Introductory Mathematical Methods for Biologists
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Lecture – 34
Fourier Transform

Hi welcome, to this lesson on Mathematical Methods. We have been discussing how to use sin series and cosin series, which is called Fourier series to represent a function. In this lecture, we will discuss a related topic which is called a Fourier transform.

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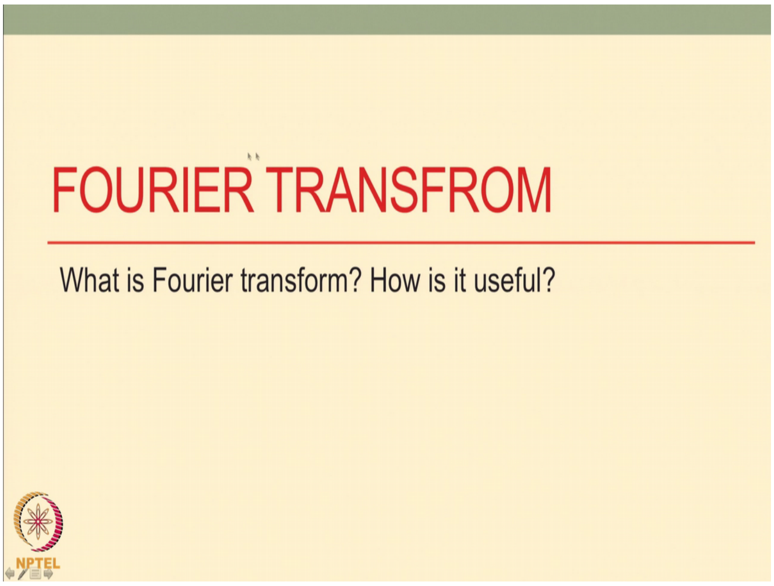

So, today's topic is Fourier transform; in this lecture we will answer two questions first will answer what is Fourier transform? And we will discuss how is it useful?

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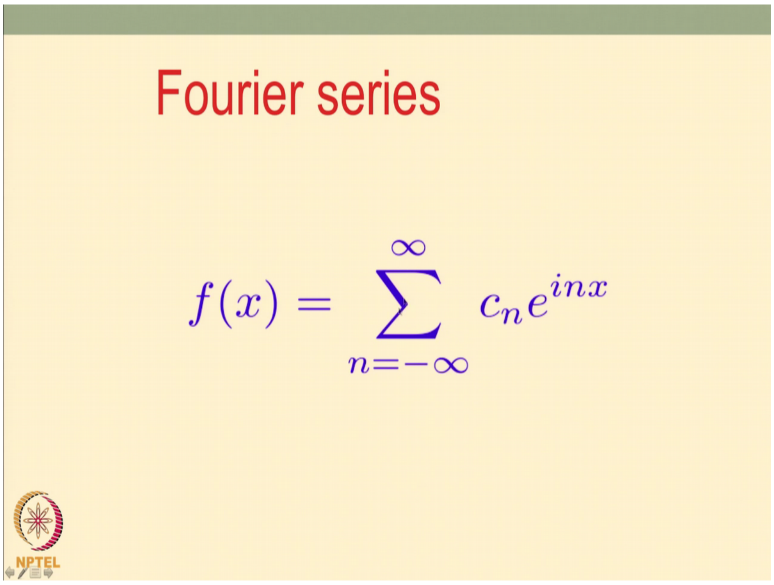

Before that let us just remind ourselves that we have this Fourier series or any function f of x, which is a periodic function can be written as some combination of; some series of cn e power inx this e power inx can be written as cos and sin. So, it could be written as the summation of cosine series and a sin series, but the idea is that any periodic function f of x can be written as sum over n cn e power inx.

Now, the cn values if I know; I know e power inx is cos nx plus i sin nx. So, if I know cn I can calculate f of x; for a function of x.

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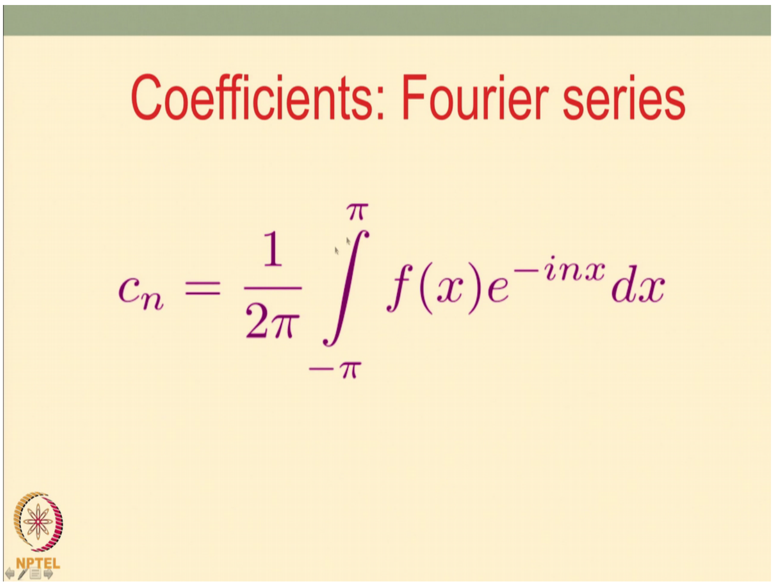

Now, how do we get cn? We can get cn by inverting this integral; that is multiplying this function with e power inx or minus inx appropriately. And by doing this integral, we can get cn this is something that we learn and what is happening here?

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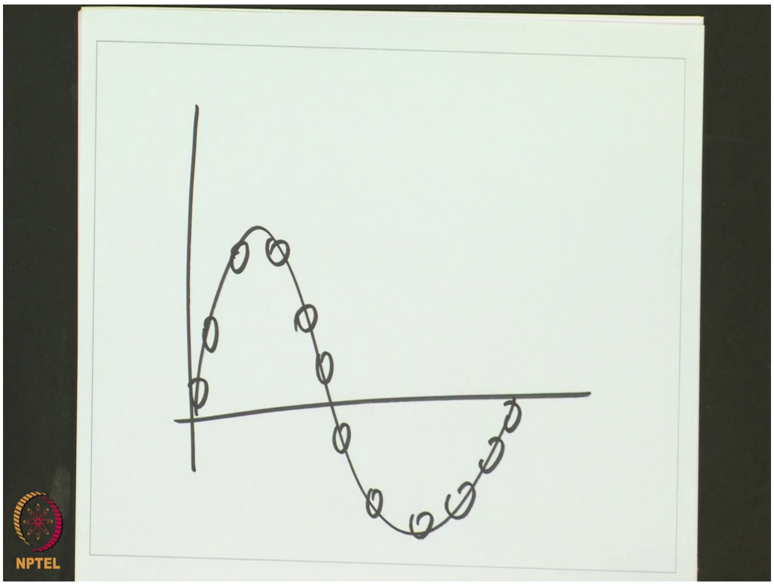

What is happening essentially is that if we want to specify some function; let us say the function which looks like the sin function; I can either specify all the points, you can specify the data like this or we could specify the frequency which is something which I will discuss now.

