Data Analysis for Biologists
Professor Biplab Bose
Department of Bioscience and Bioengineering
Mehta Family School of Data Science and Artificial Intelligence
Indian Institute of Technology, Guwahati
Lecture 10
Determinant and Inverse of a matrix

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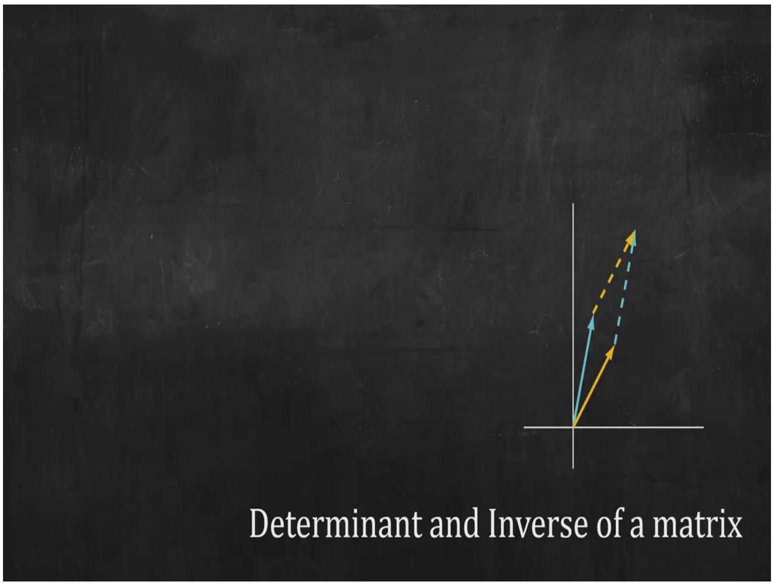

Hello, welcome back. In this lecture, we will learn about Determinant of a matrix and Inverse of a matrix. Determinant is a scalar property of a square matrix. And calculation of determinant of a 2 by 2 square matrix is very easy, you can do it by paper and pen. And you can also easily understand the geometric meaning of that.

But it is a bit difficult to calculate a determinant of a matrix suppose 3 by 3 or higher dimensional matrix and then understanding that also becomes bit difficult. But we have not to worry because, we have not to calculate the determinant by paper and pen in our data analysis, rather the computer will do it for us, but we will learn it for 2 by 2 matrix, so that we can understand how it is calculated for 2 by 2 matrix. And what is the physical geometric meaning of that.

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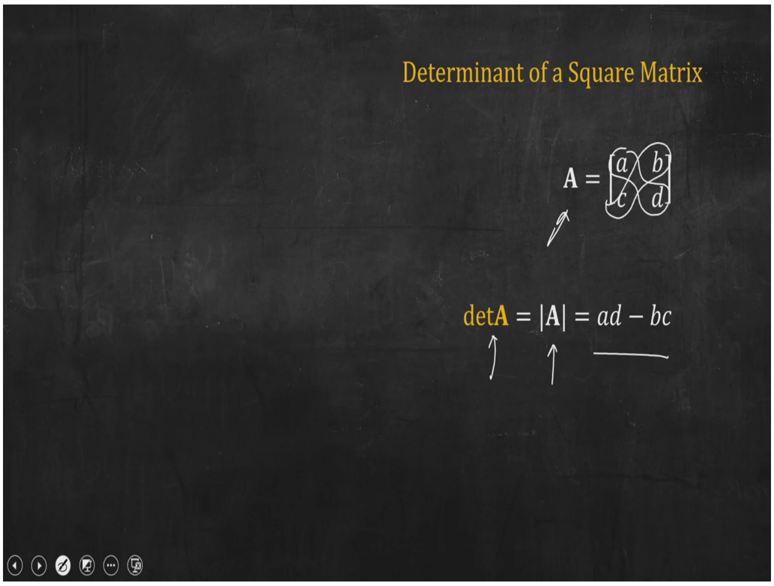

A =

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\det det\, A \;=\; |A| \;=\; ad \;-\; bc$$

So, let us start with a 2 by 2 matrix. So, I have taken a matrix A, a, c, b, d and its determinant by definition, it is written as det of A, or sometime people write it as like this between 2 straight-line, you put A. So, determinant of A by definition is equal to for this 2 by 2 system is a into d minus b into c, what is a into d? You are actually multiplying these 2 and then subtracting b into c that simple. So, you can do it by paper and pen.

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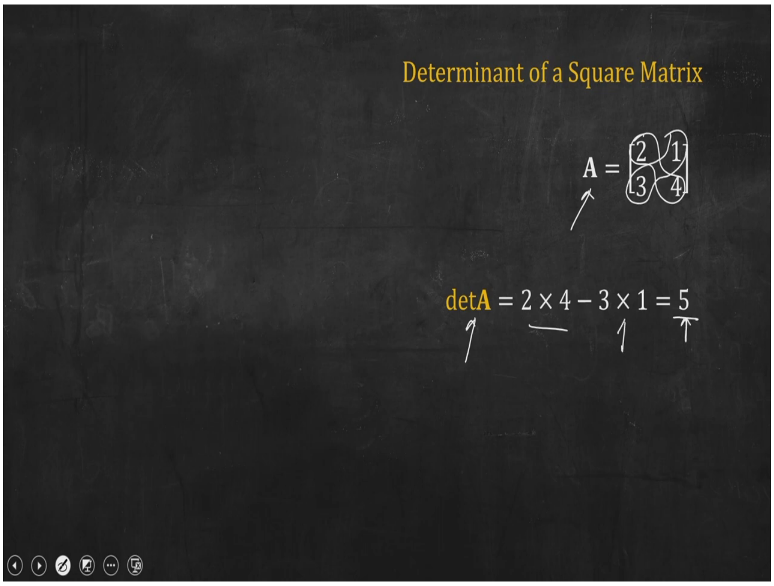

A =

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$$

$$\det det\, A \;=\; 2\times4 \;-\; 3\times1 \;=\; 5$$

Take a example, I have a matrix 2 by 2 given here. So, the determinant of this matrix det A by definition will be I will multiply 2 and 4 that will give me 8 minus 1 and 3, that is 3. So, the determinant is 5. So, now let us try to understand what is the geometrical physical meaning of determinant for a matrix

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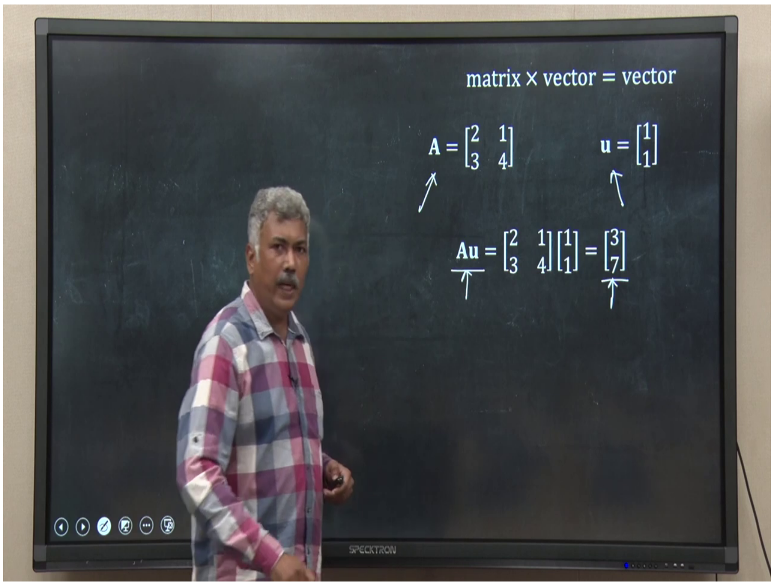

