculator to calculate Eigenvalues and Eigenvectors for a given matrix, you can try them and that will familiarize you with Eigenvalues and Eigenvector and in every result, you can check whether this property that A into V matrix into its Eigen vector is equal to eigenvalue into Eigenvector or not you can check that.

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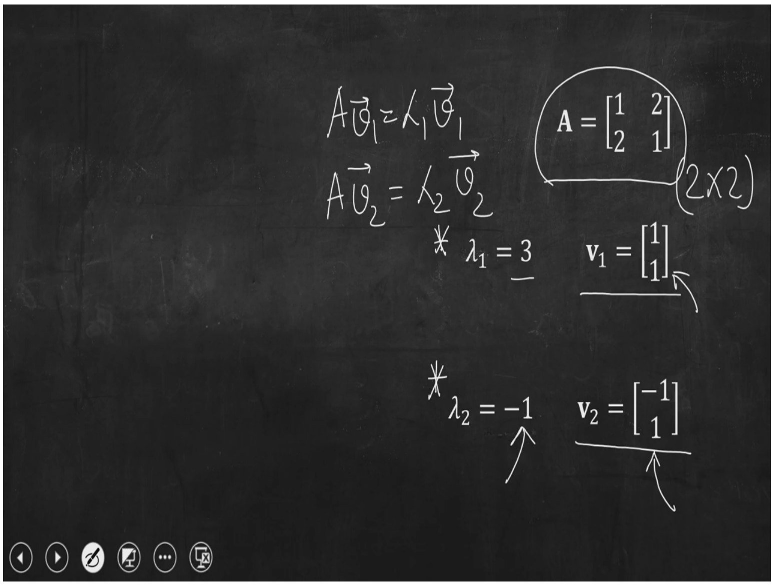

$$A = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$v_1 = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$v_2 = \begin{vmatrix} -1 \\ 1 \end{vmatrix}$$

$$A\vec{v_1} = \lambda \vec{v_1}$$

$$A\vec{v_2} = \lambda \vec{v_2}$$

So, I have already done the calculation for the original matrix A. So, for this matrix A if you take this is 2 by 2 matrix. So, this is a 2 into, 2 column 2 row matrix it has 2 eigenvalues, lambda 1 and lambda 2 and for each lambda there is a corresponding Vector, Eigenvectors. So, Eigen vector 1 is the corresponding, its corresponding eigenvalue is lambda 1 that is 3 and the vector is 1 1 whereas, for the second eigenvalue which is minus 1 the vector is minus 1 and 1. So, you can check both these Eigenvalues and Eigenvector pair will satisfy your requirement.

You will find that A into V 1 will be equal to lambda 1 into V 1. Same A you take the same A multiply with the second vector V 2 and that will be equal to lambda 2 into V 2 you can check that multiplication.

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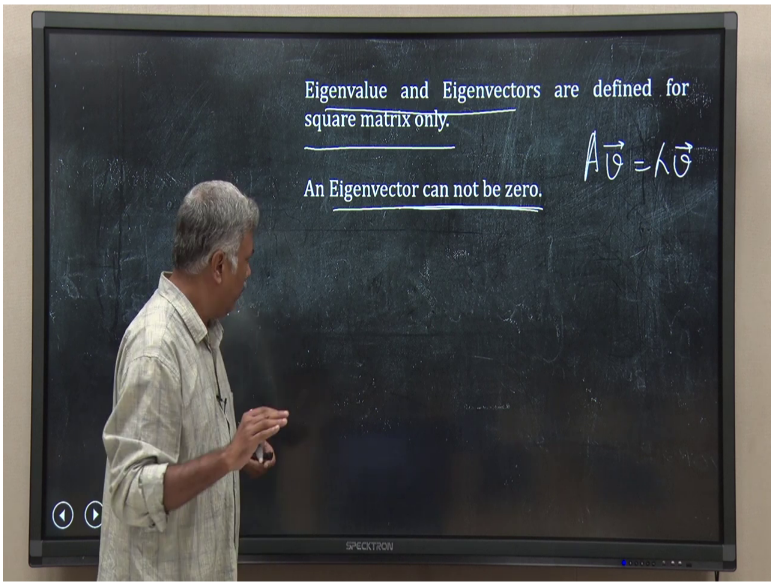