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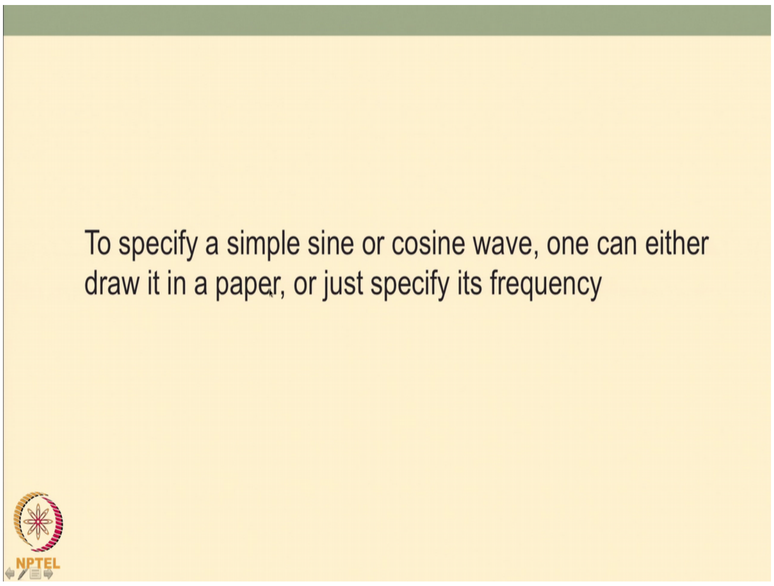

So, let us think of two function sin x and sin 2 x. So, what is plotted here is two function one is sin x and one is sin 2x; what is sin x? Is the red curve is sin x.

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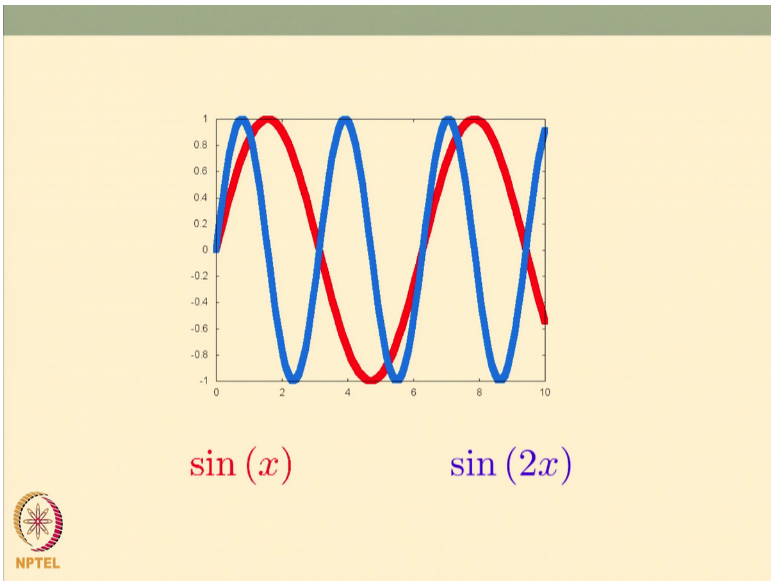

So, between 0 and 3.14 radian; so if is a pi radian which is 3.14. So, this is; so and this is 0 at 3.14 and goes and then again comes back; so, this red curve is sin x.

Now, the blue curve is sin 2 x; so by the time sin takes one full; turn sin x takes one full turn; sin x takes two full turns. So, instead of either I can specify the data points for this or I can specify just the frequency 1 and 2. If I know that it is sin function; I have to just specify the frequencies. If I specifies the frequencies, I can get the function itself; I can reproduce, if I just tell you two as a code word and you already know that the function is sin like you know the sin 2x and you can plot it.

So, either you can specify the function itself or you can specify the frequency and these can be useful and equivalent at some point.

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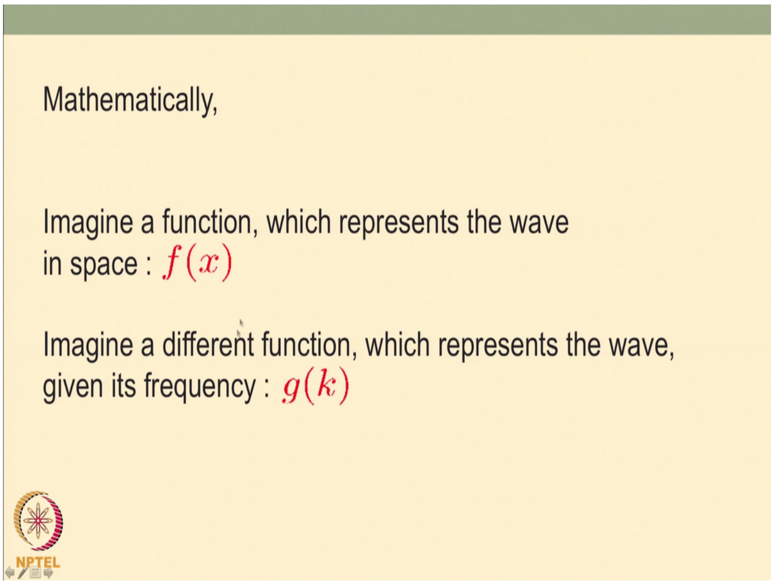

And this is very useful; so, for example, imagine a function which represents the wave in space; this is f of x. You can imagine a different function which represents the wave given its frequency, it has a function of its frequency. So, you can represent g of k or g of omega; so, you can just think of a function.

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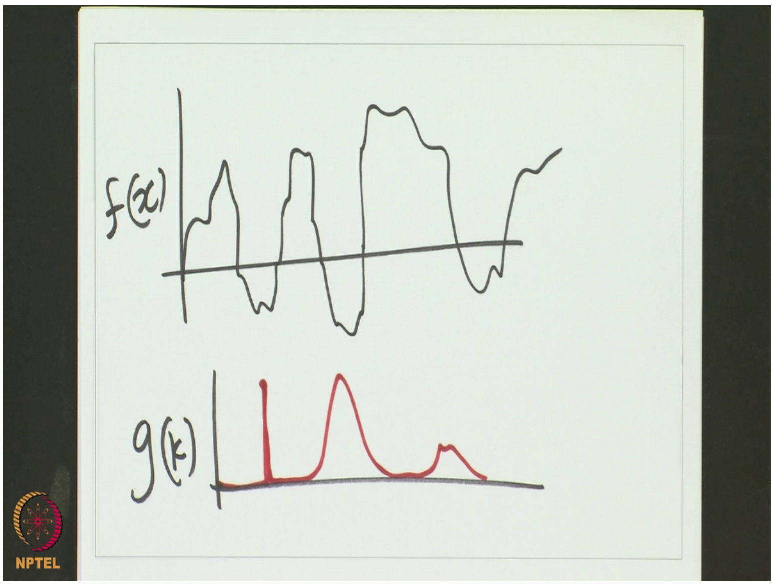

So, let us think of some function; this could be like let us say any function you like; it could be even a music. You have a music which could be like sound increasing and decreasing; now instead of specifying this because f of x; I could specify some g of k which is how often it repeats and what is its frequency wave nature etcetera.