A =

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$$

u =

$$\begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

Au =

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

=

$$\begin{vmatrix} 3 \\ 7 \end{vmatrix}$$

Now, if you remember we have discussed earlier, that matrix do linear operation on a vector, what do I mean just to remind you, so if I take a vector and multiply that with a matrix, it will spit out another matrix. So, it is just like a function it is taking vector as an input and throwing out a vector as a output. So, let us take the example. I have a matrix 2, 3, 1, 4 given here is 2 by 2 and u is my input vector. So, I multiply u with A so Au, Au gives me a new vector, 3, 7.

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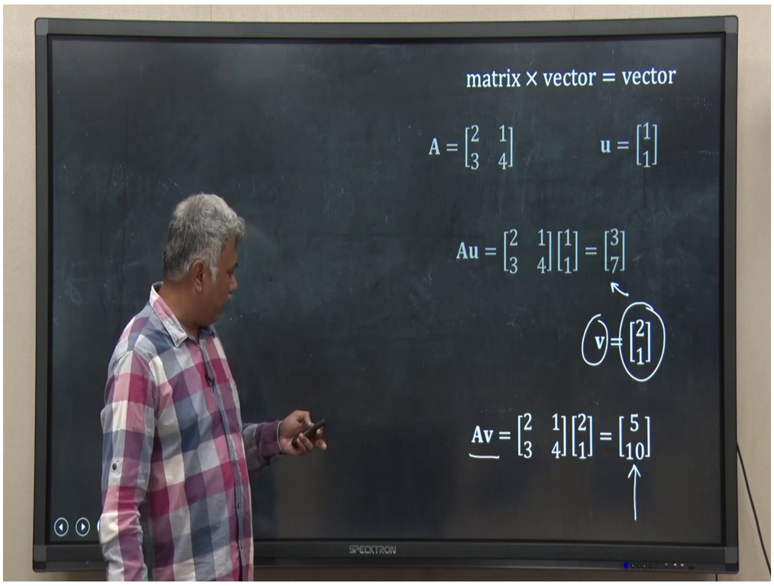

$$A = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$$

$$v = \begin{vmatrix} 2 \\ 1 \end{vmatrix}$$

$$Au = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} \begin{vmatrix} 2 \\ 1 \end{vmatrix} = \begin{vmatrix} 5 \\ 10 \end{vmatrix}$$

So, let me see this geometrically, what does that mean? But before that, let me take another vector, this vector is v, that is 2 1, and I multiply v with A again, and I get another vector 5 10. So, by multiplying u with A, I have got a new vector 3, 7, and I have similarly got a new vector by multiplying v by A. So, as I said, I will see geometrically, let us plot these vectors.

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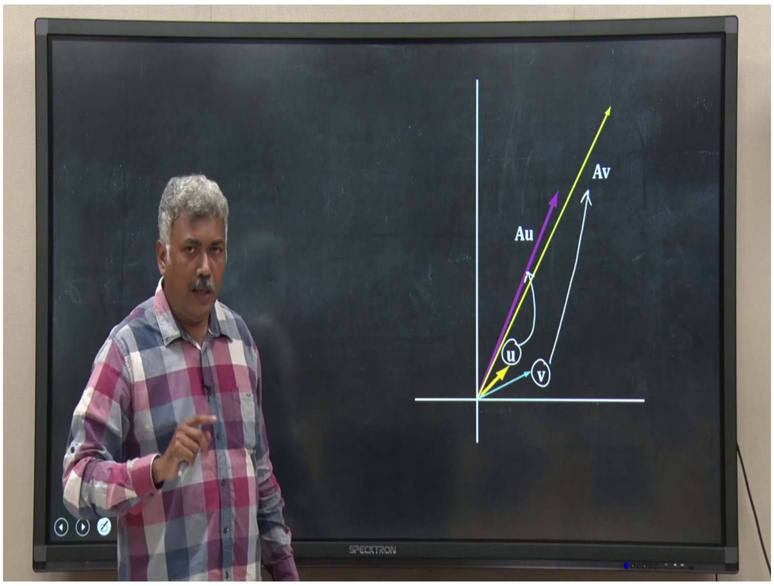

So, this is my 2-dimensional space, in which I have 2 vectors, original vector u and v. And these 2 vectors upon multiplication with the matrix A, they have changed, now u has become this pink one Au you can see it has got extended and moved. Similarly, v has got stretched and it has moved also.

So that is, what is happening? A is actually doing linear transformation of these 2 vectors. A, in essence geometrically, changing the shape of the space. So, as the shape of the space, this 2-dimensional space is changing, this vector are getting stretched and moved. That is what linear transformation is all about. And the matrix A is doing that. Now, I will change the matrix A, but I will keep the vector u and v same and see what happens.

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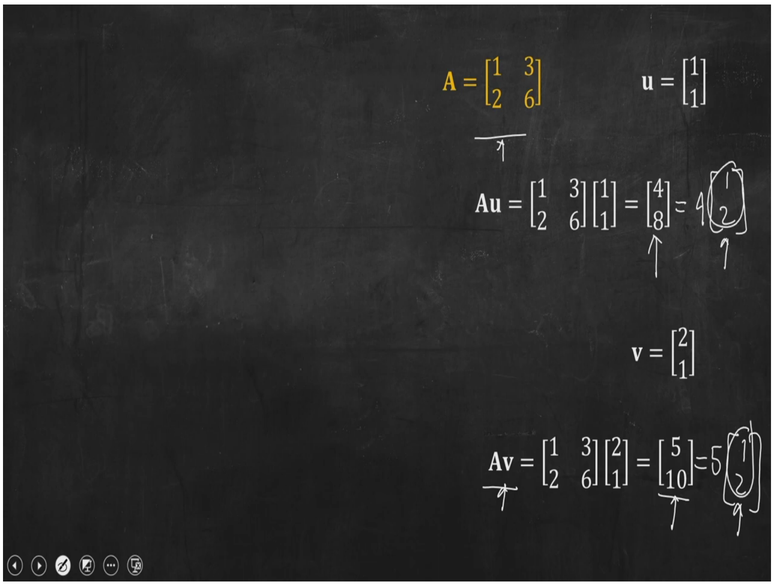

A =

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$$

v =

$$\begin{vmatrix} 2 \\ 1 \end{vmatrix}$$

Au =

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} \begin{vmatrix} 2 \\ 1 \end{vmatrix}$$

=

$$\begin{vmatrix} 5 \\ 10 \end{vmatrix}$$

= 5 ×

$$\begin{vmatrix} 1 \\ 2 \end{vmatrix}$$

So, I have now taken a new matrix A that is 1, 3, 2, 6 here, and I have kept u and v same. Now, I get a new vector, Au by multiplying u by A, which is 4 and 8, whereas multiplying v by this new A, I get a vector 5, 10. I will plot them, but before them, just check out, can't I rewrite this new vector, I can write this way. I take 4 and then it will become 1, 2 whereas in the second case for v it becomes 5 into 1, 2.

