Introductory Mathematical Methods for Biologists
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Lecture – 29
Diffusion Equation

Hi. Welcome to this lecture on mathematical methods for biologists. Today's topic is diffusion equation.

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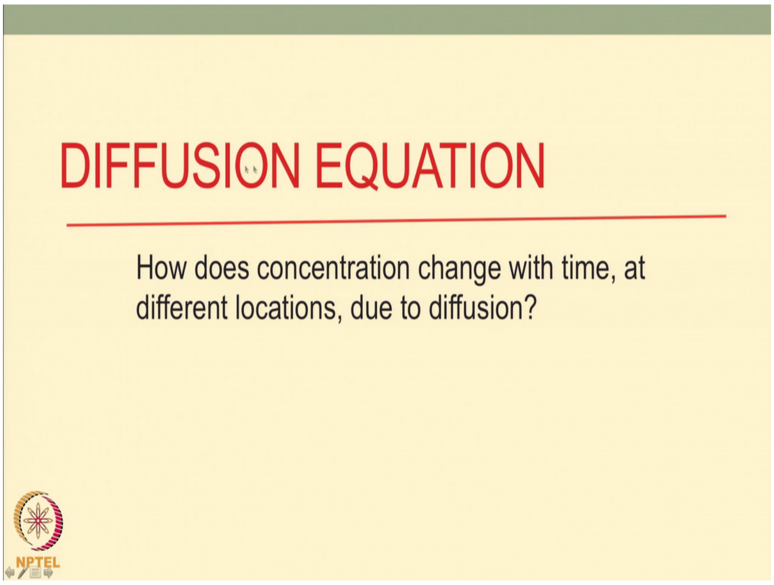

We will discuss about diffusion equation and we will answer the following question; will answer the question; how does concentration change with time at different locations due to diffusion? So, this is a question that we will answer how does concentration change with time at different locations due to diffusion; what does it; what do we mean by this question? Let us understand this a little bit and there will give us some idea about what are we talking about.

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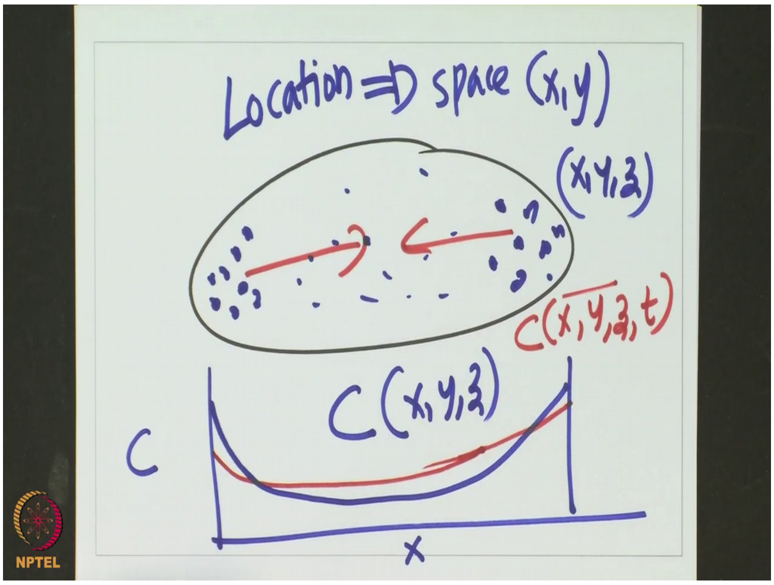

So, if we look at a cell the concentration, if we look at a cell the concentration at different places would be could be different. For example, the proteins would be very high concentration at one end and then very low concentration in the middle and there could be very high concentration here. So, the concentration will first if I plot concentration versus the x distance, it could decrease and then somehow increase.

So, the concentration could change in space according to this first. So, at different locations; so, by location I mean space location mean space. So, that is at some point at some region. So, given x, y or z in 3 D in 2 d given an xy position into 2 D like x, y, z position, we would get different positions and different-different concentrations. So, the concentration C is a function of x, y and z. It is also a function of time because things would diffuse from here to here there could be a diffusion from a higher concentration lower concentration, there will be diffusion from here to here. So, the concentration here would increase after some time.

So, the concentration increase as a function of time also something that we need to think about and we can get C of x, y, z comma t. So, the real concentration C would be a fun position a function of x, y, z and t in 3 D space x, y, z represent the 3 D space and t represent the time. So, as a function of the time, the concentration will change for example, if this is the concentration at t equal to 0. After some time, it will the concentration will come down here and the concentration would increase something like this could happen for example.