So, as a function of k maybe; it could be like 0 everywhere and somewhere k it will have some value. And then it will have again come back to 0 or it could be like; this it could, I do not know; it could have various shapes. So, this is two functions against both could represent the same thing; so, and this is something that we will tell how to do that.

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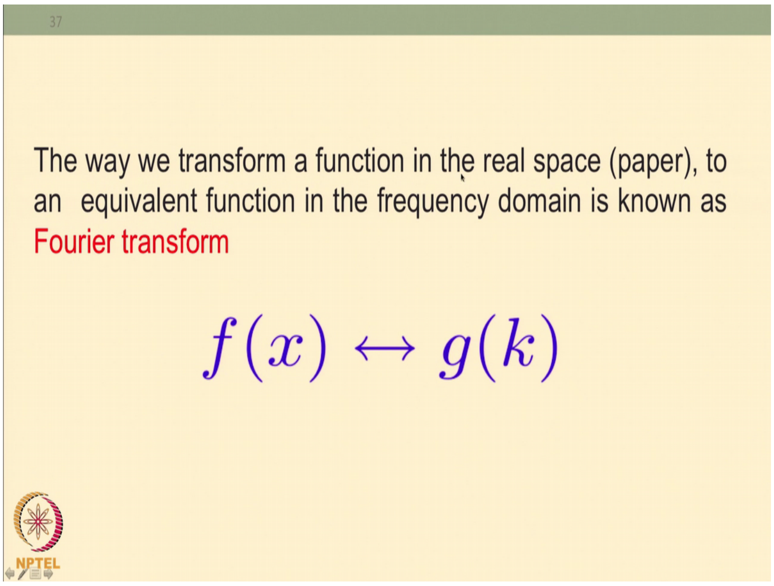

So, the way we can transform a function in real space; which is the in paper, in typical real space is a function of x to an equivalent function of in the frequency domain is known as Fourier transform.

So, Fourier transform is going from the x, y, z space to a frequency space or a k space. So, now, frequency could be spatial frequency or a temporal frequency.

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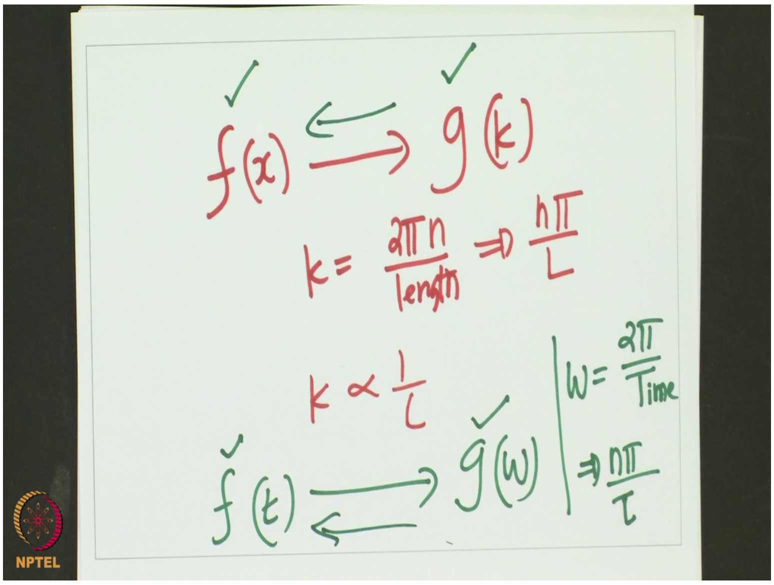

So, you could have something as a function of x; where you could get something as a function of k, where k is essentially 2 pi by some length. So, which is x is in length; so this is the inverse dimension of length. So, 2 pi by some length scale would be like your k. So, it typically will be n 2 pi by L; so often this n pi by L or n 2 pi by L depending on this way, you can define k appropriately, but k will be proportional to 1 over length.

So, this is one way of defining frequency in the spatial frequency; so the k is inversely proportional to L. This equivalently you could also have temporal function; that is f as a function of t from which you can get g as a function of omega; which is a temporal frequency where omega is some time function; it could be 2 pi by some time.

So, it could also be some n pi by some tau say some time function. So, omega and time has omega as inverse dimensional time and k as inverse dimension as lengths. So, this is length inverse, this is time inverse and you could either specify this function or specify this function, that would be equivalent.

Here you could either specify this function or specify this function; those are equivalent, but specifying this for example, can be very useful at times that is one reason to do this. Second using calculations in this space calculation of things as a function of k could be very useful. For example, if you want to solve a differential equation if you convert that equation into the k space; if you do this kind of a conversion from f to kg, g of k it is going to be very useful with something that is useful.

Some very often experimental data will be in this space; it would be in this space. So, therefore, you could go from; you need to go from here to here. So, that is also very needed; to do all this we have to learn this idea called Fourier transform. So, what is Fourier transform? Fourier transform is the following.

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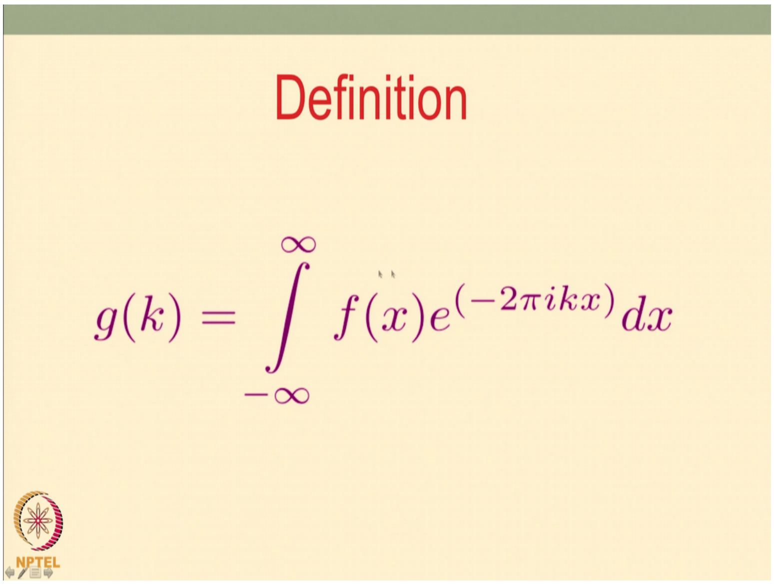

If this is the definition of Fourier transform; if I have a function f of x, I can convert that into g of k by multiplying that function by e power 2 pi ikx dx. So, I can define k appropriately; so, some very often you would also be represented you would, so here you would sometime see.