So, that means, this matrix, this new matrix A has done linear transformation in such a way that both u and v has given me some vectors, which are multiple of a vector 1 and 2, 1 and 2, this vector and this vector are same, but they are got multiplied 1 by 4, another by 5. So, let us see geometrically what has happened?

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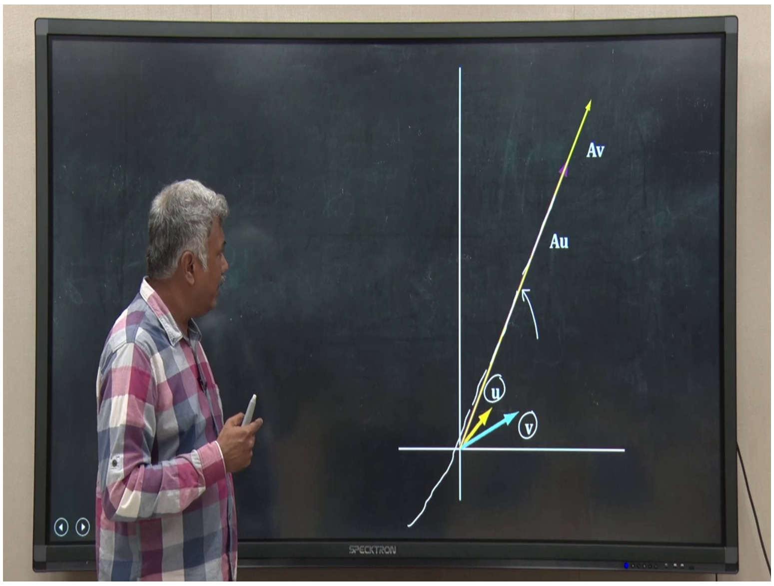

So, this is again my same 2-dimensional space, I have u and v original vector. When the linear transformation has happened by A so, I got new vector Au and Av you can see both of them are now on the same line and you can try any other vector in this space. And if you multiply that with A you will find all those vectors will lie only on the same line.

So, that is what the difference between these two matrix, one matrix has linearly transformed in such a way that still although v and u has changed, they have given rise to stretched or longer vectors, which has moved in a particular direction, but they are still separate whereas, in this case, all the vector have collapsed only on one line. And this is related to the determinant of the matrix, of the matrix that we are using, I will not go into the details of the matter of that, but you can easily calculate the determinant and see the difference.

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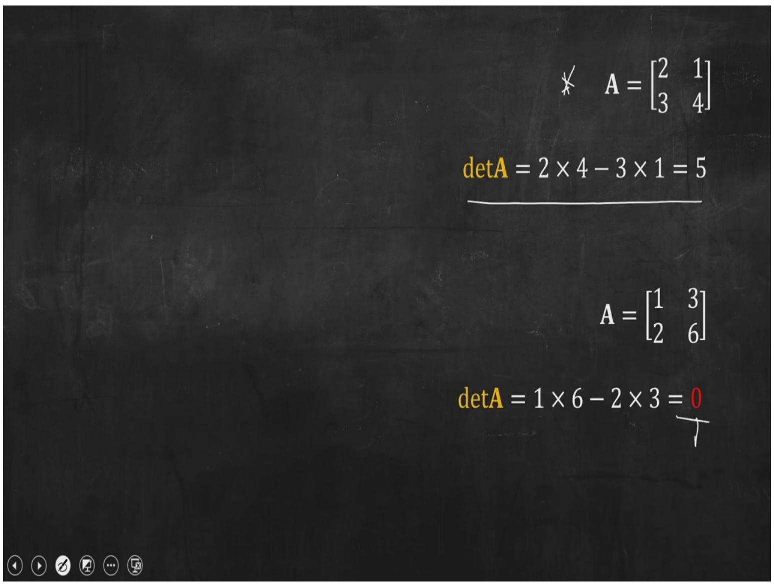

A =

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$$

$$\det det\, A = 2\times4 - 3\times1 = 5$$

A =

$$\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}$$

$$\det det\, A = 1\times6 - 2\times3 = 0$$

So, let us calculate the determinant of the first matrix, its determinant is 5, we have done the calculation. For the next matrix our determinant is 0. And that is why it can be shown that all the vectors which are transformed by this matrix are collapsing on the same line.

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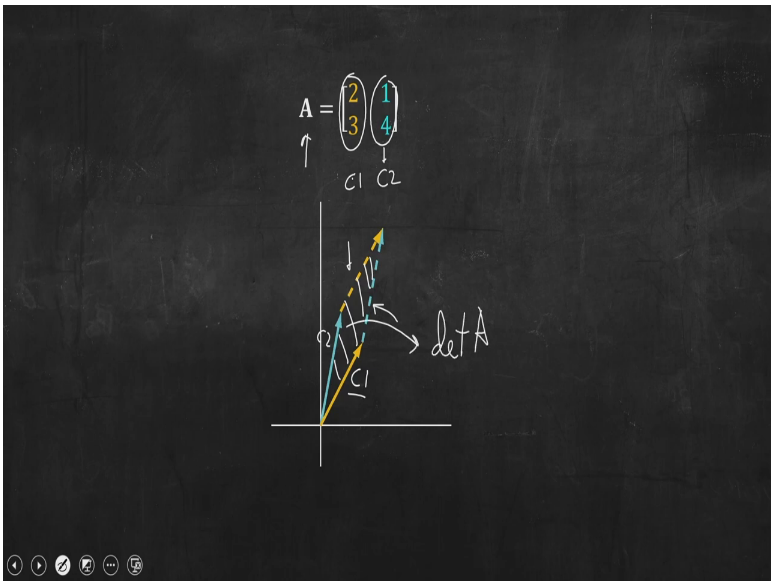

Now, there is another way to understand the meaning of determinant that is something using parallelogram. So, what I have done here, I have drawn these 2 vector, this is the vector for this column, this is the vector for this column, this is the matrix I have taken, initial matrix which has determinant 5. So, suppose I call it C2 vector, column 2 vector, this C1 vector so, this is C2 and this is C1.

And if you remember, vectors can be moved anywhere in the space without changing its direction and length. So, that is what I have done, I have moved C here in this dotted yellow line and I have moved C2 also made a copy of that here as the dotted blue line, and what I have got? I have got a parallelogram and it can be shown, I will not go in details of that mathematics, but it can be easily shown that the area of this parallelogram is equal to determinant of A. Now, let me plot the parallelogram for the next matrix for a determinant was 0.