I have not told earlier. Actually to calculate Eigenvalue, Eigenvectors by definition my matrix must be square matrix you can check that out you can easily prove it, it is not you no need to a mathematician to prove that, you remember the rule of multiplication, how the row and column has to match if you now try to use that you can easily prove that for a rectangular matrix you cannot define Eigenvalue Eigenvector

the way, we have defined that Eigenvalue, Eigenvector relation is something like this while lambda is a scalar. So, you need a square matrix. The second thing by definition, Eigenvector should not be zero. If I multiply a matrix with a zero, obviously, I will get back zero, that is trivial, there is nothing in that. So, you are not doing that. So, we are not bothered about a vector which is zero. So, Eigen vector cannot be zero by our definition, we are dealing with nonzero vectors and when people calculate these Eigenvalues and Eigenvector this is used as a condition to get, solve the equations and get the value Eigenvector and Eigenvalues. As I said earlier, we do not need to go into those for this course.

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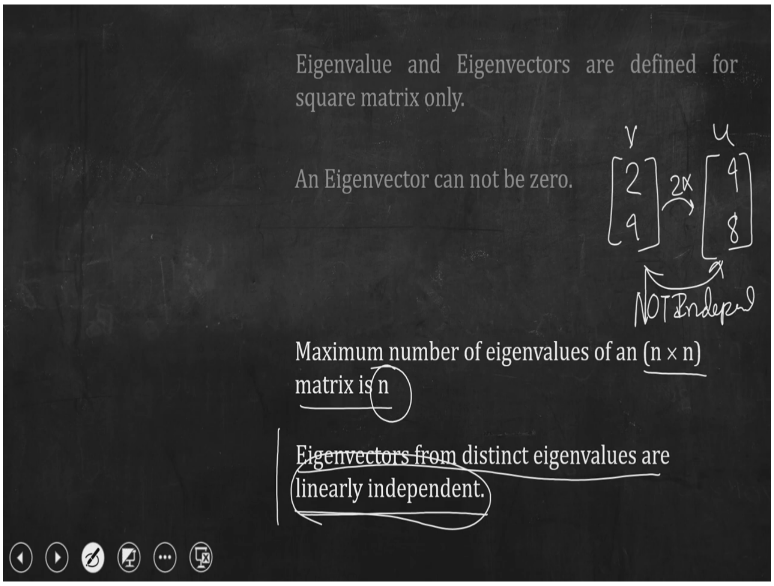

Now, if I have an n by n matrix, if I have n by n matrix, at maximum we can have n number of Eigenvalues and remember Eigenvalues can repeat. So, that means, suppose I have three Eigenvalues, does not mean the three will be distinctly different numbers I can have reputation of two numbers also. Now, so, if I have an n by n matrix the number of Eigen values will be n and the interesting thing is that you look into those n eigenvalues and find out that distinct Eigen vectors.

Sorry, the Eigenvalues. So, you check the distinct Eigenvalues, the Eigenvectors from those distinct Eigenvalues are linearly independent. Now, this is tricky, I have not discussed earlier what is linearly independent. Let me briefly say, explain what is linearly independent. So, suppose I have a vector 2 4, I have another vector 4 8. So, suppose this is V and this is U. So, you can easily see actually I can get from V to U by multiplying with 2 this is scalar multiplication.

So, that means V is nothing but actually a form of U, just a scalar multiplied right. So, I can convert V to U just by simple multiplication with a scalar term. So, that means U and V are not independent, they are dependent on each other. Similarly, you can have a set of vectors and suppose I have four vectors and out of four one is such that I can create them by scalar multiplication and vector addition of rest of the three vectors. So, that means, I can combine rest of the three vectors in such a way by addition and multiplication with scalar values numbers then I can get the fourth vector. So, then I can say there is no linear independence in these vectors there is dependency in the vectors.