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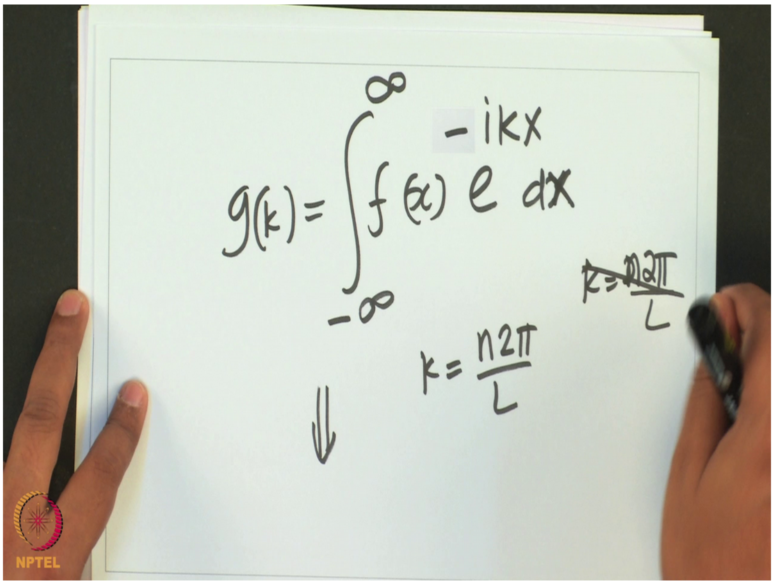

You would see g of k is f of x; you want to convert f of x to g of k you to multiply with e power minus i kx; divide a dx; integrate over dx, minus infinity to infinity. If I integrate this I will get g of k.

Now k can be appropriately defined like k can be defined 2 pi by L; n 2 pi by L and all that. So, k if I define k as n 2 pi by L and I can write this as a sum if I want. So, then if I do that; this has a connection for Fourier series. So, you might remember you have an f of x, you multiply with e power ikx or sin x or a cos x because I can write this as sin x and cox x.

So, let us write this; so, this thing is called a Fourier transform, this conversion f of x to g of x this integral, this transformation, this mathematical transformation is called a Fourier transform and this is lot of use we will discuss how this is useful. But immediately one thing you can see is that; this is the connection with Fourier series of course.

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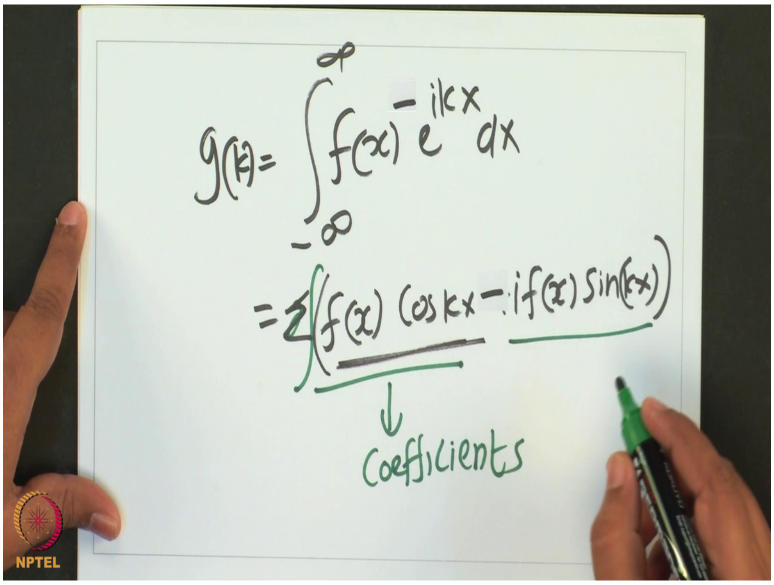

So, if I just write g of k is equal to f of x; e power minus ikx integral dx minus infinity to infinity, I can write it as f of f of x cos kx minus i sin kx and of course, it is an integral can be written as a sum. So, I can represent as sum of the whole thing.

So, now depending on the limit of the sum; so you can see that this is like the integral one would do or I could do and. So, this is like an integral if this was an integral; this integral is something that you would do to calculate the coefficients, in the Fourier series to calculate the coefficients we would we did this kind of integral. So, these are the integrals we did to calculate the coefficient the Fourier series and therefore, there is a connection.

In the Fourier series, we needed a periodic function; we needed a repeating periodic function. Here this is some the analogue here would be is like period or the length going to infinity limit. In that limit that is equivalent of a Fourier transform; I will not go into the detail this you could read and understand the details, but for this purpose of this course is just enough to understand that; if I have a function f of x it can be converted by doing an integration to a frequency space and you can convert back.

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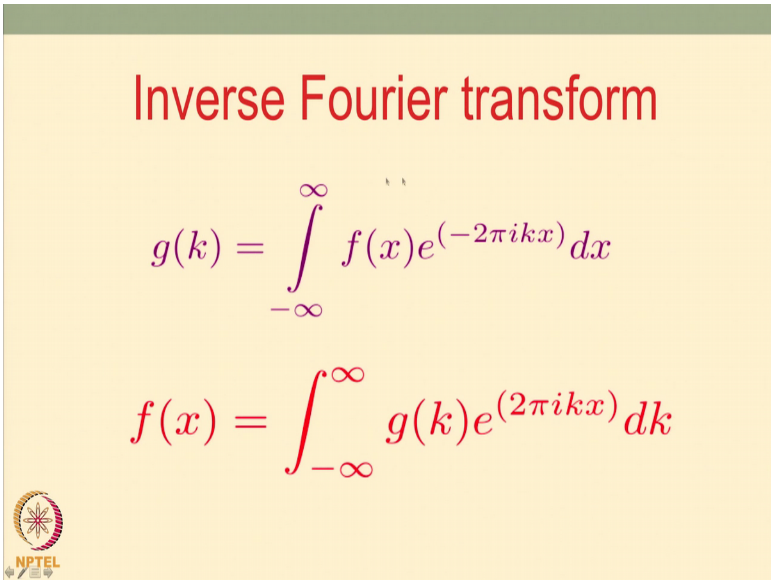

So, if I you can do also an inverse Fourier transform. So, if I do have an f of x; I can convert g of k by doing a mathematical operation which is an integration. If I have a g of k, I can get an f of x by doing an integration. Now doing this integral in reality could be difficult because this f of x could be some complicated function or g of k could be a complicated function. But nonetheless, at least we can calculate the area under this function f of x e power i; 2 pi kx. So, we can calculate area and that is nothing, but the integral.