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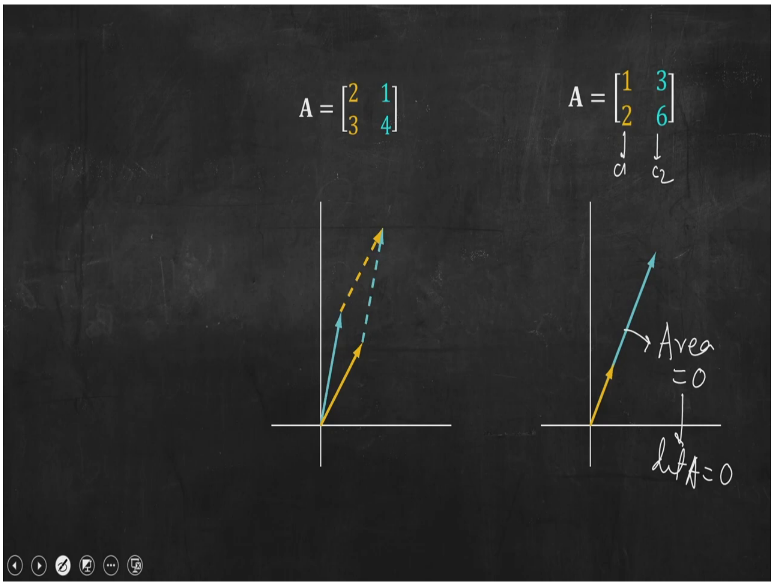

This one is C1, this one is C2. And as you can see, both these column vectors are on the same line. So, that means, if I try imaginary parallelogram there, the area will be equal to 0, that is why is determinant of this matrix is also 0. So, what we have discussed till now? That the difference between these 2 matrices is that they are, they're determinants are 0. Now, you can ask the question now that what type of matrices will be there, whose determinant will be 0? Can any matrix have a determinant equal to 0?

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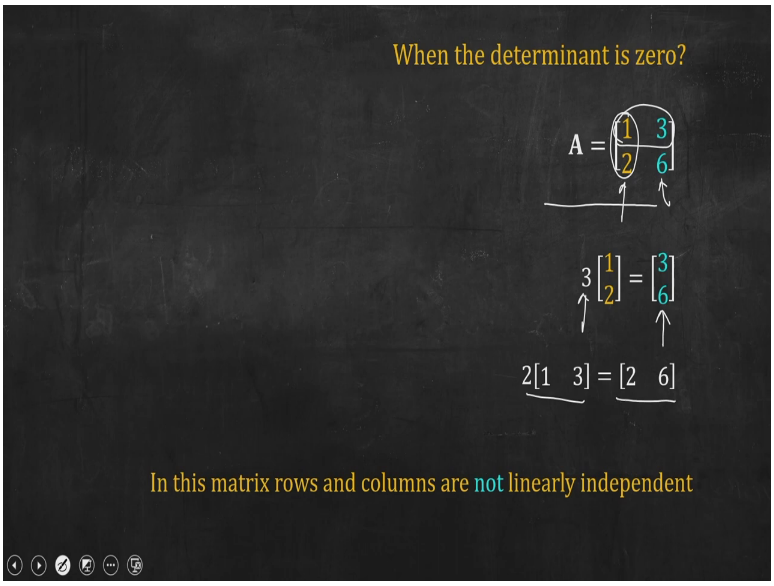

A =

$$\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}$$

3 x

$$\begin{vmatrix} 1 \end{vmatrix}$$

$$| 3 |$$

$$=$$

$$\begin{vmatrix} 3 \\ 6 \end{vmatrix}$$

$$2\,[1\ \ 3] = [2\ \ 6]$$

Let us check that. So, understand that which type of matrix will have determinant 0, let us look into the matrix that has given us determinant equal to 0. So, for this matrix det A equal to 0. So, if you check carefully, what you can see, this is the first column 1, 2 and this is the second column is 3, 6. So, I can multiply this first column by 3, then I can get the second column or you multiply the first row, this 1, 3 and you will get the second row.

That means, in this matrix for which determinant is 0. The rows and columns are not independent. You can linearly transform one column into another by simply multiplying with a constant scalar term, whereas you can similarly convert one row into another row by a multiplication of a scalar. So, in this matrix, the rows and columns are not linearly independent.

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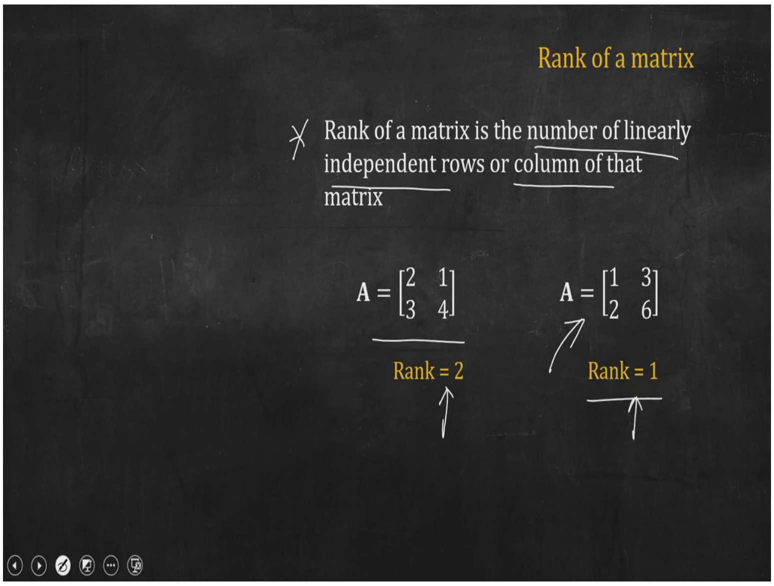

A =

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$$

Rank = 2

A =

$$\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}$$

Rank = 1

And that is brings us to a new interesting concept called rank of a matrix, what is the rank of a matrix? By definition, I have written it here by definition, rank of a matrix is the number of linearly independent rows or column of that matrix. So, if you took the first matrix for which determinant was non-zero, we got 5, determinant is equal to 5, you take that matrix and you can see the rows and column are linearly independent, you cannot convert one row into another one by simply multiplication or addition of something.

So, these rows and columns are independent that is why its rank is 2, it is a 2 by 2 matrix it cannot be rank cannot be bigger than 2. Whereas for the second matrix for which the determinant was 0, the rank was 1, because I can convert one column to another column or another row, one row to another row, that means one row or one column will suffice for this matrix. So, it has rank 1.