So, what we are saying here is that if I have suppose m distinct Eigen values, there are m Eigen vectors also and these M Eigen vectors will be linearly independent. This is very important when we analyze data and I remind you this one when we will go into those sections where this principle will be used to analyze the data.

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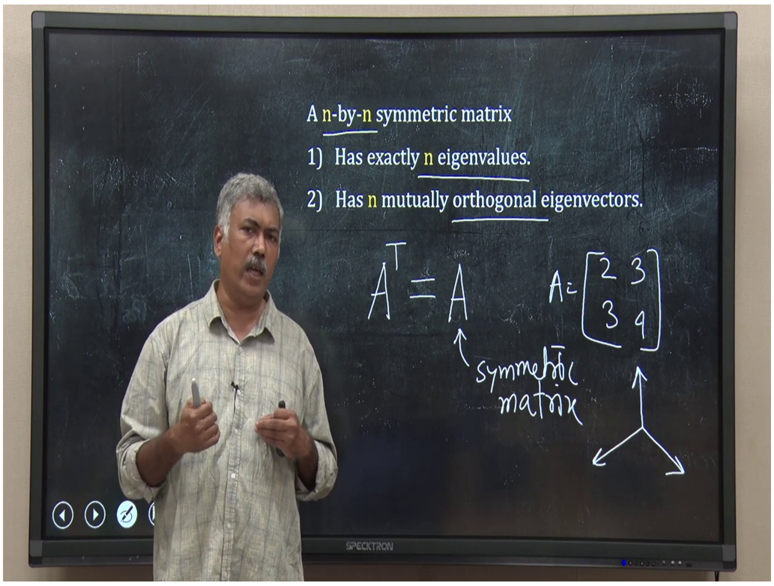

$$A^T = A$$

$$A = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}$$

Now, another interesting thing which is very useful in our data analysis is the property of eigenvalue and eigenvector of a symmetric matrix. Now, before I go into the Eigen value, Eigen vector property, what is symmetric matrix? Symmetric matrix is something very simple, if I take a matrix and if I transpose it, remember, I swapped the rows with columns, then I suppose get back the same matrix.

I have a matrix whose transpose is same as the original matrix then I say that matrix is a symmetric matrix. So, A is a symmetric matrix, let me give you an example. Suppose, A is equal to 2 4. this is the diagonal and I have 3 here and 3 here now, you swap the rows and rows with columns you will get back the same matrix. So, this is called a symmetric matrix.

Now, the interesting thing is that if I have n by n symmetric matrix and even encountered them, for example, when you will calculate the covariance matrix from your data, as I said that will be useful in PCA and also useful in other techniques, the covariance matrix is symmetric. So, that's all for the basic properties which will be useful for us to understand the use of Eigen value Eigen vectors.

There are a lot, as I said, there are lots of other properties I will not go in details of those, as and when we required those properties of Eigen value Eigen vector, I will discuss them in those particular lectures.

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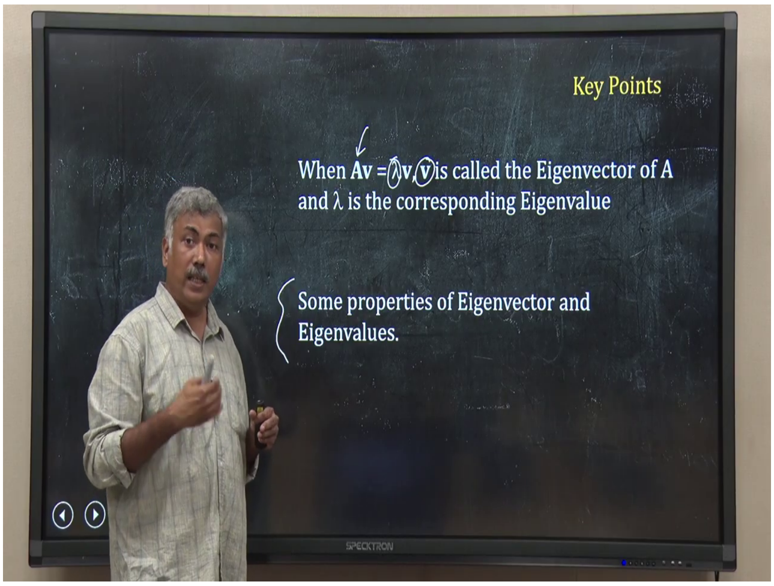

