Introductory Mathematical Methods for Biologists
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Lecture – 32
Fourier Series: Part I

Hi. Welcome to this lecture on mathematical methods, today we are going to discuss a new topic which is called Fourier series.

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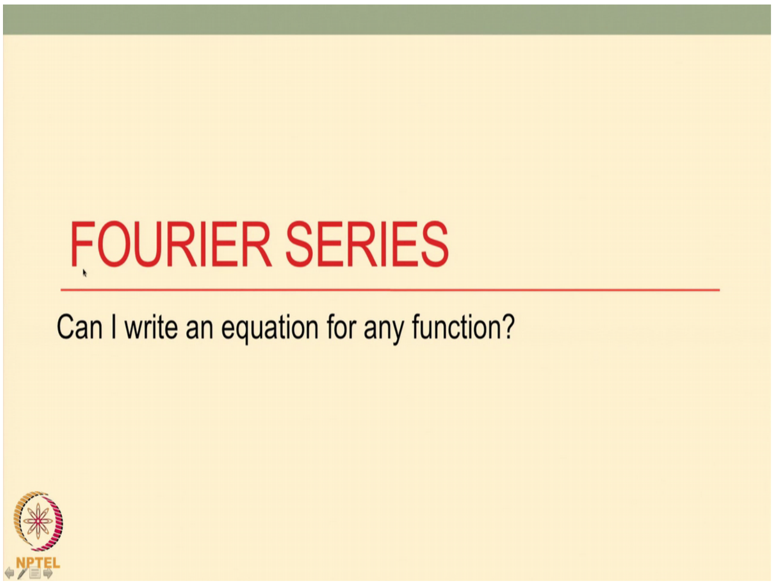

So, the title of the lecture is Fourier series the question we will answer is can I write an equation for any function this is the question that we would ask if you think about doing an experiment and you collecting some data. So, let us first think about your collecting data as a function of time, let us say you are measuring concentration as a function of time in a particular cell or you would want to measure some specific property as a function of time there are 2-3 ways this can happen and one of the ways is that.

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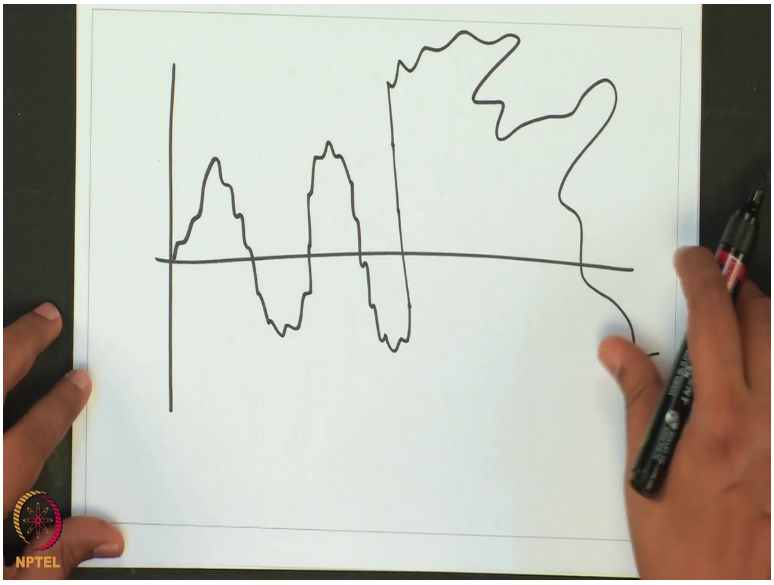

This could this function this could kind of increase and decrease we around some average value or it could it could it could be it could show some behavior, it could show some arbitrary behavior.

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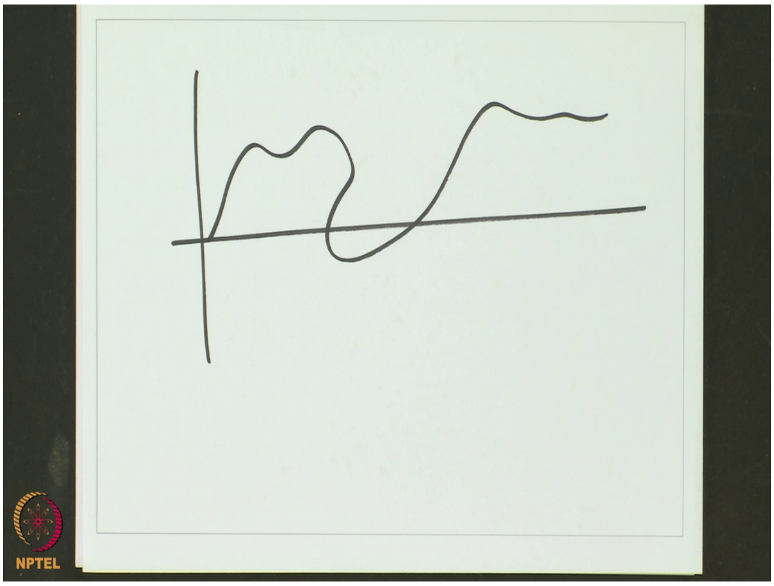

So, if you have a data which is kind of arbitrary. So, if whenever you have a data which is has some shape arbitrary shape, we do not know what equation to write for this what is the equation for such a curve, we do not know can we write an equation for such a curve that is first question that comes to everybody's mind and we will see what we can do if you have some data which looks kind of arbitrary that is one thing.

Second thing is if you have we will have various structures for example, we will have structure of a protein we will have structure of DNA organize chromatin we would have concentration in a particular manner. So, or we would have cells organized in a tissue; if and when we have such structures can we write an equation for this for example, I have a protein.

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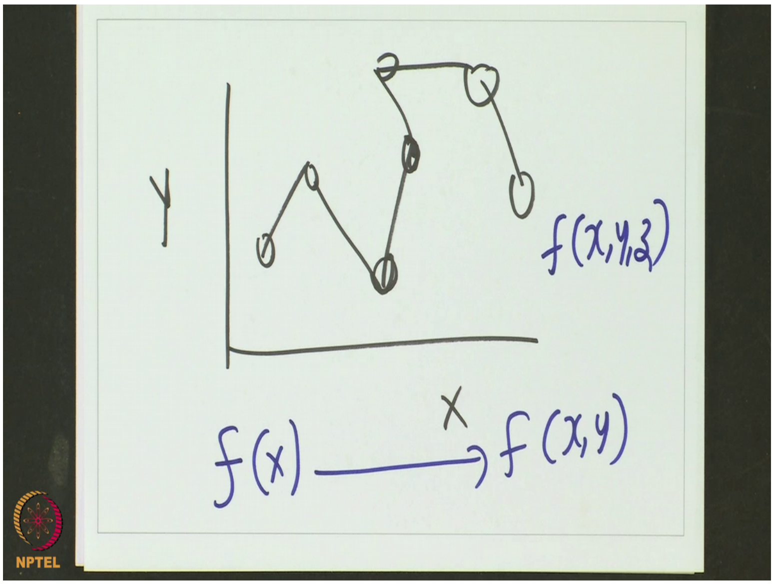

I have some protein. So, we have basically atoms organized somewhere in a 2 d space here, but in general in a 3 d space.