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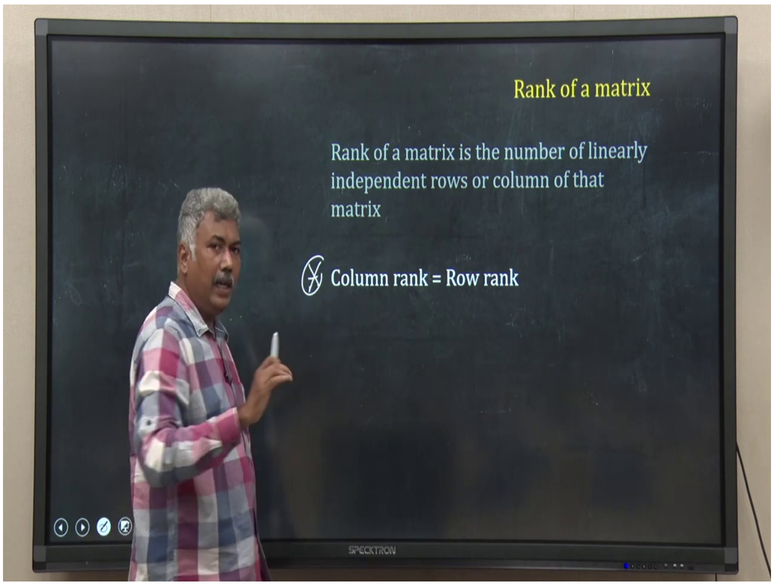

So, now one interesting thing just like magic, it can be shown that actually the column rank and row rank of a square matrix are same.

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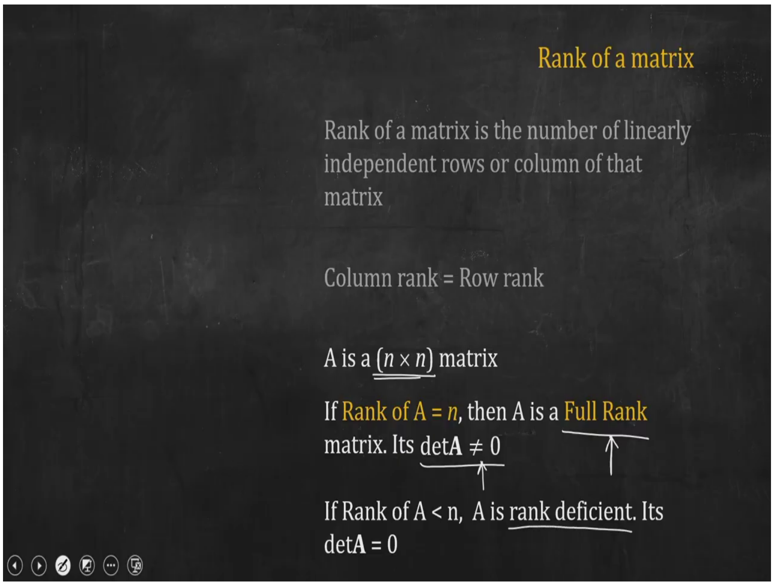

Now, a square matrix if it is n by n, the maximum rank it can have is 1n sorry n. So, if I have a square matrix of n by n, then the maximum rank it can have is equal to n. So, when that particular n by n matrix has a rank n that means, all columns or all rows are linearly independent, I will call that matrix as a full rank matrix. And you can be assured that its determinant will be non-zero.

But if the rank of that matrix is less than that maximum number n in this case, then it will be called the rank deficient matrix. And in this case, the determinant of that matrix will be equal to 0. So, now we have discussed till now about determinant, when determinant becomes 0, what is the physical meaning of determinant, at least for 2-dimensional system and I hope it is clear to you and you have got the idea of rank of a matrix which will be very useful in subsequent discussions.

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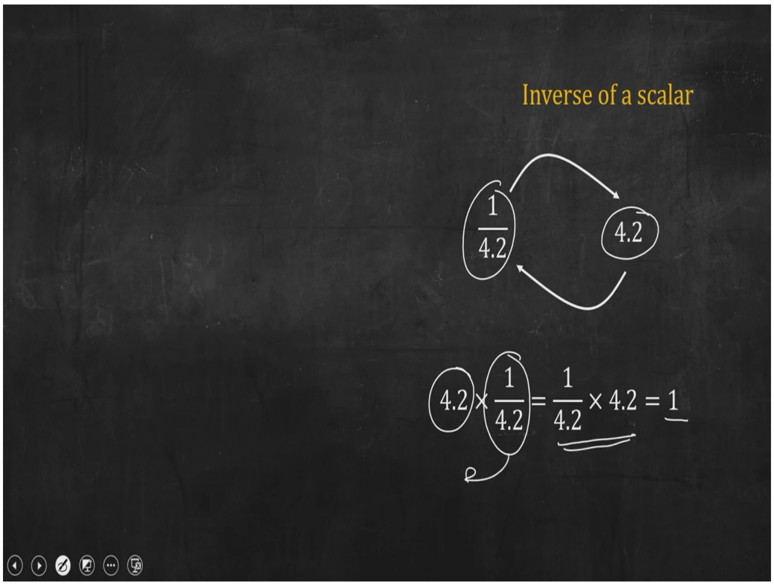

$$4.2 \times \frac{1}{4.2} = 1$$

Now, I will enter into the second topic of today's lecture, that is called the inverse of a matrix. We regularly do inverse of a scalar thing, isnt it? For example, suppose, I have a number, is a scalar term, 1 divided by 4.2. I can invert it by simply dividing in 1 by that 1 and I will get 4.2. And you know, the properties of those inversion.

So, if I take a number and multiply with the inverse of that, that will be equal to 1 and actually, you can swap the position of these 2 terms, and that is what I have done here. So, the position does not matter, you can swap the position of a number and its inverse and you can multiply them together you will always get 1. So, is there anything similar for matrix? Yes, it is there, it is called inverse of a matrix.

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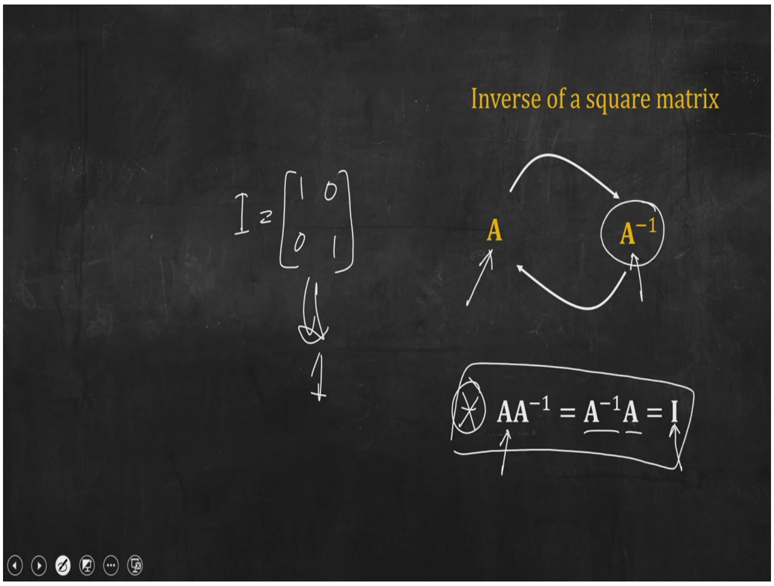

$$AA^{-1} = A^{-1}A = I$$

So, suppose A is a square matrix, then by definition this A to the power minus 1, we will call it A inverse . So, A inverse is by definition, is the inverse of a square matrix if and only if, this relationship is true. What is this relationship? This relationship says that, if I multiply A with A inverse and if that is equal to A inverse into A and both of these are equal to identity matrix I, then this A to the power minus 1 that I have written will be called the inverse of matrix A.