So, numerically one can do the integral always; analytically doing this integrals might be very difficult. But if I can plot this f of x e power i 2 pi kx and calculate the area under this; that is going to be this integral and I would get for different values of k; if I do this I would get g of k. So, I can numerically compute this integral and compute g of k; now why is it useful? Let us think of; so now we know that this has a connection with Fourier series and we know how this is done in the case of Fourier series at least.

Now let us think of some simple example; so, simple example is if I have a wave like function, so if I have just sin 2x or cos 2x.

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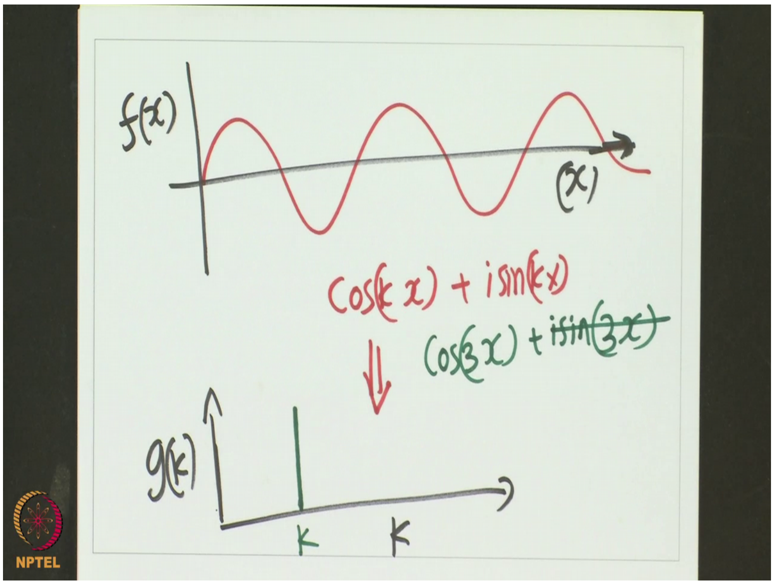

So, if I have some function let us say it is some wave like function. So, if I have a function like this; so, this could be some combination of sin and cos. So, in general this can be written as cos kx plus i sin kx, but does not matter you could you take cos kx alone. If I do the Fourier transform of this, so if I do a Fourier transform of this; this is my f of x versus x; this is a function of x and you will get a curve like this as a function of x.

Now, if I just want to plot g of k versus k; how would it look like? For this particular function, if I do the integral that we like the answer will turn out to be just one peak at some particular value of k. So, whatever be the value of the k here; at that k, if k is 3 this will be 3; if k is 4; so, this k would be 3; if this is cos 3 x; plus i sin 3 x. If this is my function then I will have just a k or if we just cos 3 x; I would have some function k equal to 3; I will have just peak at one particular point.

So, this whole thing is now converted to just one peak, one function which is called a dirac delta function.

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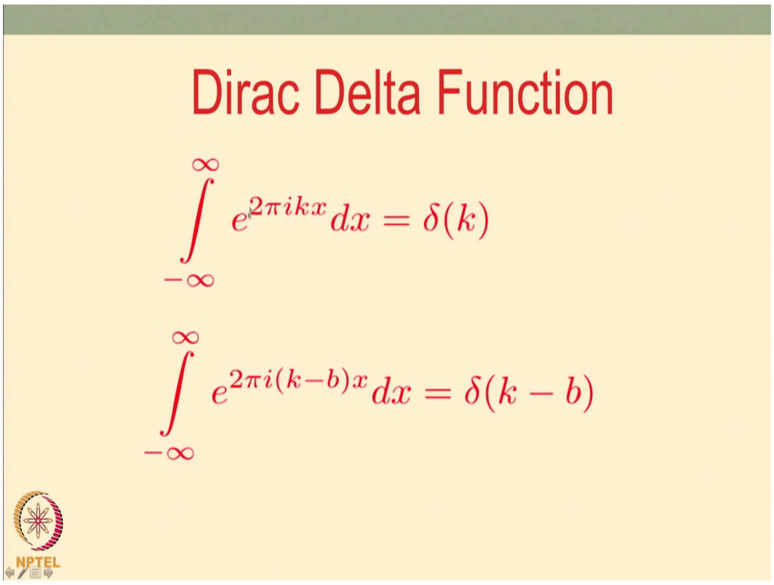

So, mathematical definition is the following e power i 2 pi k x is delta of k; if e power 2 pi i k minus bx this is delta of k minus b.

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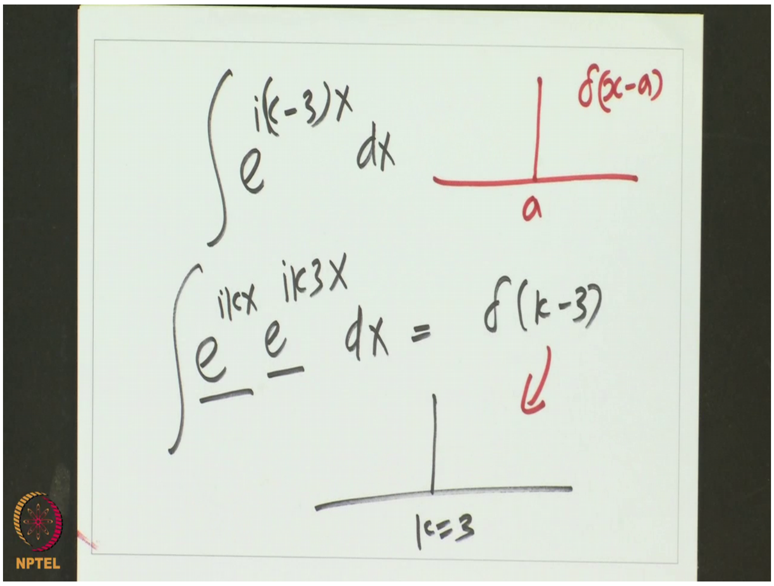

So in this example in the earlier example, if we have b equal to three for example, if I have; if I have some particular sin kx or some particular value, then I would have if I have e power i k minus 3; x dx if I have integral. This is just like e power ikx; e power ik 3 x; these are two functions. So, both are and I do integral of this, dx I would get delta of k minus 3, which is like a peak at k equal to 3.

So, that is what this is says; so, if I have a function e power 2 i pi k x; this is delta of k and delta of k is plotted; this delta of k is called a dirac delta function which is has value only at that point; everywhere else it is 0. So, such peak functions; single peak functions are called dirac delta functions and delta of k minus 3 means; a peak at k equal to 3 alone delta of x minus a means a peak at a alone. So, this function is what is plotted here; so, if I plot this function it would look like this. So, functions with just one peak are called delta functions.