So, this is my x and y, I am just showing a protein shape in a 2 d paper and these are molecules. So, these are molecules. So, the position of these molecules along this varies. So, this could be written as some function some function of x and y like. So, x in general a 3 d thing can be written as some function of x and y or some function of x, y and z.

So, any shape of a protein or any molecule could be written as some function of x, y and z. Now can we write an question for this if you know the data if you have the data can we write an equation for this the answer to this question is the following. So, there is something called. So, this Fourier series will help us in this context.

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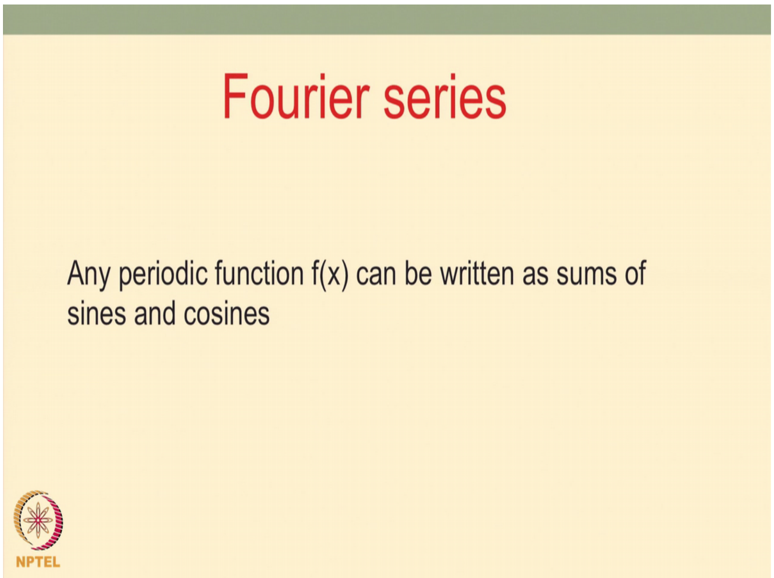

So, let us first understand; what is the Fourier series Fourier series is basically any periodic function f of x can be written as sums of sines and cosines. So, you have some function some function which is periodic function. So, what is it? What do I mean by periodic? Periodic means repeating after intervals.

So, if you have a function which is repeating after some interval then that function that data if it is repeating I can use sines and cosines and some series of sines and cosines I can use and write this function. So, this is the idea here now let us think about it little bit what happens when you have were what are the context where you will have some data. For example, you would know in crystallography; what happens in X-ray crystallography you have a protein you have to crystallize it what is crystallizing means you have to form a crystal what is crystal? Crystal is repeating unit of the protein itself. So, you have to have you have many proteins arranged in a lattice in a periodic manner.

So, if you put lot of effort and pack this protein as a crystal then you have the molecules of this protein repeating in space many times this is like a periodic function if you wish now one can use these ideas from Fourier series and Fourier transform to understand this positions of this molecules in your protein right. So, you have let us think of a particular protein having 2 atoms.

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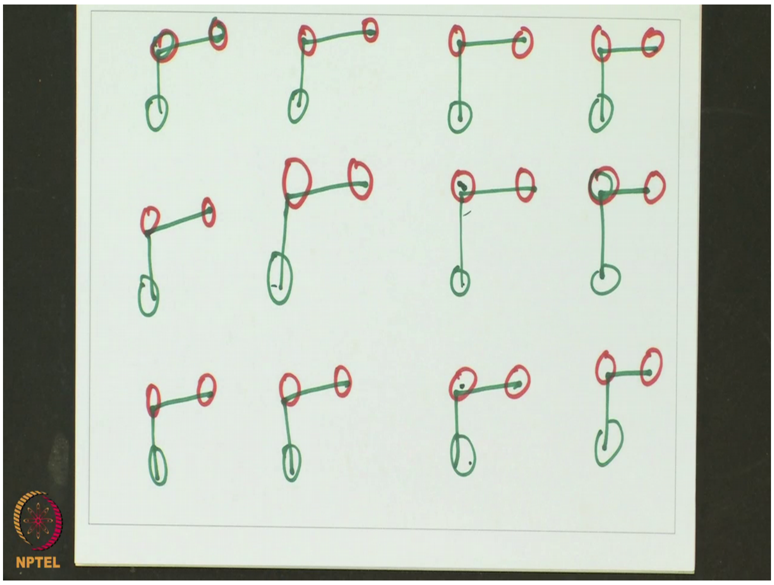

So, let us think of a simple. So, you have I am just going to think of a simple protein. So, which has some shape like this. So, this is a shape like this which is like a inverted L.

So, I could have a repeating function of this I could have a I could pack this protein in a repeating fashion here I could pack this protein in 3 d that would be forming a crystal. So, so I would basically. So, I would form a crystal arrangement let me show this here. So, now, here and here, I would draw this and this and here and here, here and here and let me quickly draw this this is I am just quickly drawing the repeating.

So, what do I have I have this protein which is like an inverted l let us say this is the shape of my protein and this is the folded shape of my protein I have arranged this protein into nice lattice. So, if I just take this green molecule if I go from some distance to the right I can again get this green if I go some distance to the left, I can get again to this green if I go since this in top I again see this green circle, if I go some distance down I again see this green. So, this green atom here or green molecule here is periodically arranged.

Similarly, this red this is periodically arranged. So, each of this if I just take an atom or a molecule which is a circle here it is periodically arranged in space. So, if I this whole thing is like a periodic function if you wish. So, if I have this periodic function which is like a crystal if you wish. So, you have a nice crystalline arrangement a periodic arrangement then we can use sine and cosine that we learned what are sins and cosines they are periodic functions they repeat as a function of x right. So, this is the idea that we can use some combination of sines and cosines to describe any periodic function, how do we do that it is the following.

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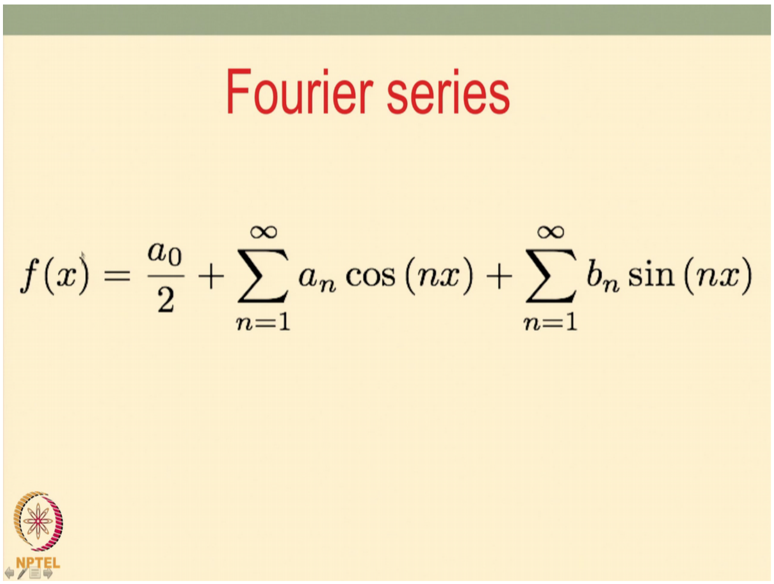

So, it says that any periodic function f of x. So, first we will do in one dimension f of x is a function which is like a repeating function of a crystal like we saw for example, this can be written as basically a 0 by 2 which is a constant plus a n cos n x sum over n I will explain what does it mean b n sin n x sum over n. So, this is the series and let me explain this what does it mean what are we saying here we are saying.