So, let me jot down what we have learned in this short video. So, I have introduced you to eigenvalue and eigenvector. If a matrix is given to you, then that matrix is suppose A, then Eigen vector and Eigen values are, v is the Eigen vector and lambda is the Eigen value such that A into V, matrix into Eigen vectors is equal to the Eigen value, corresponding Eigen value into Eigen vectors.

We have to remember that if I have a n by n matrix then I can have n eigenvalues and then I can have multiple Eigen vectors. Then we have discussed also about some basic properties of Eigen values and Eigen vectors. And we have also discussed the, in at the very beginning discuss the geometric meaning that how Eigen values and Eigen vector is actually related to linear transformation by matrix.

So, that's all for this lecture on Eigen values and Eigen vector. We will discuss further on similar topics in the upcoming lectures. Till then I leave you with a problem to think over.

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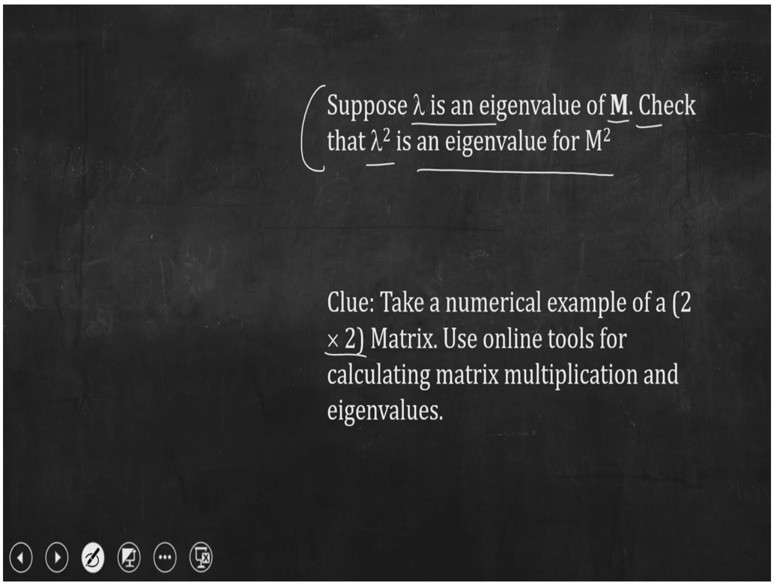

Suppose lambda is an eigenvalue of M, M is a matrix. So, what you have to check, not prove, I do not want you to prove it mathematically, this is not a mathematics course on linear algebra, but I want to check you that, check that lambda square is an Eigen value of M square, the square of the original matrix.

So, what I am asking you to check is that, if I have a matrix M, and one of the eigenvalue of this is lambda, then if I take a matrix which is M squared, then lambda square will be also an eigenvalue of the M square matrix, new matrix M square. How will you do it? As I said it is not a mathematic course on linear algebra, so, you do not need to go and make an analytical prove of it, what you do you take a matrix you take a small 2 by 2 matrix put any numbers, reasonable number don't put very arbitrary number.

Put 1 2 5 10 Something like that. So, make a 2 by 2 matrix and there are lots of online tools by which you can do matrix multiplication and calculate Eigen value Eigen vectors. So, use those tool to create from M to M square, for each M and M square you calculate the eigenvalues and eigenvectors and see whether this property can be checked, is whether these properties correct for your example of M or not. So, I hope you will try this one. We will meet again in the next lecture to discuss some new topic of linear algebra useful for data analysis, till then Happy Learning.