So, delta of x minus a; would mean at x equal to a there is a peak. So, if we have a nice oscillating function it is likely to have a Fourier transform, which is just one peak. Now if you have more complicated function, if you think of a music; what is a music? Music is essentially some kind of a combination of sines and cosin with some complicated manner.

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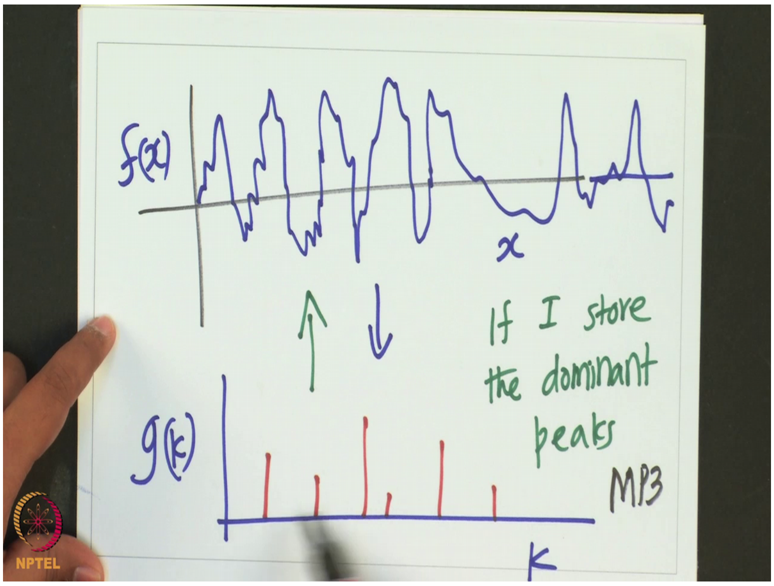

So, if I just if you just many often you would have seen this patterns of music; what would you get? You would be getting some complicated pattern like this; the sound amplitude and its frequency would change and all that; lot of some music like this.

Now, this is your f of x; I can do as a function of x. I can do a mathematical operation and convert it to a g of k and the mathematical operation is what we described; it is an integral that is multiplied with this e power ik x and find the area under this; that is all you have to do. Multiply this with sin and cos and find the area under this. So, my first multiply with cos; find the area then you multiply with sin find the area under the curve that is like integration and from that you can get this g of k.

Now, imagine that if the g of k looks like this something like this; I am just. So, if this g of k looks like this. If I store this much information that is this peak positions and the height of the peak only two less information here. So, I have to just store the peak positions and the height of the peak. If I store this g of k information, I can construct this back and if I store only the dominant peaks. So, I might have very small peaks which is negligible, but if I store only the dominant peak; I can reconstruct approximately this music.

So, this is what is typically done for example, in the MP3 music like roughly this is what is done. You might have heard this thing called MP3; so, MP3 is a format which many of you might have heard, MP3 format which is some kind of compressing format. So, if I have this complicated music; I can do some mathematical operation which is the Fourier transform.

I can convert this to this and I would get just g of k which is much going to be much simpler than that. If I only store the dominant peaks in this function and then I can throw away all the other peaks; that would help us compressing this and storing much lesser information. And whenever we want we can reconstruct this back; how do we do this? This is what we just discussed, how do we do this is what is discussed.

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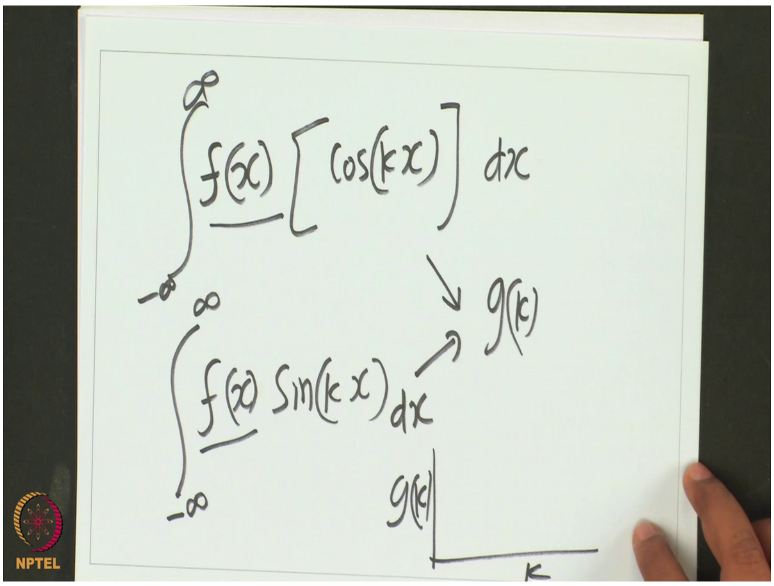

If I have this function f of x; I can multiply with this cos kx and then integrate, I can multiply that with sin kx and integrate minus infinity to infinity and combining this two cos kx plus i sin kx; would give me g of k.

So, basically this is all I have to do; if I do this and sum; I would essentially get I can get g of k. So, if I do this integral and this integral I can essentially calculate g of k. So, for every k; I do this integral first I take k equal to 0 I do this integral, then I take k equal to next value.

So, k would vary is a continuous variable again I could take various values of k and I do this and I plot g of k versus k. And I can store this information which is equivalent of storing f of x so that is what is happening in many of the; this is also true for j peg figure; so if you have a figure, if you have a photo what you have essentially is a two dimensional information.

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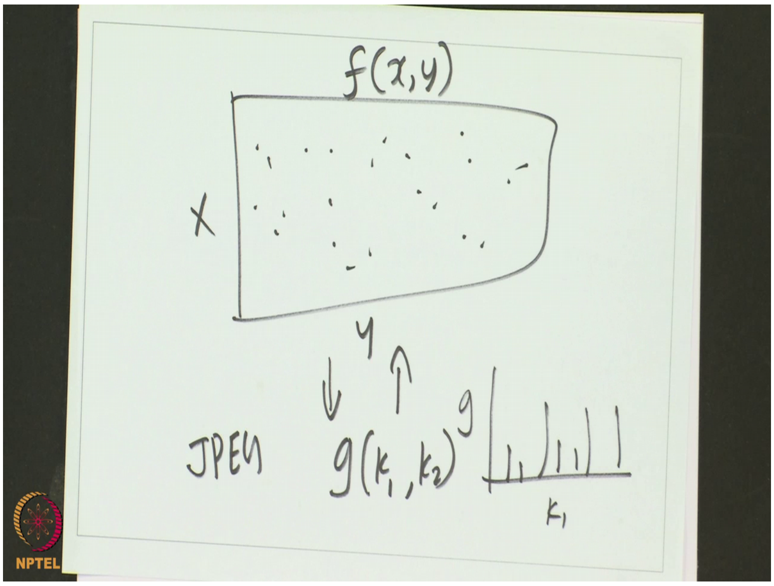

You have some picture which is as we said its some two dimensional pixel information which is x and y. And what you have is f of x comma y and this you can convert; this to g of k x; k 1 comma k 2; I can convert this into a g space by doing a similar transformation both for x and y separately; I will get k 1 and k 2 and then this you can convert back.