So, the concentration profile will change as the protein molecules or the molecules of our interest diffuse from one place to the other. So, given that there is diffusion how does this concentration C change with xyz and t that is the question that we would ask. So, what are we going to ask?

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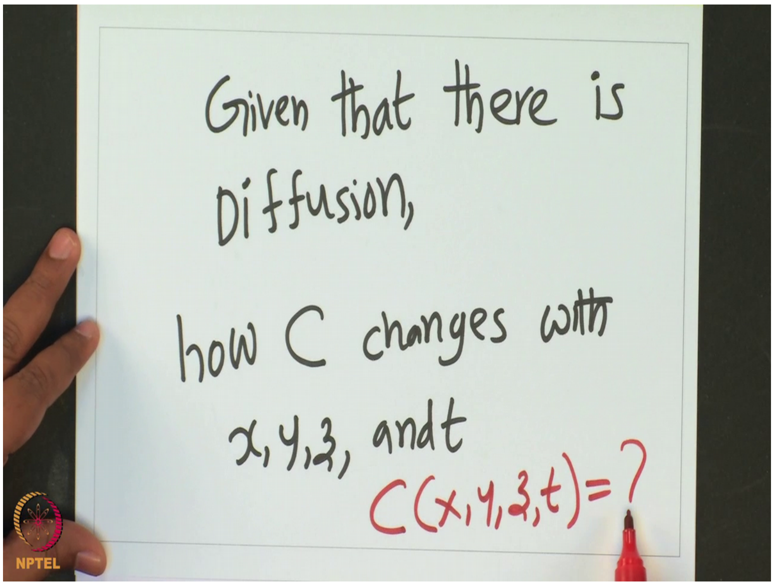

We are asking going to ask the following question given that there is diffusion given that there is diffusion given that there is diffusion how concentration changes with x, y, z and t how does this change as a function of x, y, z and t. In other words the question is what is this function can we calculate this function C x, y, z comma t given the diffusion how do we calculate, it can we get this function if we know some initial boundary conditions can we get this function diffusion what is the question that will be obeyed by this C is a question that we would want to address.

Now, what do we know we know that the flow the current is something that we have already computed? So, what we know is that if there is a concentration gradient if there is high concentration at.

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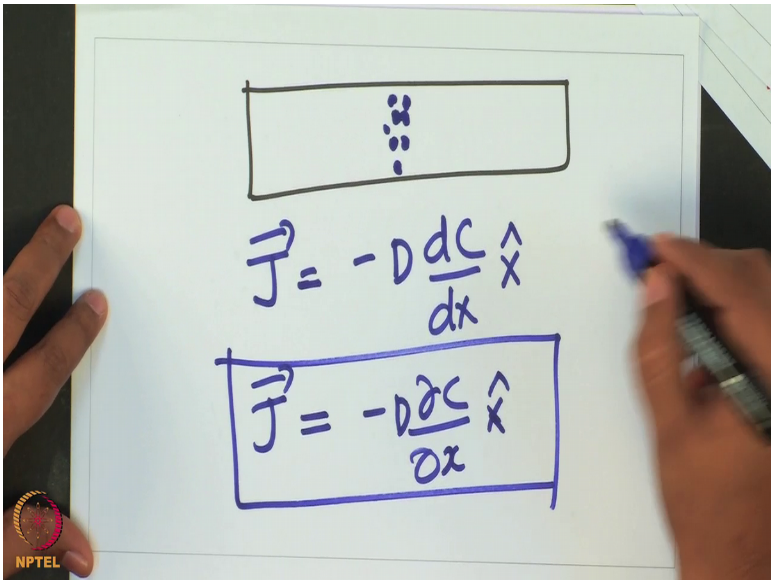

So, if there is high concentration at one place. So, there is a very high concentration here. So, there is high concentration and very low concentration at this locations we know that there will be a current change and this current is given by minus D in 1 dimension dc by dx x cap.

So, this is the thing that we know since C is a function of x, y, z this is J is actually the diffusion current is actually minus D del C by del x x cap. So, this is something that we already know now given this J knowledge that we have; what do we know about the change in concentration. So, let us think about this just to remember the J is a function of x. So, if we have different concentration the J would be a function of x.