So, what is identity matrix? I hope you remember identity matrix for a square matrix suppose 2 by 2 will be 1, 1, 0, 0. This is equivalent to 1 in scalar is not it? We have discussed that earlier. So, this is the definition, if these criteria is met, then I will say this is the inverse of the square matrix A.

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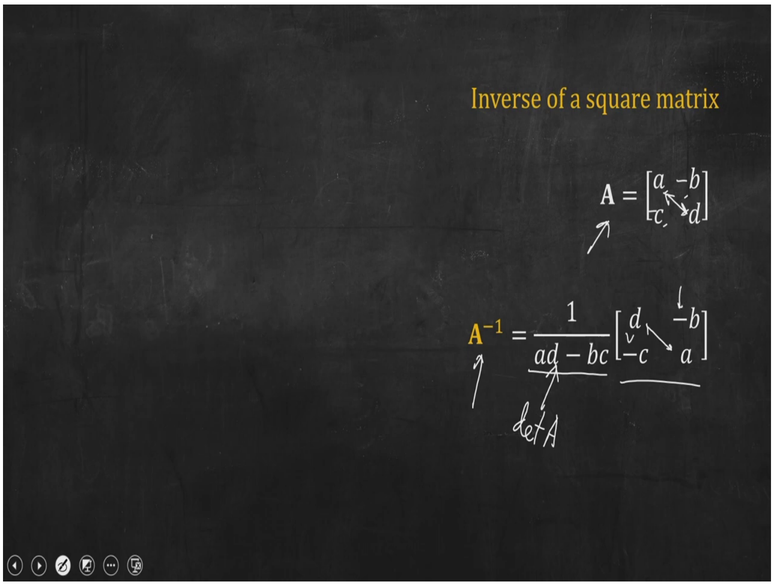

A =

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \times$$

$$\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$$

Let us take an example to calculate inverse and check this property. So, I have to do it manually because it is doable you can do it easily for a 2-dimensional system. For higher dimension system it is not trivial to calculate inverse, so I will not go into it. So, I will take an example of a 2 by 2 system and I will show you how we do the calculation.

So, for 2 by 2 system like this one I have to write in a generic matrix, for a 2 by 2 system the inverse of the matrix, A inverse will be equal to 1 divided by ad minus bc, what is ad minus bc. So, you can easily recognise this, just few slides back we have discussed this is the nothing but determinant of that matrix.

So, I have got 1 by determinant of that matrix, the first term into, I have a matrix here, what is this matrix? This matrix nothing, but I just swap the space of some terms. So, a and d will exchange their place. So, a and d has exchanged their place and we got a minus sign in front of b and c. So, one can show that the inverse of a square matrix will have this formulation. So, now let us take a concrete numerical example and use this formula to calculate the inverse of a square, a 2 by 2 square matrix.

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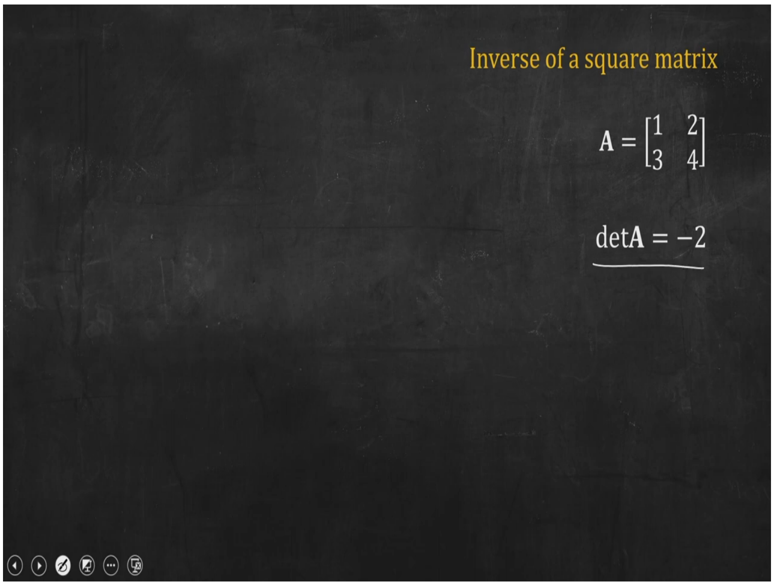

A =

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$\det det\, A \ \ = \ \ -\ 2$$

So, here is the example 1, 2, 3, 4. So, what will be the inverse of that? First, we calculate the determinant. Determinant is minus 2.

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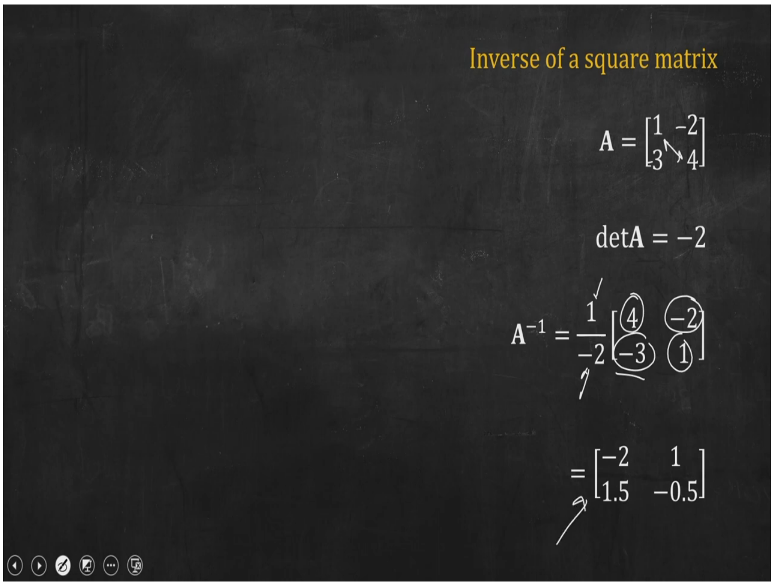

$$A^{-1} = \frac{1}{-2} \times \begin{vmatrix} 4 & -2 \\ -3 & 1 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1.5 & -0.5 \end{vmatrix}$$

Now, I calculate the inverse of that matrix A. So, 1 divided by determinant into now, 1 and 4 will swap their position. So, 4 is here, 1 is here and I will put minus in front of these 2. So, I get minus 2 and minus 3 here. So, do the multiplication and I get the matrix minus 2, 1, 1.5, minus 0.5. So, this is the inverse of the given matrix A. So, if you remember I said the condition that you have to make is that A into A inverse would give me an identity matrix. Let us check that.