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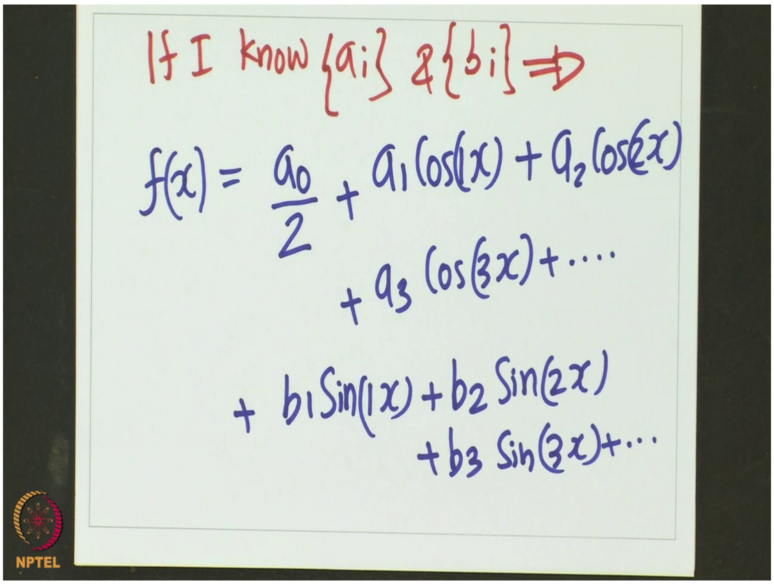

If you have some function if you have some function f of x this can be written as some constant which I could be call anything, but I am calling a 0 by 2 for convenience, but I could call this anything plus a 1 cos 1 x plus a 2 cos 2 x plus a 3 cos 3 x plus dot, dot, dot is an infinite series up to infinity plus I have b 0 sin 0 x. So, b 0 sin 0; since sin 0 is 0, I will have a; I will not have a b 0 term, I will have a b 1 sin 1 x plus b 2 sin 2 x plus b 3 sin 3 x plus dot, dot, dot.

So, i have basically an infinite series of sines and cosines I do not know what are the values of a 1, a 2, a 3, b 1, b 2, b 3, etcetera, but if I know this a 0, a 1, a 2, a 3, etcetera, and b 0, b 1, b 2, b 3, etcetera, I will get this for; I will get a formula for f of x. So, if I know. So, a 0, a 1, a 2, a 3 are unknowns and if I know a i's these are some set of values and b i's these are another set of values we can get we can get this f of x if I know a i's and b i's, we can get f of x; this is the basic idea that we would learn. Now we let us see how to how do we do this.

So, let us look at this again. So, let us look at this again once more if you look at this series here if I know let us assume that we have only 2 numbers which is a which is a constant and all other numbers are 0.

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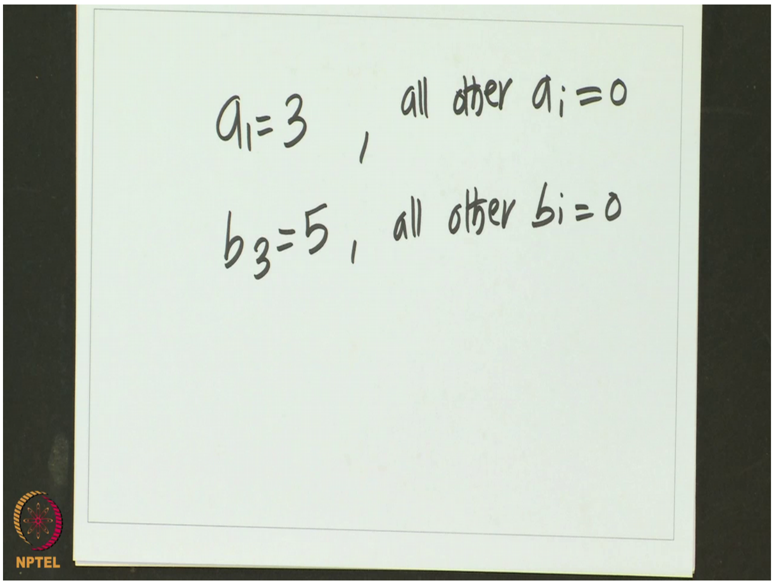

So, let us assume that. So, our function let us assume I know a 1 is equal to 3 and b 3 is equal to 5, all other a all other as are 0 all other a i's equal to 0 all other b i are 0. So, other than this a 1 and b 3 all other coefficients are zero.

So, if I take this equation f of x, I have a 1 which is 3 according to me. So, if I have this 3. So, I have 3 cos x and this is everything else is 0 and I have b 3 which is 5. So, I have 5 sin 3 x. So, I have my equation my f of x my f of x, if I have only a 1 is equal to 3 and b 3 equal to 5.

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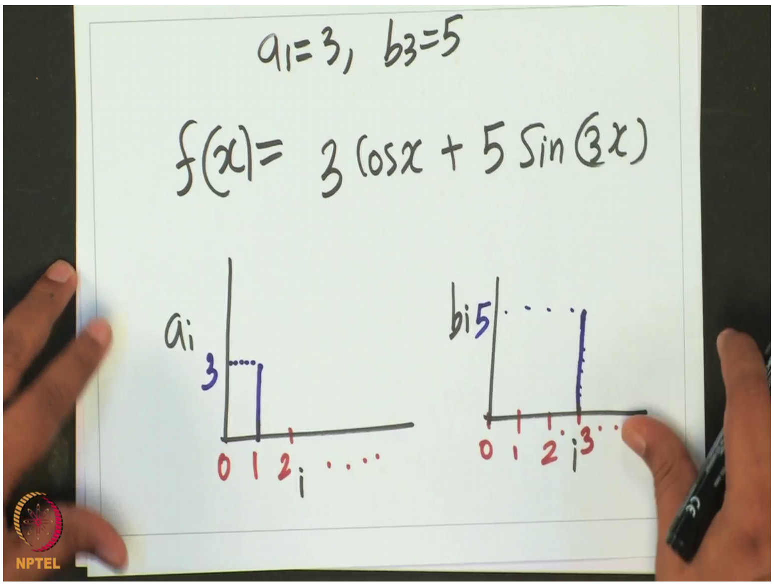

If I substitute in that expansion what do I get I would get f of x. So, please go back and look at this expansion I am saying a 1 is equal to 3 and b 3 is equal to 5 everything else is 0. If I specify these 2 numbers, I get a function which is essentially 3 cos x plus 5 sin 3 x. So, I have an equation by specifying these 2 numbers.

So, if I specify these 2 numbers, I have an equation for a function this could be my protein for example, the protein expression could look like this this is some function I can specify this. For example, I can plot a graph of a i's versus I and b i versus I and I have only a value. So, only when a one. So, this is 0 1 2 etcetera I would go here also I would go from 0, 1, 2, etcetera. So, when a 1 is 3. So, what does it mean a 1 here is this value is 3 all other values are 0 only this is 3 all other values are 0 here the next one is b 3; b 3 is here. So, this is b 3; this is my 3. So, b 3, b 3 is 5. So, b 3 is 5.