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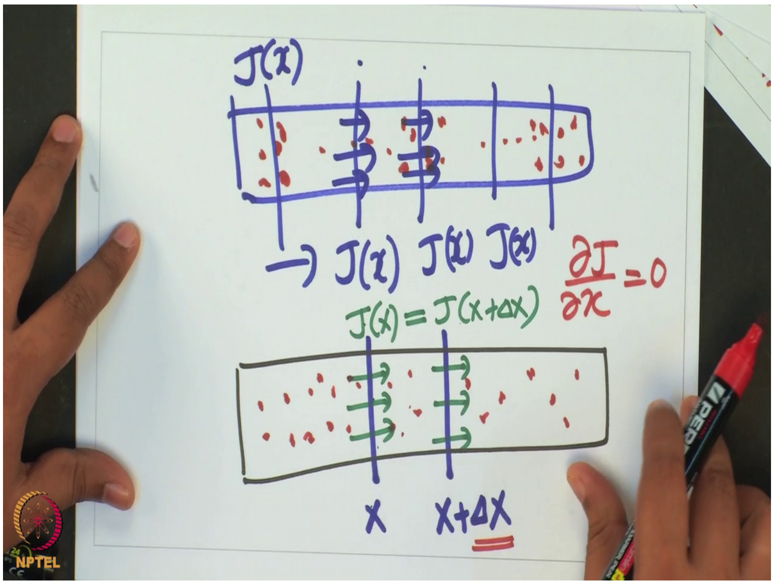

For example, if we have a tube like this and if you have lot of molecule here and very few molecule here and lot of molecule here and very few molecule here and lot of molecule here the current here the current here would be very large because there is high lot of molecules. So, they will flow while current here would be very small current here would be large current here could be small and current here could be large. So, J at this x point would be different from J at this x point would be different from J at here J at here.

So, J at different function different positions of x would be different. So, J is a function of x. So, this is something that we know now let us think of J; what is J? J is the flow. So, if the flow at this point and this point let us consider these 2 points. So, let us consider a small. Let us just consider a tube like this and we are marking 2 points x and x plus delta x these are the 2 points that we are considering and there are more protein molecules everywhere which we are not drawing that is there is lot of protein molecules. Let us mark protein molecules everywhere with some concentration, I do not know; what is the concentration there is some concentration and there is some protein gradient. I am not saying anything about concentration, I am only saying that if the flow here.

Let us assume the flow at x if the flow at x is along this let us draw the flow at x let us say there are 300, I drew 3 arrows; 300 protein molecules cross every second this line and here also 300 molecules cross every second from here to here.

So, these 2 lines 300 molecules, let us say are crossing this line in every second that is the flow here the flow here is also 300 molecules crossing every second; that means, the flow here and the flow here are the same. So, the J at x is same as J at x plus delta x. So, if we have J at x is same as J at x plus delta x. So, the 300 molecules are crossing this line in every second.

So, that is the flow here 300 molecules are crossing this line every second these are the flow here if they are equal if the flow here is equal to the flow here in this in between region 300 molecules are coming in 300 molecules are going out therefore, the net number of proteins net concentration of the protein will remain unchanged because whatever came in went out. So, the flow at 2 nearby locations are equal the concentration in between will not change this fact if we write mathematically that becomes the first part that becomes an interesting equation which we will see how to write this.

So, the physical fact we are writing is let us me repeat we have a cell like pipe or a tube in which there is a concentration gradient of this red particle proteins; protein particles and I am clay taking 2 points x and x plus delta x and I am saying that at x if I look at it I will find I am finding that the flow is along the plus x direction and the flow is some number which is lets us middle a 3 arrows which would let us let us say this means 300 protein particles are crossing this line every second that is the flow here that is J of x J at x plus delta x let us assume it is exactly the same as these in the same direction that is here also 300 particles are crossing in the plus x direction if this current is equal to this current in both direction and magnitude the concentration of the red proteins in between will remain unchanged.

So, the concentration here will be 0 so; that means, will not change with time; that means, 300 particles came in 300 particle went out . So, if this is equal; same; this implies that the dj by dx is 0. So, this would this would imply the del J by del x, the change in flow J of x plus delta x equal to J of x. Therefore, the derivative will be 0, if these 2 are equal there is no change in flow as we go along the space therefore, del J by del x is 0 let us the assumption here is of course, is that these 2 are very close to each other delta x is very small then I can write del J by del x is 0.