So, this is what is typically j peg and all that the compressing formats of pictures; they convert this lot of pixels into some peaks; in the as g function of k 1 and g function of k

2; I make some peaks and then that would give us this; that will help us to store this information, there are many applications. Another application is in scattering experiments.

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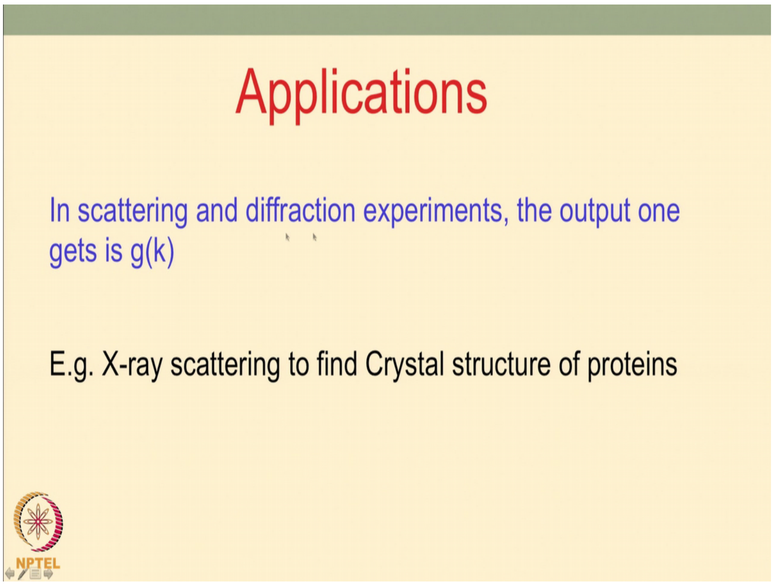

So, scattering and diffraction experiment the output one gets is g of k; example x ray scattering which would find us the crystal structure.

So, this is what is very often used in all many kind of experiments as we said; they also briefly mentioned in the last class that I have the shape of a protein which is my function. I can send light or I can send some kind of electromagnetic waves and that would scatter and you would get some measurement, which essentially will be like the g of k. And by doing some mathematical operations to that one can get f of x the function which is the shape of the protein itself, so this is the application that we have. So, let me just remind you the basic principle.

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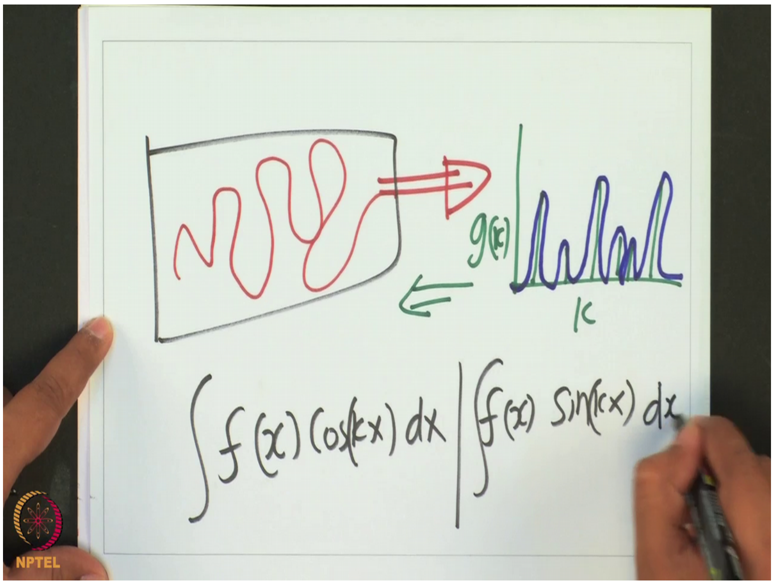

So, you have some shape of a protein or shape of any structure that you would like; some structure. And you do some experiment and what you would get? You would get some other function which is g of k.

So, which would be like; which I would draw as some kind of peaks it could be more continuous function like this, it could be some continuous function like this. So, I would get g of k as a function of k; now how do we get and one can get back, get this back by doing this operation. What do you do? You have to either to go from here to here mathematically as I said; I would take this f of x; multiply with cos kx and integrate.

And then multiply with sin kx and integrate and these two integral together will give us this. And if I do this and integrate back; I would get this back, I would take g of k and integrate with cos kx and sin kx with appropriate signs.

So, if I have a function g of k how will I switch? How will I calculate f of x? I have this experimental data g of k.

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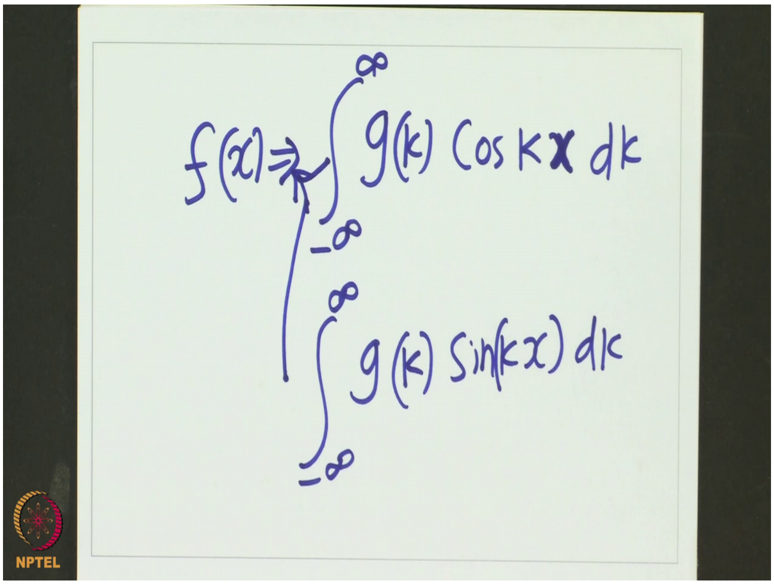

And now I want to calculate the shape of the protein in the real space. I would first multiply with cos kx; cos kx and then integrate over dk minus infinity to infinity. So, this if I do this; I would get for a given value of x; integrate over all k I would get f of x. So, this is one thing I will do, then I would do g of k; sin kx, I will do both. If I do both and from this together this and this combined; I can get this f of x.