So, if I have this 2 peaks like this. So, there is this and this 2 peaks. So, what I have I have just some what I have essentially.

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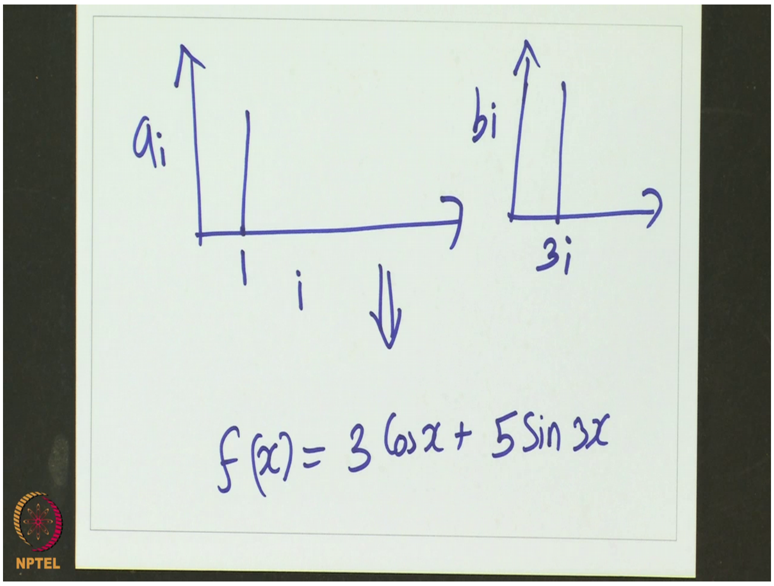

I have basically a graph which would specify a i versus i and b i versus i; I can specify some peaks here I know; if I specify just this for a particular i value, here i is equal to 1 and i is equal to 3 here if I specify this kind of a graph correspondingly this would imply that my f of x is 3 cos x plus 5 sin 3 x for example, if this is the thing that we just described.

So, basically what am I saying if I specify a i and b i; I can specify a function f of x, I could specify many a i's and many b i's, then I would get some function f of x there are infinite there are infinite numbers, but if I specify only the most important coefficients the dominating coefficients that function. So, if I just specify only most important dominating coefficients.

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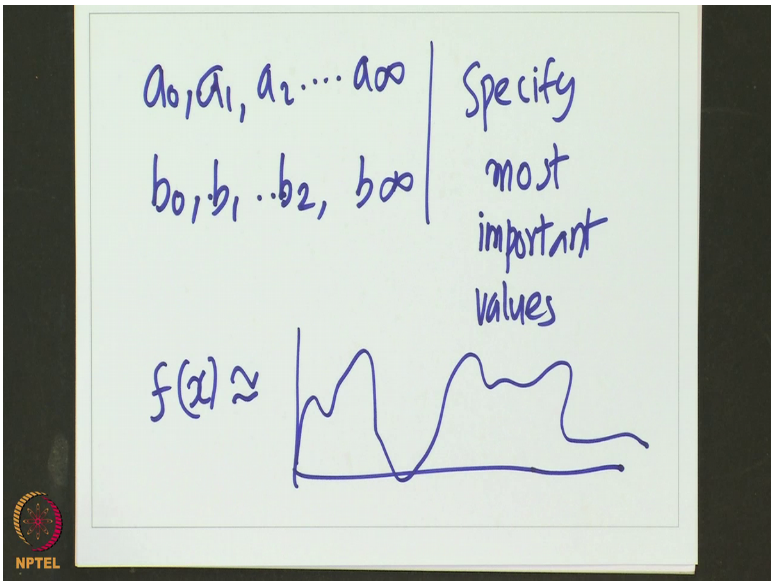

So, I have this full number set of a 0, a 1, a 2 up to a infinity b 0, b 1, b 2, up to b infinity if I specify only some of them I specify most important values how do we know important we will discuss that later if I specify some important values then my f of x can be approximately written I am specifying only some values I am not specifying the infinite values i only specify important values and I can get approximate expression for f of x and that upon approximate expression could look roughly like your real curve.

So, this is the first idea that you should keep come to your mind this also has another connotation instead of specifying the whole curve I could specify some 2 numbers let us say I have a data which looks like. So, we had this curve example equation. So, let us say we had this equation which we just describe.

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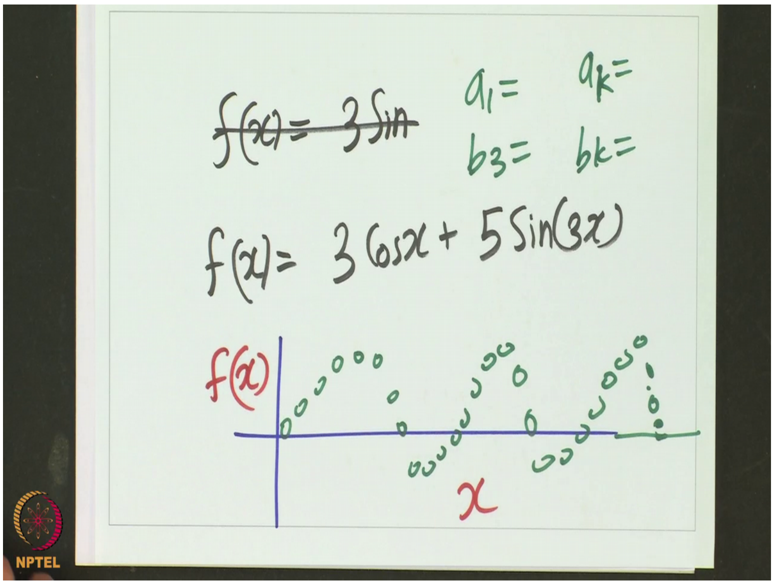

Which is f of x is equal to 3 sin x cos x. So, what we wrote f of x is equal to 3 cos x plus 5 sin 3 x. So, this is the equation that we wrote and this if I have this as a data. So, if I plot it this would look like some curve i in space. It would be some oscillating curve if I have some data. If I want to specify this as a data I would need various numbers which is varying in various ways; I would; I do not know i; let us say I have some data like this; I have to specify lot of data points to describe this instead of specifying all these data points I could just specify some numbers like a 1 b 3 and some numbers like this if I specify like it could be a i some number some ak whatever k could be something and if I specify some pk that would be equivalent of specifying this data.

So, I could either specify this data which is a function of x or I could specify this ak and bk or a i and b i. So, I could either specify this or specify this function both are equivalent and one would give us the other this is the first point to remember. So, we have an expansion we have a series expansion where if I specify some coefficients a case and b case a i's and b i's a ends and b ends; I would have some function specified.

Now, the reverse question we can ask if I know the function can I get a i and b i. So, this is something that we will quickly learn if I specify a function can I get a i and b i, it turns out that you can get. So, just to understand quickly how do we get it you can imagine this little bit like a vector?