So, if del J by del x is 0; that means, the flow here is same as the flow here then the claim is that the concentration will not change with time the concentration here will remain constant over time . So, let us write that. So, let me write this once more.

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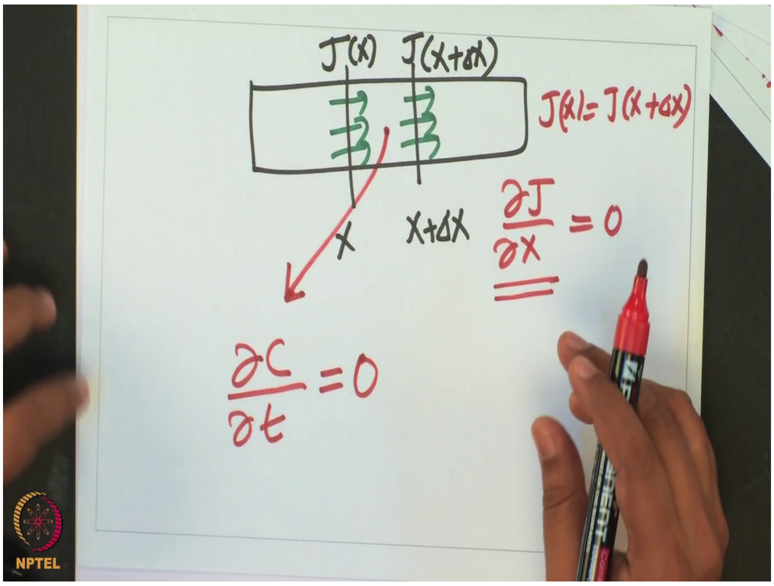

Let me draw again we have this we have these 2 points x and x plus delta x and we are worried about the flow here J at x and flow at x plus delta x and we are saying that the flow here is exactly equal to the flow here equal in magnitude equal in direction. Therefore, J at x is equal to J at x plus delta x this means the derivative is 0 this means del J by del x is 0 and because of that if same number of particles come in and same number of particles go out here the concentrations change with time how does the concentration change with time this will be 0 because this will not change either.

So, del C by del t will be 0 del J by del x will be 0. So, this is the first point that we want to make the second point is let us imagine a case where this is not equal to this. So, that is the second point that we will consider.

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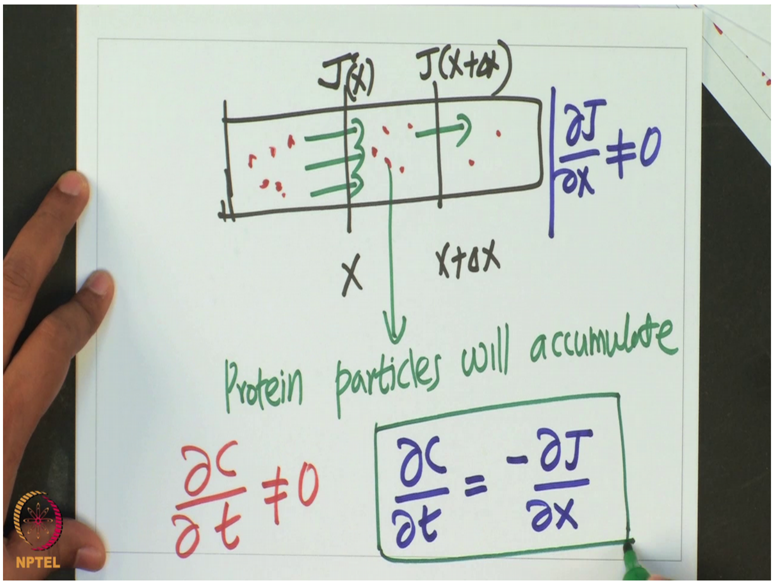

So, let us consider again a tube; tube like cell let us consider 2 point x and x plus delta x and the flow here x and this is the flow at x plus delta x.

Let us say the flow at x is 300 particles in every second and only hundred particles go out there is a flow out is only one line; that means, this is less compare to this. So, del J by del x is not 0. This whatever comes in lot of particles are coming in 300 particles are coming in, but only hundred particles are going out what does that mean; that means, here the proteins particles will accumulate this means that protein particles or protein molecules will accumulate at this point accumulate there will be an accumulation of protein particles because there is lot of protein particles are flowing in and only few are flowing out.