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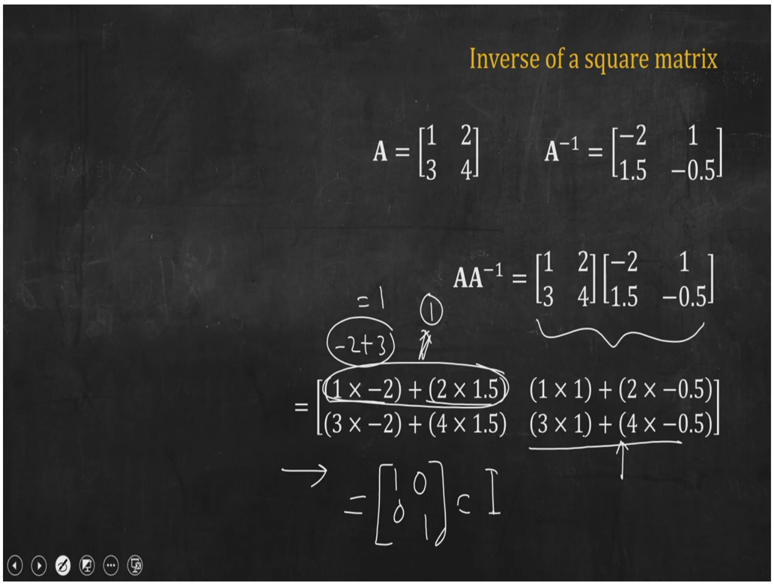

$$A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$A^{-1} = \begin{vmatrix} -2 & 1 \\ 1.5 & -0.5 \end{vmatrix}$$

$$A\,A^{-1} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \begin{vmatrix} -2 & 1 \\ 1.5 & -0.5 \end{vmatrix}$$

$$= \begin{vmatrix} (1 \times -2) + (2 \times -1.5) & (1 \times 1) + (2 \times -0.5) \\ (3 \times -2) + (4 \times -1.5) & (3 \times 1) + (4 \times -0.5) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

So, I have the A matrix, the inverse of that, we calculated using the formula. So, now A into A inverse will be multiplication of these 2 matrices and I have done the, shown the matrix multiplication using the matrix multiplication rule, you can use that and try it, you can easily see what do I have? 1 into minus 2, this is minus 2, plus 2 into 1.5, that will be plus 3. So, this will be equal to 1 so, this 1 is whole first term is equal to 1.

Similarly, you can see this whole thing will become 1 and if you do the whole calculation and I will insist please do the calculation yourself for practice you will see it will become 1, 1, 0, 0 and this is a identity matrix done. So, now looking, let us look into a particular property of this inverse. Let us look into the formula of you calculating the inverse of a 2 by 2 square matrix and look into the particular issue.

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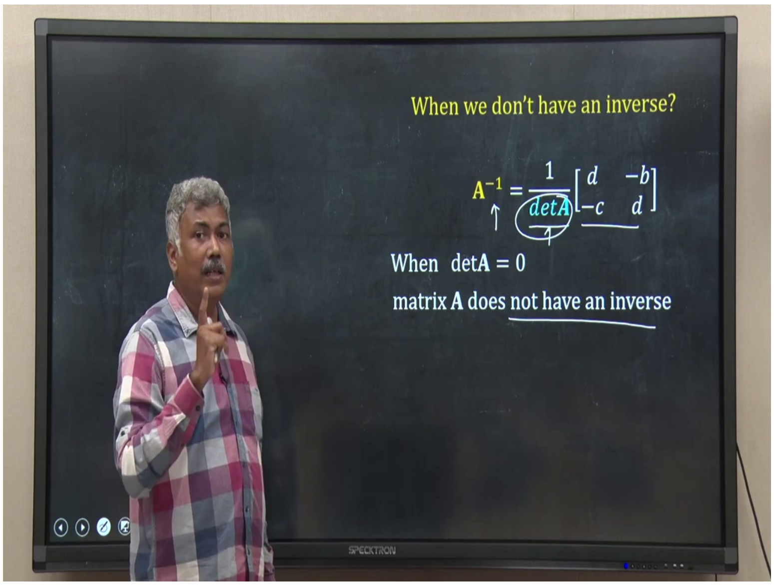

$$A^{-1} = \frac{1}{\det det\, A} \times$$

$$\begin{vmatrix} d & -b \\ -c & d \end{vmatrix}$$

So, for a 2 by 2 matrix by definition this inverse, A inverse is equal to 1 by det A into this new matrix. So, I can ask, can every 2 by 2 matrix will have A inverse using this formula? Not really, because if determinant of that matrix is equal to 0, then you cannot calculate the inverse because 1 by 0 you are getting 1 by 0 so, you cannot calculate the inverse.

So, that means, if I have a matrix, a square matrix 2 by 2 or a higher one whatever it is, if its determinant is equal to 0 then we cannot have a inverse of that matrix. And few slides back we discuss when we can have determinant of a matrix equal to 0. A matrix will have determinant 0 if it is rank deficient. So, the number of rank is less than the number of, maximum number of rank possible that means, the columns or row has linear dependency among themselves, then we will have determinant equal to 0. So, in that case I will not be able to invert the matrix.

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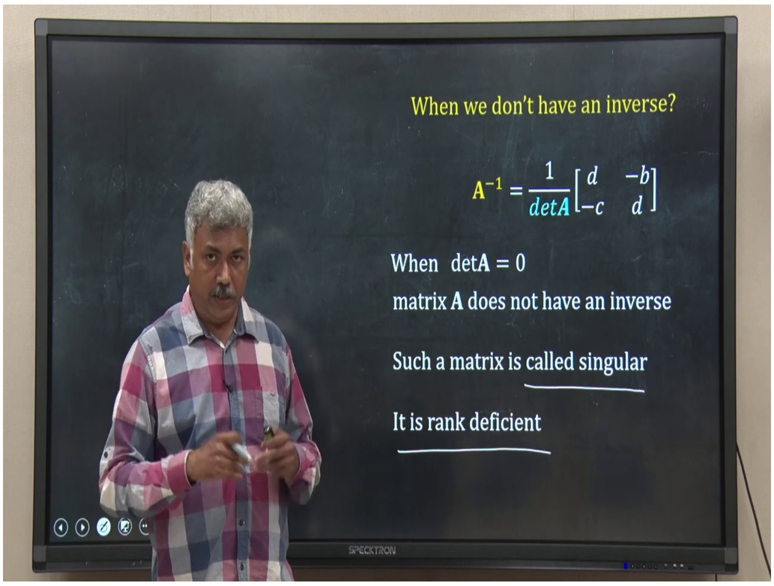

$$A^{-1} = \frac{1}{\det det\ A} \times$$

$$\begin{vmatrix} d & -b \\ -c & d \end{vmatrix}$$

So, that type of matrix, where the rank is less than the maximum one that means is the rank deficient one and we have determinant of A equal to 0, and that is why we do not have any inversion of that matrix will be called a singular matrix. And the singular matrix is by rule is rank deficient. So, we have discussed about determinant, we have discussed about rank and now we have completed discussion on what is the inversion of a matrix.