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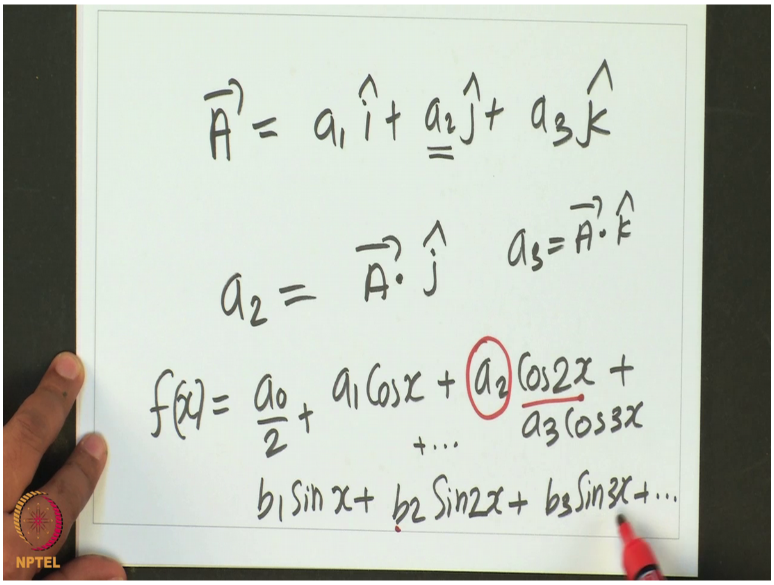

So, we had think of a vector a which is a 1 i plus a 2 j plus a 3 k if I want to know the value of a 2 given a I can write a 2 is a dot j this is something in vector one would learn and school also if I take a product of this dot product of this a vector and j, I would get this function a 2 it is value a 2.

Similarly, a 3 I can get a dot k. So, a 3 would be a dot k the; so, similarly I have now a function f of x where I have a 0, I would write a 0 by 2 here plus a 1 cos x plus a 2 cos 2 x plus. So, on and. So, forth a 3 cos 3 x plus dot, dot, dot, dot similarly b 0 is 0 b 0 is irrelevant b 1 sin 1 x plus b 2 sin 2 x plus b 3 sin 3 x plus dot, dot, dot.

Now, if I want to find out a 2 here this a 2 if I want to find out this a 2 what you have to do is multiply this f of x with cos 2 x and integrate if I want to find out a 2 multiply this f of x with cos 2 x and integrate that is if I want to find out b 2 I would multiply f of x with this sin 2 x and integrate. So, if I know f of x and if I want to find out any of this coefficient take the corresponding sin value or cos value multiply this and integrate think about it.

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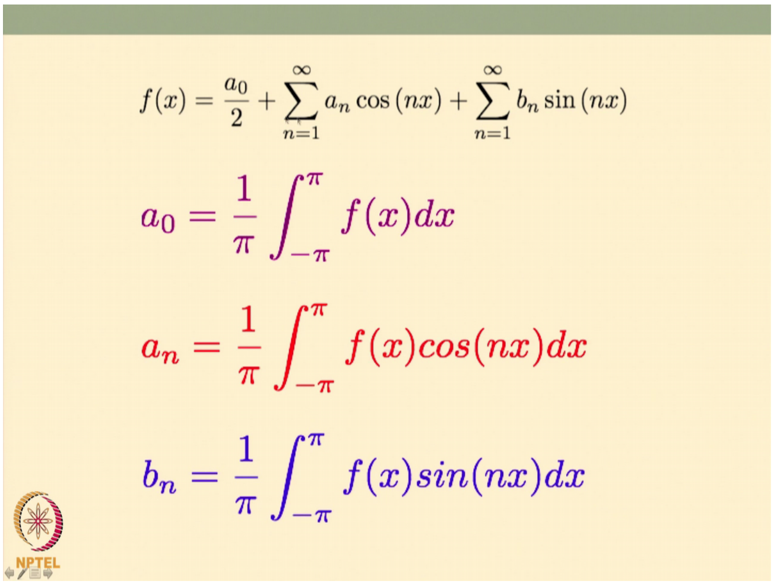

So, if I write it as a equation this is what it means to be precise definition f of x is written as this way a 0 by 2 plus sum over n an cos nx plus sum over n bn sin nx if I want a 0 I would multiply this x f of x with cos 0 x cos 0 is 1. So, this just integral of f of x will give me a 0; if I want a 1, I would take f of x multiply with cos 1 x and integrate if I want a 3, I would take f of x multiply with cos 3 x and integrate integral is from minus pi to pi divide this is a typical definition. Similarly if I want bn let us say b 3, I take f of x and multiply with sin 3 x and integrate if I want b 7, I will take f of x multiply with sin 7 x and integrate.

So, if I want any bn I multiply; take f of x and multiply sin nx and integrate for an I multiply with cos nx and integrate.

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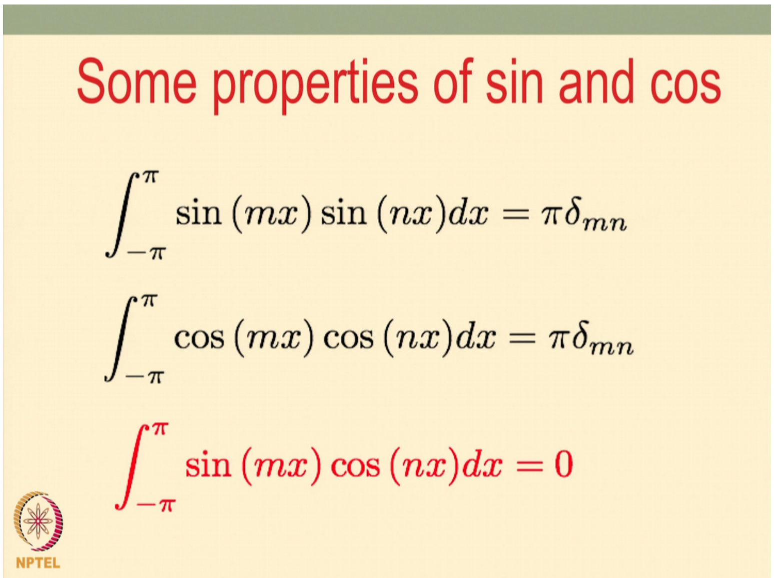

This works because of some particular property of sin and cos which we would not go into the detail now, but just to say that this sin mx sin nx if I multiply; they are 0 if m is not equal to n this what this delta this delta mn means delta mn is 0 if m is not equal to n this is one if m is equal to n. So, this have some value only if m is equal to n that is why this works similarly cos mn cos n if I find the product and multiply and integrate this integral is 0 if m is not equal to n.

Similarly, sin m and sin cos whatever be the mn is always 0. So, this is the property that helps us to un to do this trick and help us to solve and understand to obtain this. So, what did we say we said 2 things if I have a periodic function f of x, I can write that function as a series of sines and cosines if I know some coefficients which is a 0, a 1, a 2, a 3, etcetera and b 0, b 1, b 2, b 3, etcetera I can precisely write this f of x, if I do not know this a 0 a 1 a 2, but I know the f of x if I know the f of x; I can also back calculate this a zeros a 1 a 2 a 3 and a 4, etcetera.

Now, the question is if I specify a function can I get basically the if I specify a function can I get a 0 and a 1, a 2, etcetera. So, we would start with some very simple periodic functions. So, we would start a function which looks like which is like a straight line. So, I have a function which is just like a straight line this is my function and this function would repeat again, then this is repeat.