So, the concentration change here del C by del t will not be 0 it will be a non 0 will not be equal to 0. It will be a quantity and this will not be 0 this is we know because the part the concentration will change with time as we wait more and more particles will diffuse in and they will start accumulating here and the concentration will change with time del C by del t will not be 0 what will del C by del t be and the claim is that the del C by del t will be equal to del J by del x.

So, let us look at del J by del x. So, there is a del J by del x is also not 0. So, if I look at del J by del x this is also not 0 because there is a higher here and lower here. So, the J changes with x J is decreasing with x. So, del J by del x is negative and the claim is that del C by del t magnitude is minus del J by del x this is the claim this equation is called the diffusion equation. This equation is also called a continuity equation because the only thing we said here is whatever comes in goes out or not there is the only thing we checked.

We checked essentially the continuity and this continuity will give us this phenomenon of diffusion will describe the phenomenon of diffusion and this will give us diffusion equation. So, this is the magnitude we will worry about the direction later. This is a scalar this is a. So, if I put J is the magnitude of the vector and I only want to put the magnitude here we will worry about the direction later. So, this is the magnitude and now let us think about this little bit more carefully what does this mean what does this del J by del x.

So, let here we can see that J at x plus dx minus Jx. So, the flow here is less compare to the flow here. So, this minus this will be a negative number. So, this will be a negative thing and this would means that the del C by del t is a positive thing was negative and del J by del x is negative. So, this will give us a positive number. So, let us look at it little bit carefully let us look at the both the cases.

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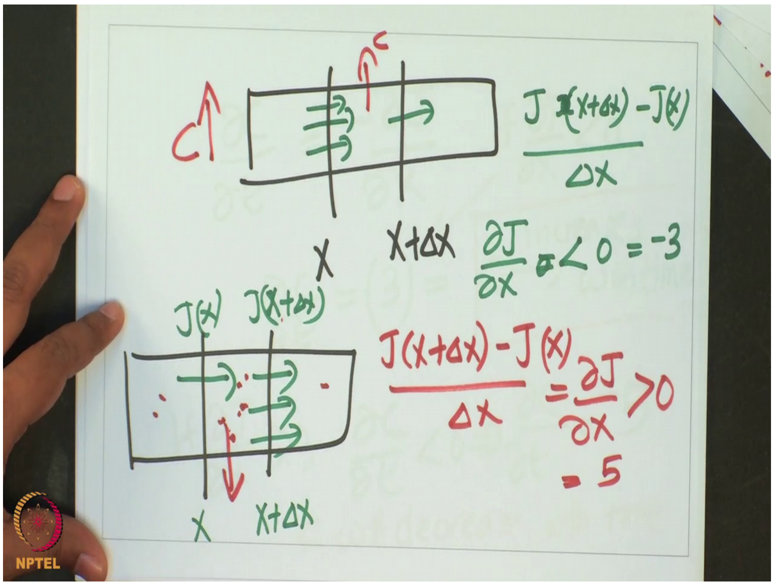

So, let us look at the first case which we describe just now we have 2 point x and x plus 2 points and flow in is higher than flow out if that is the case we have J at x plus delta x minus J at x divided by delta x this is our del J by del x and this is going to be negative number this is going to be less than 0.

So, this is let say minus some number minus 3 this is going to be some negative number if you look at the other case which is let us look at this case where you have this 2 location x and x plus delta x and this is J at x and J at x plus delta x and the flow in is less flow out is more if this happens the concentration of protein here will deplete the con the proteins molecules here will go away and this concentration here will decrease because all of these are going out a lot of molecules are going away and only few are coming in.

So, if this is greater than this J at x plus delta x is greater than J x. So, this will be a positive number by delta x which is del J by del x; this will be greater than 0 because the flow here is more than the flow here. Therefore, this will be greater than 0, let us say this is going to be 5 or some bigger number now if we write our diffusion equation which is saying that dc by the change in concentration here is equal to this difference.