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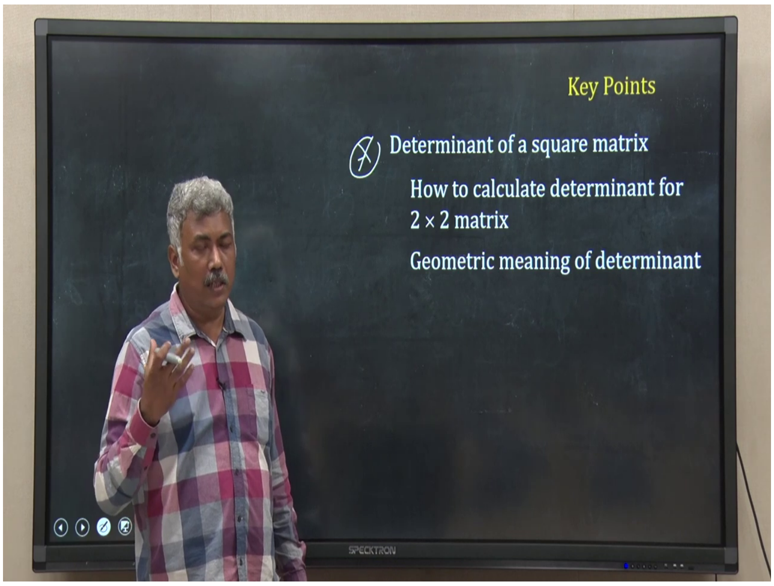

So, let me jot down what we have learnt in this lecture. A determinant of a square matrix is a scalar property of a square matrix. And we have discussed about how can I calculate that for that, at least for the 2 by 2 system which you can do by paper and pen. And also we have tried to understand the physical or geometric meaning of what is the determinant of a matrix.

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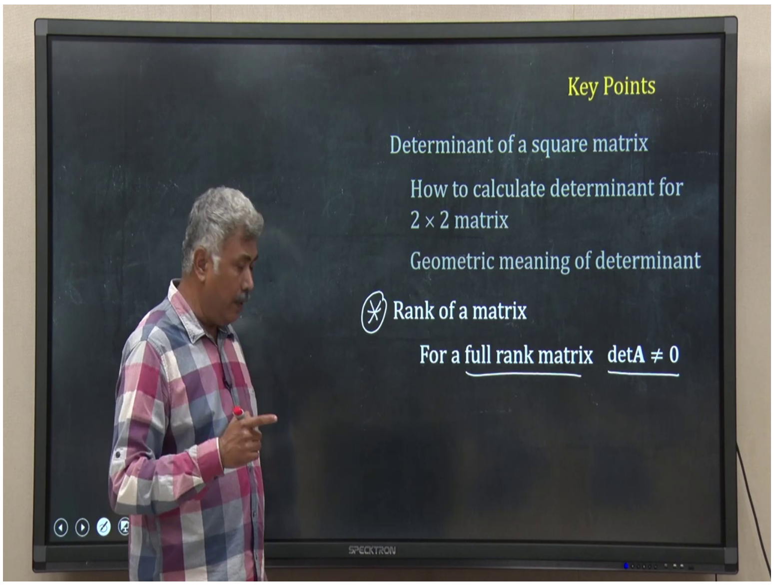

We have also learned about the rank of a matrix, and we have learned full rank matrix, rank deficient matrix and we know now, that a full rank matrix will have determinant not equal to 0. Whereas a rank deficient matrix will have determinant equal to 0. At the last, at the end of this lecture, we have also learned about the inversion of a matrix.

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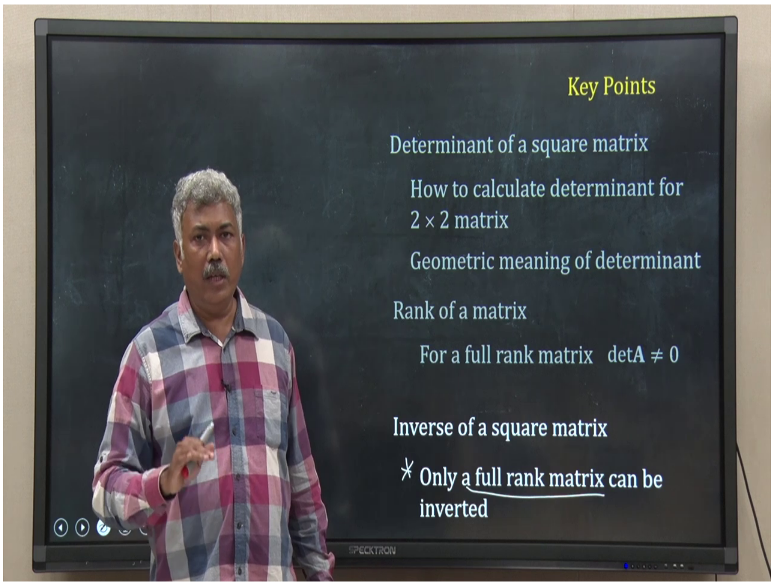

We have understand the definition of that, we have done the calculation for a square matrix and we have shown that a rank deficient matrix, whether it is a 2 dimension or higher, it does not matter, if it is a rank deficient matrix, it cannot be inverted. So, only a full rank matrix can be inverted. That is all for this lecture, but before I end, I will leave you with a problem, a simple one.

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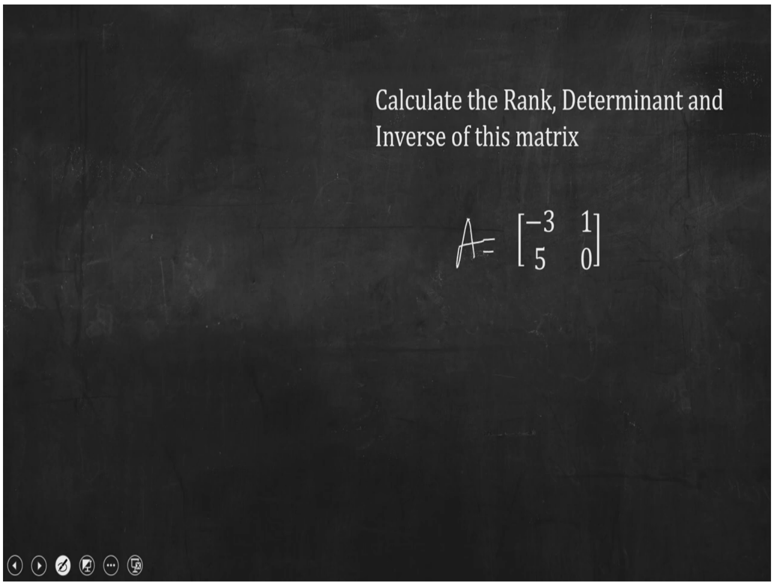

$$A = \begin{vmatrix} -3 & 1 \\ 5 & 0 \end{vmatrix}$$

So, I have given a matrix say and let us call it A. So, what do you have to do? You have to calculate the rank, determinant and inverse of this matrix. Now, if you will do the search on the web, you will find lots of online calculator which can actually calculate determinant, inverse and rank of a matrix. You can try those, no problem in that.

But I will insist why do not you try yourself by paper and pen. This is a 2 by 2 system and we know the formula for calculating determinant and the inverse of a matrix for when the dimension is 2 by 2 and you can actually also check them rank manually by trial and error. So, try it yourself and then the ideas of determinant and rank as well as the inversion will be much clearer to you. So, try it and see you in the next lecture. Till then happy learning.