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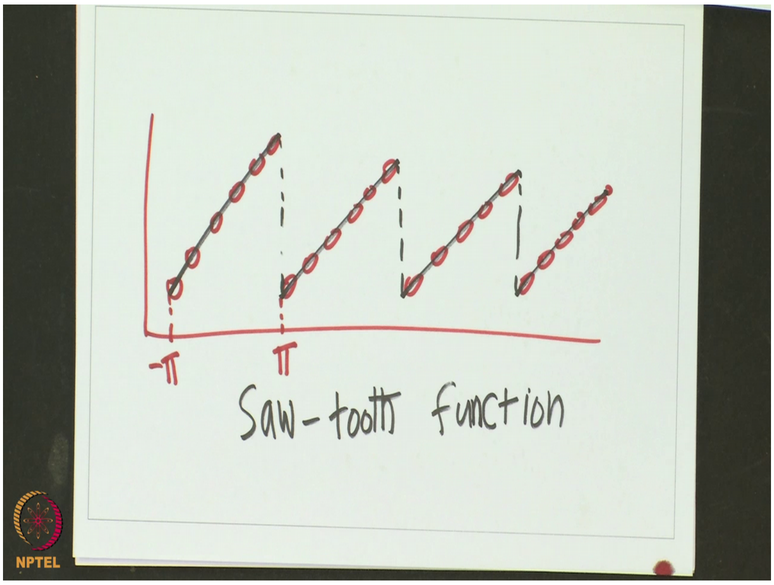

So, I just take a straight line and make it repeat many times so such function is called saw tooth; this is like a saw tooth this is a saw tooth function.

So, this is what I would describe. So, we would take a saw tooth function this function have this form.

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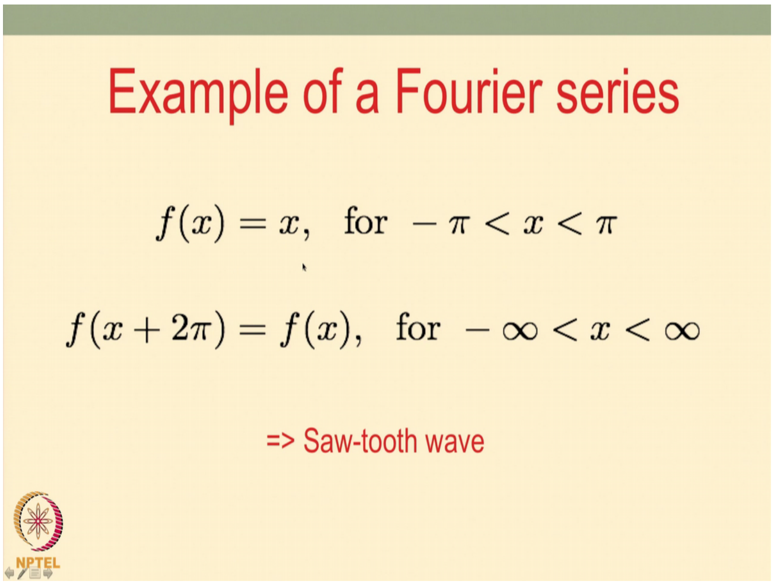

F of x equal to x f of x equal to x for value from minus pi to pi. So, I would take. So, if I just take this, this I would call as minus pi and this point; I would call as pi minus to pi to pi. This is like x and otherwise it has no value and again minus pi to pi repeat again. So, this is a repetition of this function a straight line this is a reputation of a straight line function. So, imagine that your protein is just like a rod which is like a straight line.

Now, we are repeat we have a many copies of this you form a crystal like this. So, you have a crystal now how do we write a mathematical equation for this right that is the question that you would ask. So, it is done the following way. So, our function f of x equal to x and f of x plus 2 pi is f of x itself it will repeat that is what it means.

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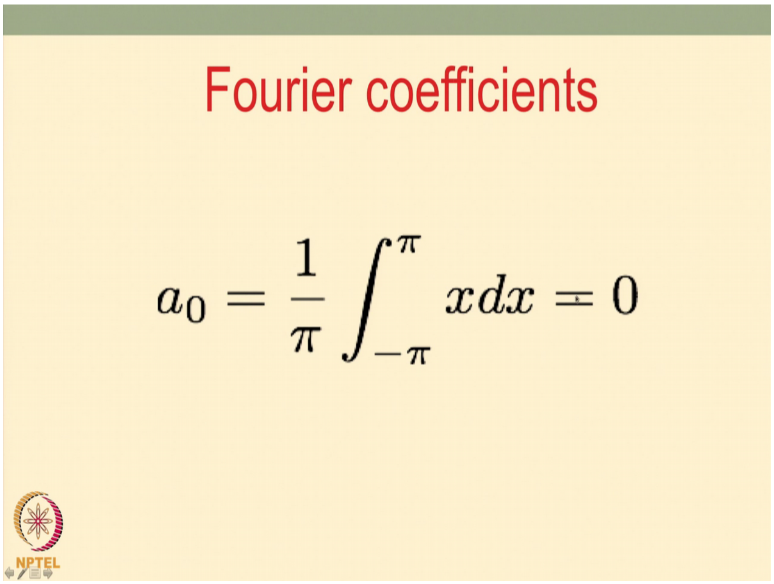

Now, I can calculate it; you take this x and integrate for a minus pi to pi, I will get a 0 which will turn out to be 0.

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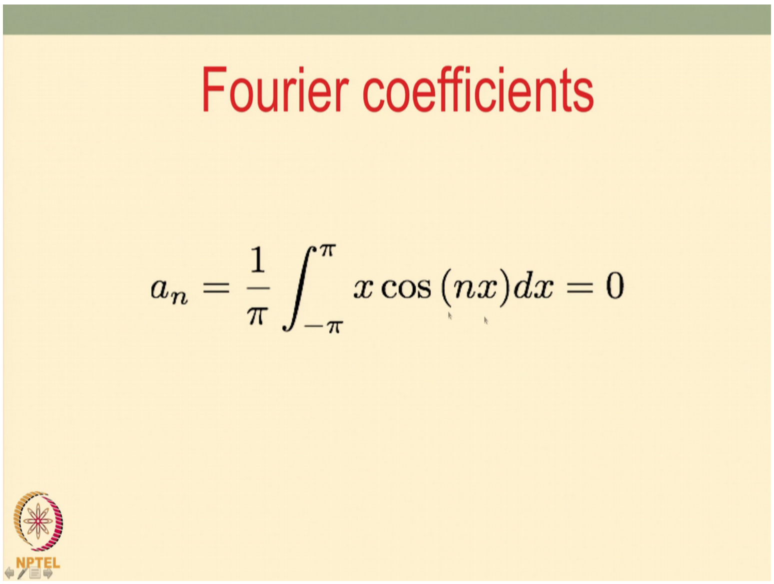

I can calculate an by doing this integration.

So, specifically look at integration I can integrate x cos nx dx and you will get 0s; all a n's are 0.