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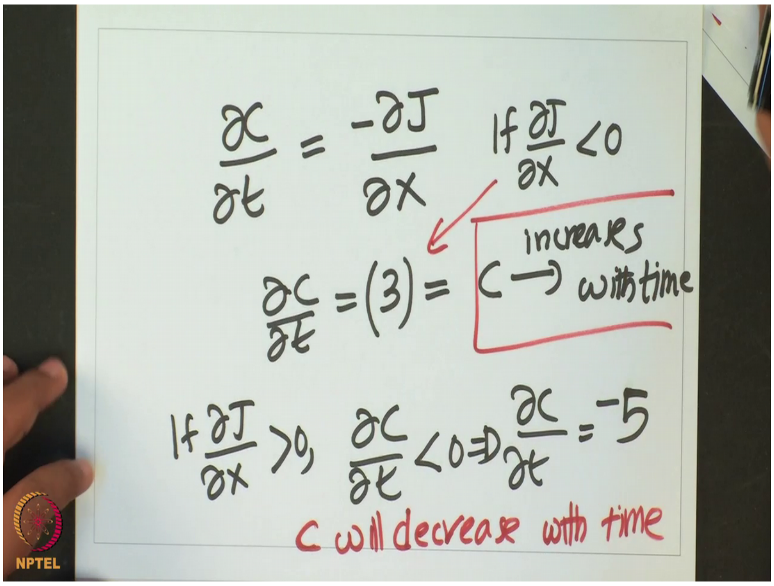

So, the claim we are making is that del C by del t is minus del J by del x this is the claim that we are making if this is the case in the first case if del J by del x is less than 0, then that would imply that del C by del t will be some positive number 3 or some if this is less than 0s this will be negative times negative is 3 positive will be some positive number like 3. This would means that C increases with time C will increase with time at this location.

If del C by del t is negative if del J by del x is positive del C by del t will be this is positive. This is negative this will be less than 0 that is del C by del t will be some negative number negative 5, this would mean that C will decrease with time C at that middle location will decrease with time. So, if del J by del x is greater than 0 C will decrease with time if del C by del t is del J by del x is less than 0 C will increase with time. So, this is this something that we can check here. Here in this case del J by del x is negative so; that means, here C will increase with time because the concentration here will increase more or coming in. So, the concentration here will increase.

Here the concentration will decrease because del J by del x is positive and less coming in more going out. So, I want you to check this and test yourself and convince yourself that this equation will indeed give us what we are wanting. So, what we want is the phenomena of diffusion which is basically description as a flow J and this by demanding if you take a point and if you if we discuss the flow which is coming in and the flow that is going out, if we balance those flow del J by del x how much is that flow itself changing.

So, that is at any point in the cell; what we are worried is any point in the take any point in the cell at any point.

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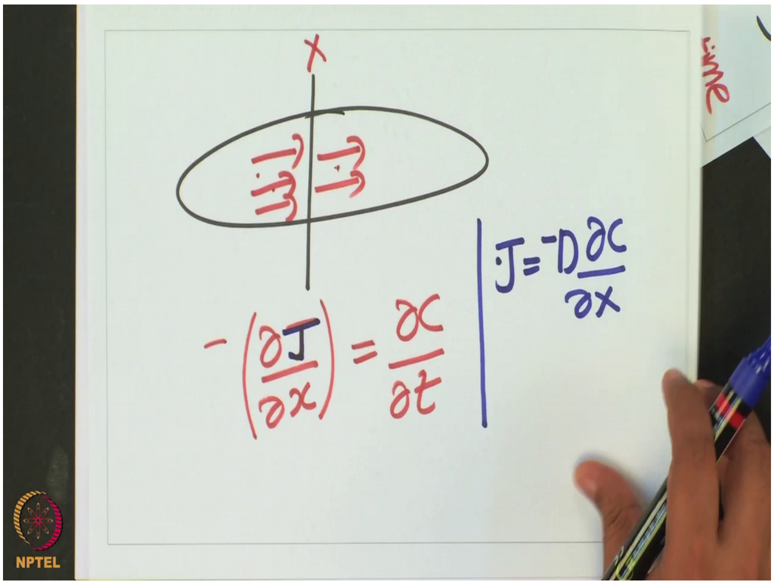

You look at the flow that is coming in and the flow that is going out. So, J here and J here this del J by del x depending on this value of del J by del x if the del J by del x is 0 this means whatever comes in goes out if del J by del x is positive; that means, this flow is more than this flow if del J by del x is negative this flow is more than this flow. So, the claim is that minus del J by del x is del C by del t. So, this is the understanding this understanding is just simple simply coming from take any location x and around that you will look at the flow which is coming in and going out and by studying this considering 2 points very close to delta x away we can serve we can discuss and write down such an equation.

So, this is possible now if we also already know also that J is d minus D del C by del x if you take just take the magnitude this is the magnitude and x cap is the direction. So, the magnitude is minus D del C by del x. So, if I substitute this J here. So, let us substitute this J in this equation. So, I will get. So, I am going to substitute this J here. So, do substitute this and if I substitute what do I get.