So, I do this two integrals multiply with sin and cos and I do this integrals; I can get f of x. So, that is the procedure that we would follow.

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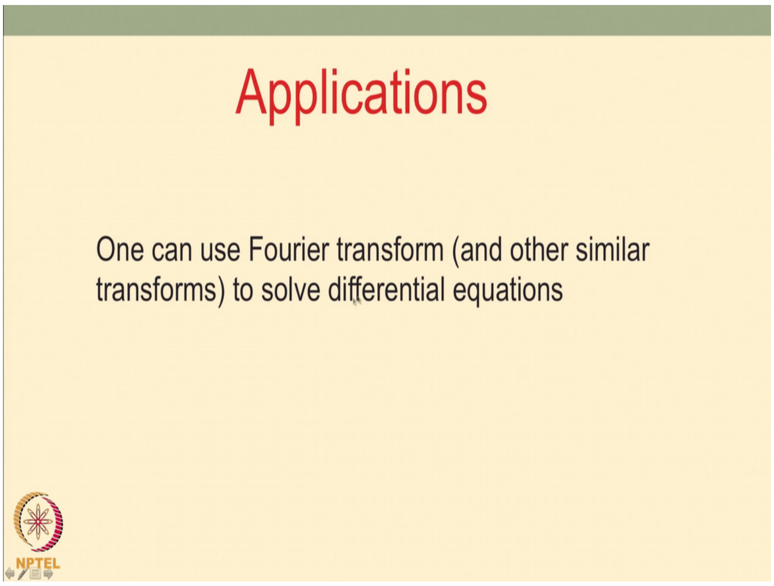

So, one can use Fourier transform and other similar transform to solve differential equations. So, this is also very useful to solve some differential equations which I would not discuss now. But in an advance topic, you would learn that how to use this idea to solve some differential equations is very useful.

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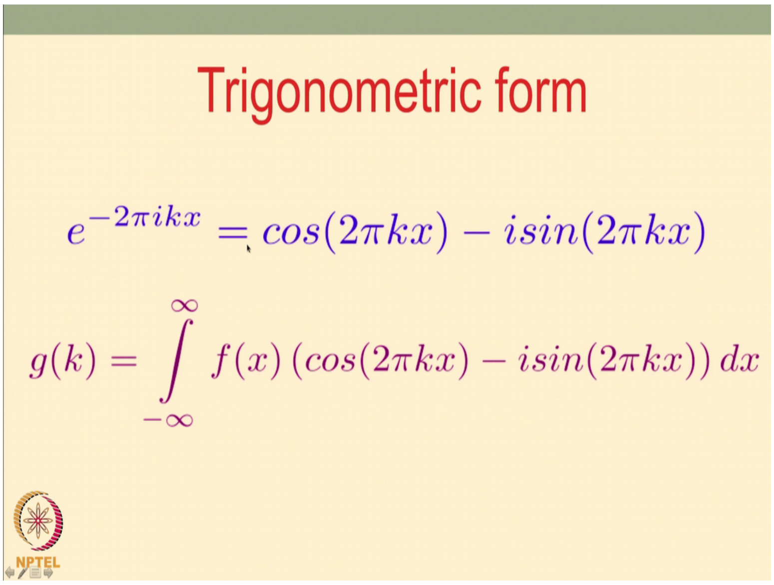

As we said just now, I can write e power i kx as cos 2 pi kx minus i 2 pi kx and then this can be written as a cosine or sin transform called trigonometric transform some time.

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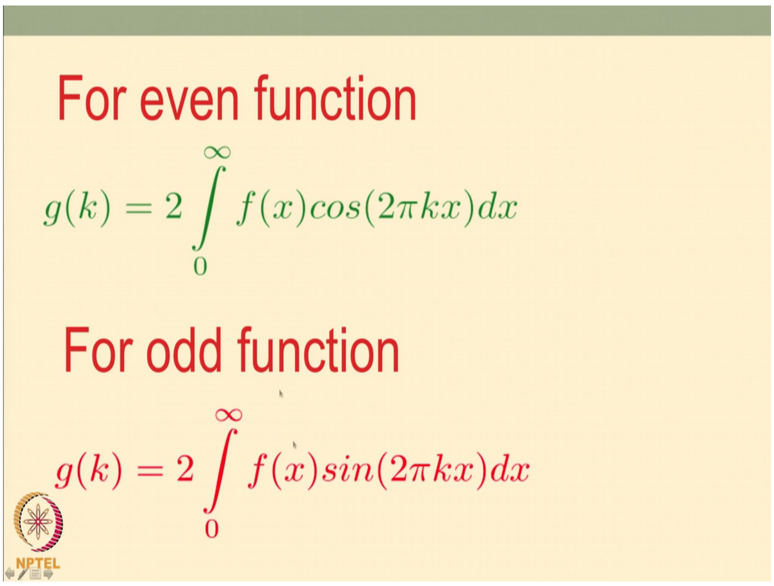

And if it is odd function and even function; some of this sin integral and cos integral will be 0 and appropriately we could just write only one of them.

So, these are some simplifications which you can think about it, which are mathematically useful, but physically what is important is that if I have some function in real space, which is the space that we see; which is f of x.

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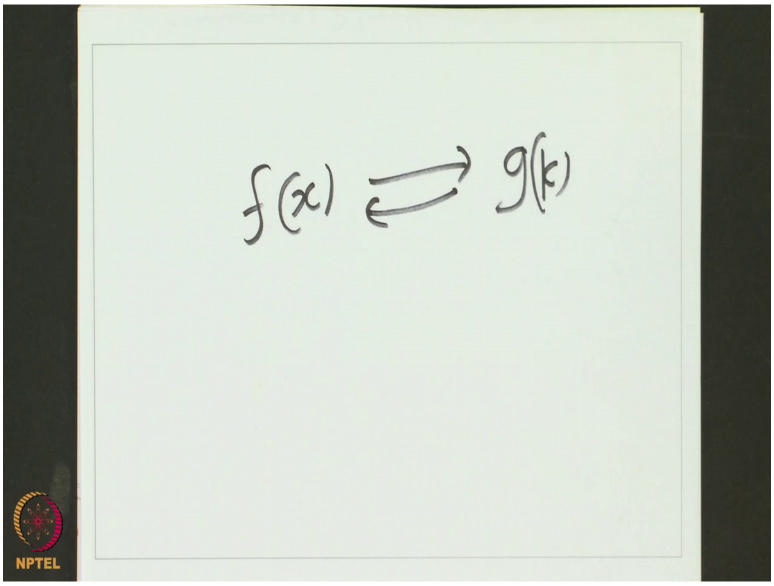

I can convert that into g of k and back; this is very useful and that is where we use the Fourier transform.

With this, I will stop this lecture and continue in the next lecture. Bye.