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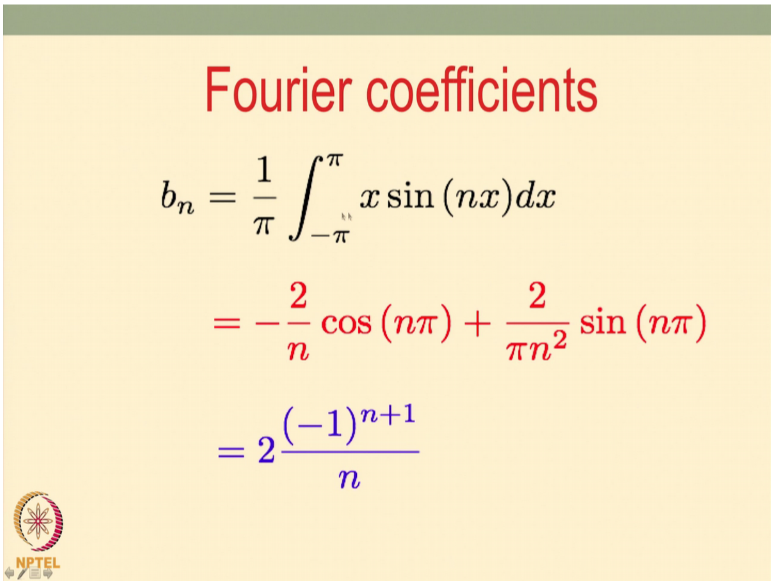

I can get bn by doing this integration and when I do this bn, I would get minus one power n plus 1 by n times 2, this is the answer of this integral you will get check this carefully if I; so, I only have bn values now.

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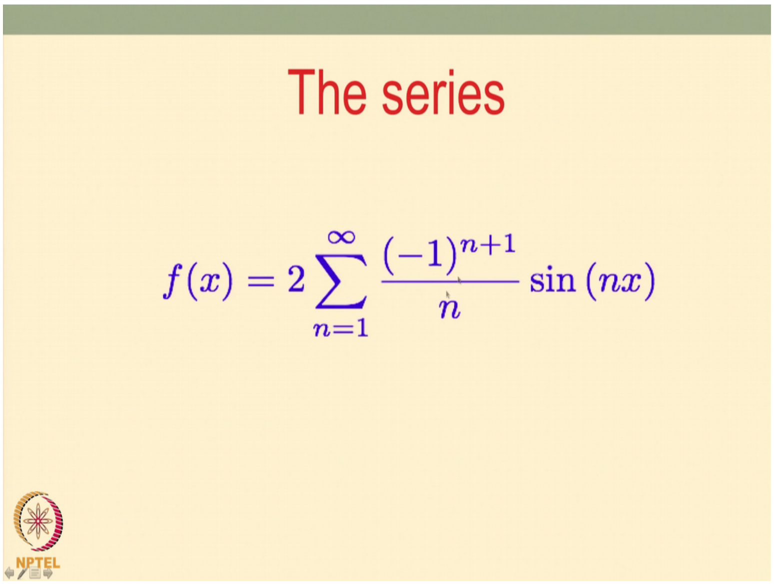

So, I can write the series only b n's. So, only sine series. So, series of sines. So, I would write the series and I have a infinite series now.

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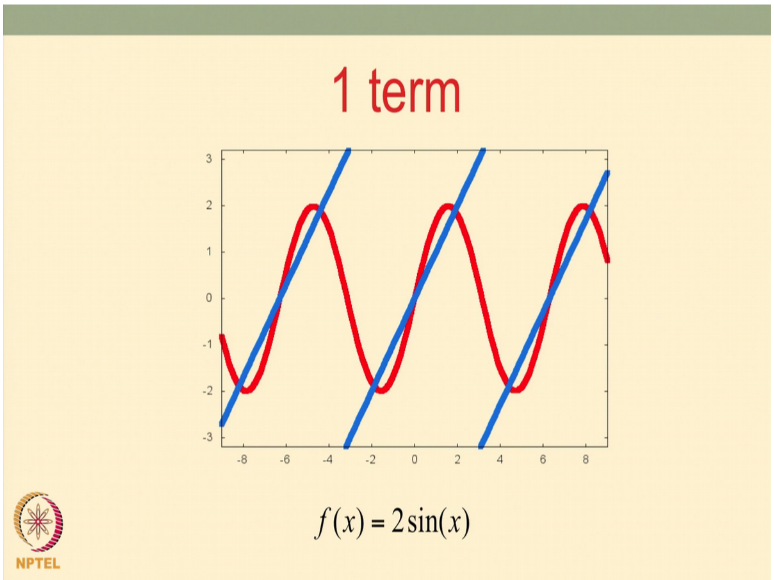

I take one term by one term and plot, if I take the first term which is 2 sin x, I plot the saw tooth function is in blue and the first term is red.

So, I would get a function which does not look like a blue, but somewhat like that.

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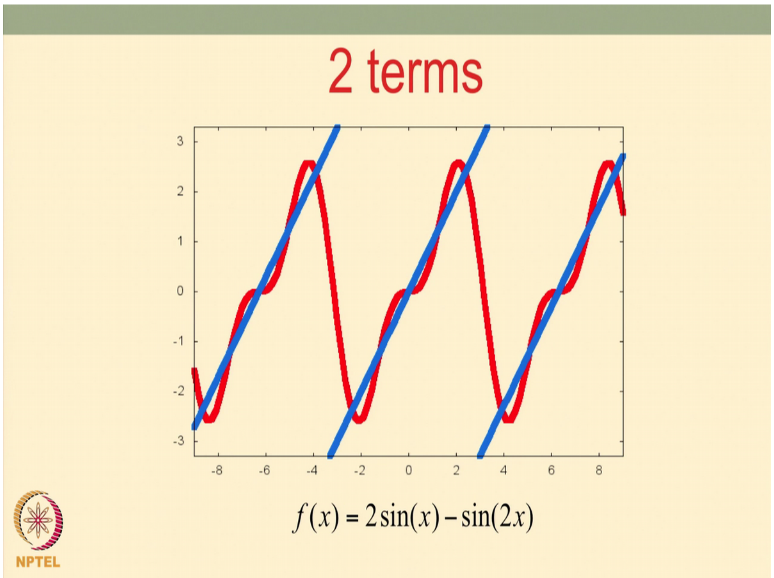

If I add 2 terms, it is 2 sin x minus sin 2 x, I would start getting somewhat slightly better.

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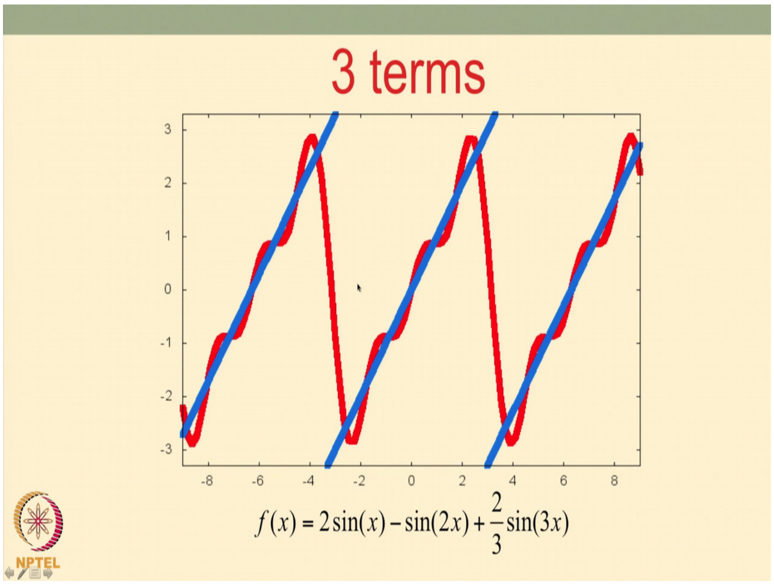

I add 3 terms; I would get slightly better, I would add 4 terms.