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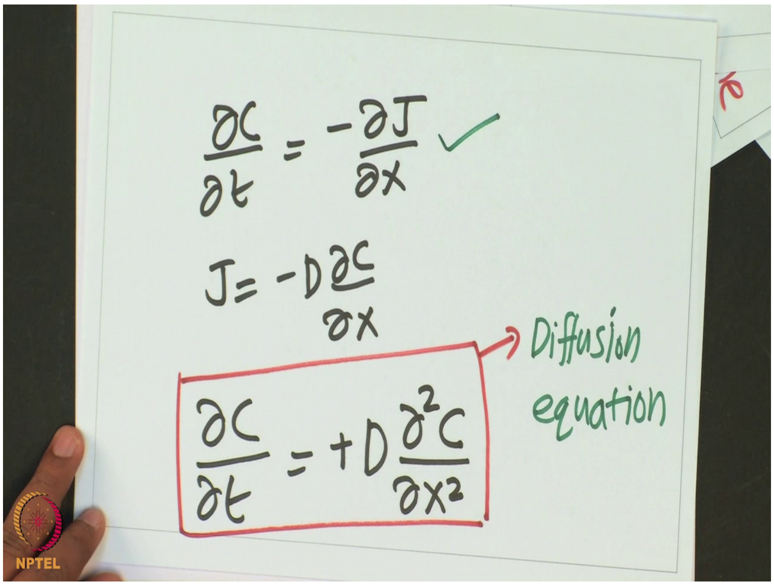

I will get that del C by del C by del t is equal to minus del J by del x J is equal to minus D del C by del x I substitute this J here I will get del C by del t is equal to minus D and there is a minus. So, it will become plus plus del square C by del x square.

So, this is the equation we will get and this is the; another form of diffusion equation del C by del t is equal to D del square C by del x square. So, this is the form of the diffusion equation. So, this equation is called the diffusion equation. this is called the diffusion equation which says del C by del t is equal to d del square C by del x square the equation is same as this equation. Exactly the same I just substituted this J here and I got this.

I wanted to convey just one more thing about the we discuss the magnitude we I also wanted to talk about direction.

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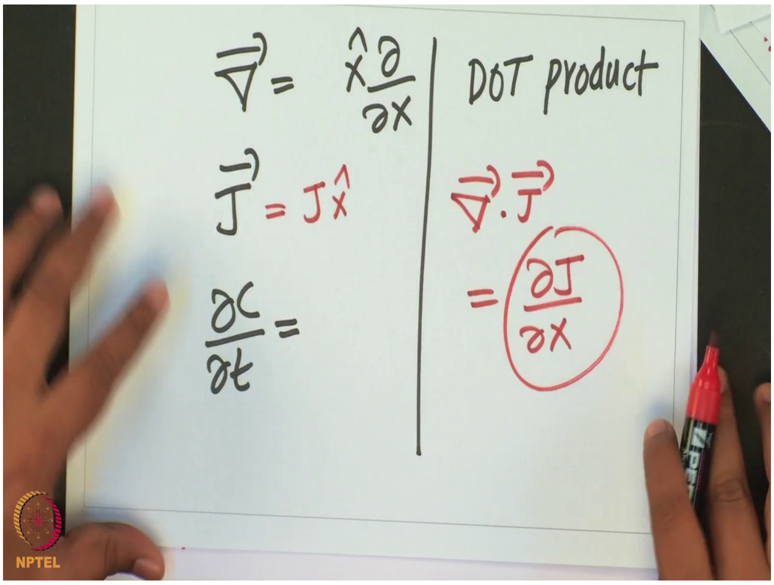

So, in vector analysis, if you have 2 vectors del is a vector this del vector is x cap del by del x is in one dimension del vector is del by del x x cap and another vector we have is J now if you want to get is and we know that dc by dt is a scalar this is a scalar and therefore, this is C as not a vector t is not a vector this whole thing is a scalar, therefore, we want to get a scalar here also.

So, making a scalar is called a dot product. So, if I find a product between this and there is something called dot product in mathematics which is basically del dot J del dot J is dot product this means that this dot this. So, this will have a component let us call it J magnitude x cap. So, J magnitude let us say this J is J magnitude x cap. So, the product is this and xk x dot x dot x dot x will become one. So, this will this is nothing, but del J by del x in one dimension this will be just del J by del x. So, this is the del J by del x that we are writing here in other words the diffusion equation the diffusion equation or the continuity equation is actually del C by del t C equal to.