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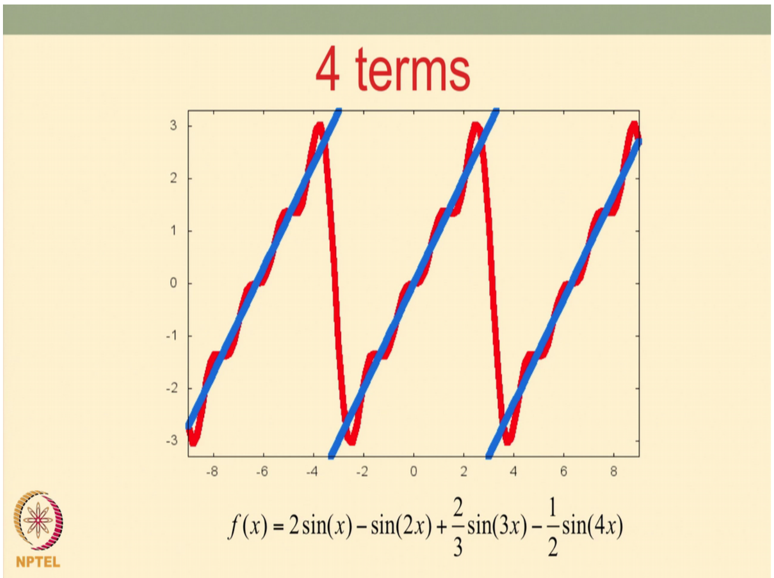

I would start getting closer and closer to the blue; I like a saw tooth it is coming.

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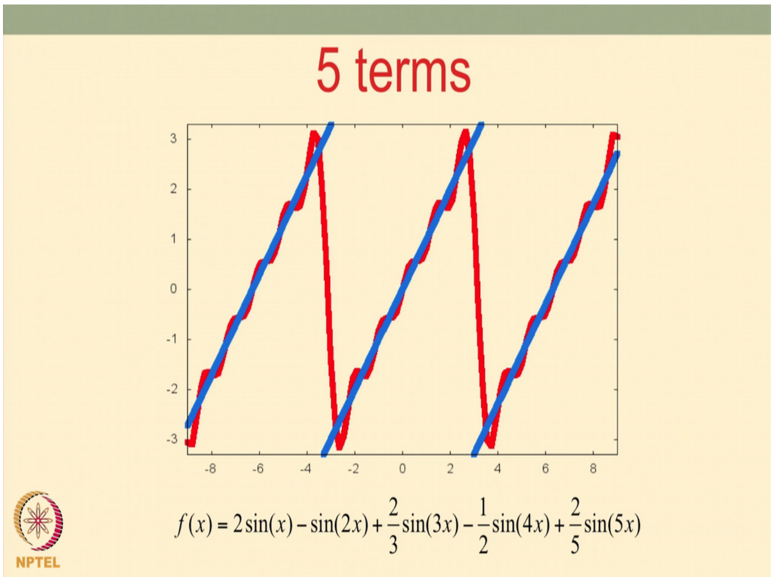

If I add 5 terms. So, these are the 5 terms described below these are the 5 first 5 terms of that infinite series. So, write down these infinite series and check if these are the 5 terms of their infinite series you would get this nice saw tooth like function.

So, if I add 5 terms, I would get this nice function which is coming closer and closer to saw tooth. If I add 6 terms, 7 terms, it will be nice flow saw tooth, if I have infinite terms it will be a perfect saw tooth function. So, if I take first 5 terms, I would get this. So, I can write an equation for the protein which is a repeating function like this, I could write 2 sin x minus sin 2 x plus 2 by 3 sin 3 x minus half sin 4 x plus 2 pi 2 by 5 sin 5; 5 x. This equation written here is an equation which is a function which can be approximated to a saw tooth like function.

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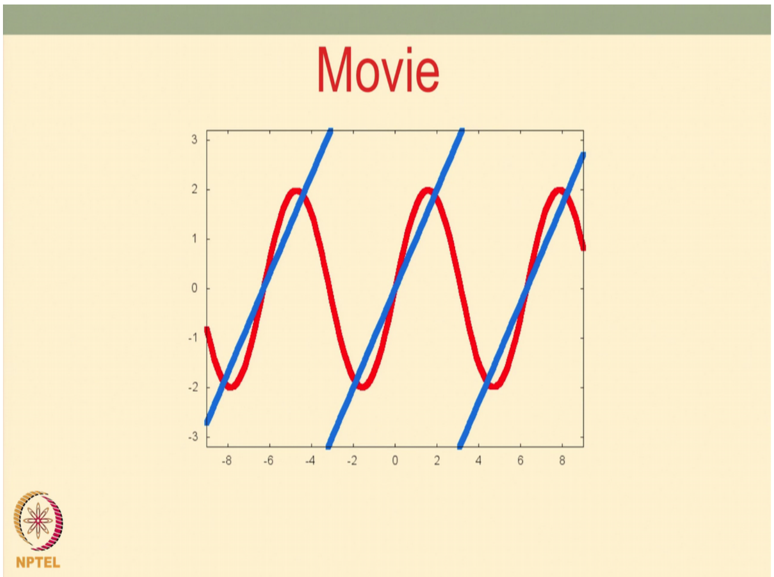

I can write more terms and get more precise function, I will move play a quick movie here which would be adding more and more terms, which would make it closer and closer to the saw tooth function.

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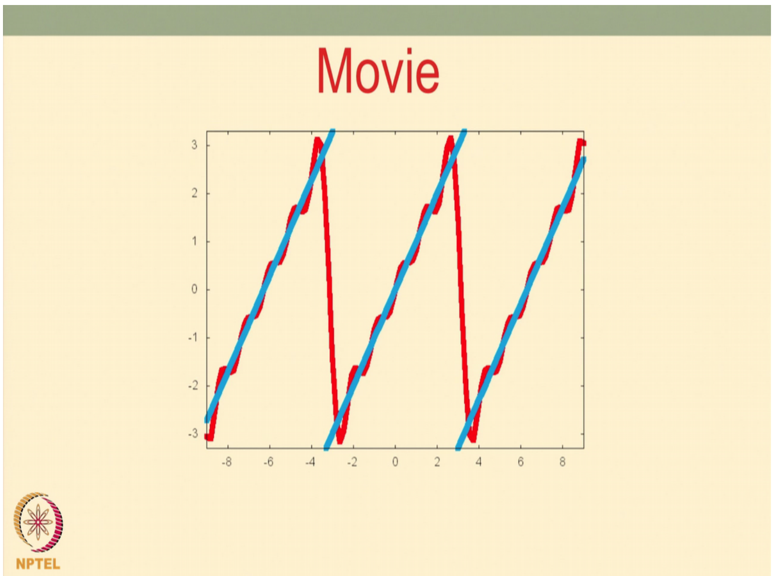

So, let me play this again first term second term 3 terms, 4 terms, 5 terms.

So, with this; I will summarize, we learnt a Fourier series; any function periodic function can be written as a combination of sines and cosine that will help us to write equations for any function think about this view this carefully read about this this will help you to think about this. And in the next class, we will say little bit more about this.

With this I will stop here. Bye.