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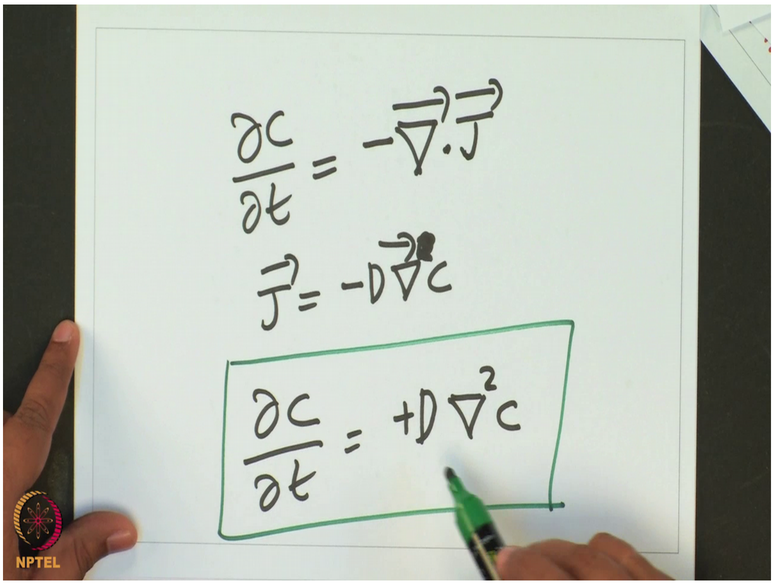

In the vector form, it will be del C by del t is minus del dot J, this is the dot product I would want you to understand this by studying the dot product of vectors which we have studied little bit in school.

So, this J is a vector del is a vector the dot product of this will give this now here you substitute the vector J is equal to minus D del C where del is a vector if I substitute here I will get what we just discuss del C by del t is equal to plus D del square C del square is a second derivative. So, this is the diffusion equation that we are saying in 3 dimension the del square will be des xyz will be there and in one dimension just x will be there.

So, let me write this in 3 dimension and summarize the what we did so far.

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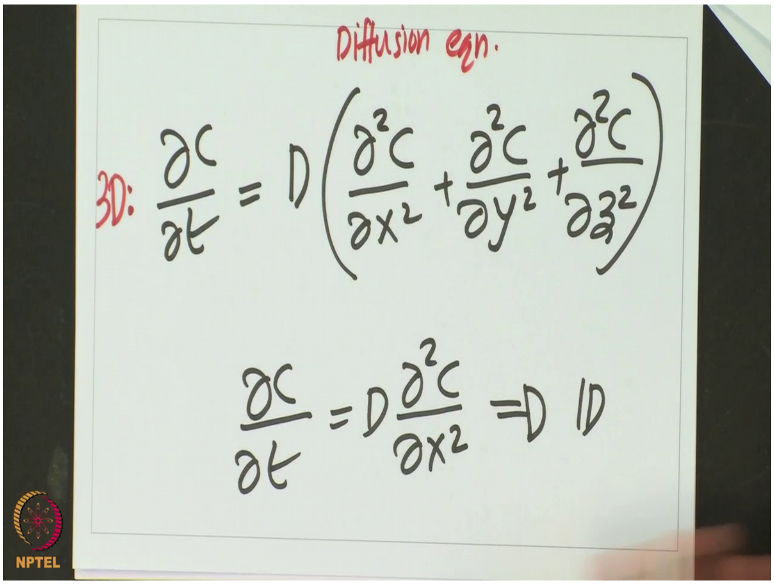

So, if I write this in 3 dimension del C by del t will be equal to d del square C by del x square plus del square C by del y square plus del square C by del z square this is the equation in 3 dimension in one dimension is just del C by del t is equal to d del square C by del x square in 2 dimension. So, this is in one dimension and this is in 3 dimension and 2 dimension, there will be only 2 terms.

So, the bottom line is that if we have a flow if we think about flow and if we take any point and look at the flow that is coming in and the flow that is going out how the protein molecule flow at any point x account for the whatever coming in and account for whatever going out and the difference between them will give us the concentration change with time and that will give us this diffusion equation and that will give us this equation which is call the diffusion equation. So, this is the diffusion equation diffusion equation in 3 D or 1D.

So, with this I will stop this lecture and continue in the next class. Bye.