Introductory Mathematical Methods for Biologists
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Lecture - 21
Exponential Growth and Decay

Hi. Welcome to this lecture on Mathematical Methods for Biologists. In this lecture we will discuss some applications of calculus; like integration and differentiation that we learned, how do we apply that to some real biological problems.

And, one of the most important biological problems which is like across different areas in biology itself would have is growth; like there are many things that grow. And, very often things grow exponentially. So, the title of today's lecture would be Exponential Growth and Decay.

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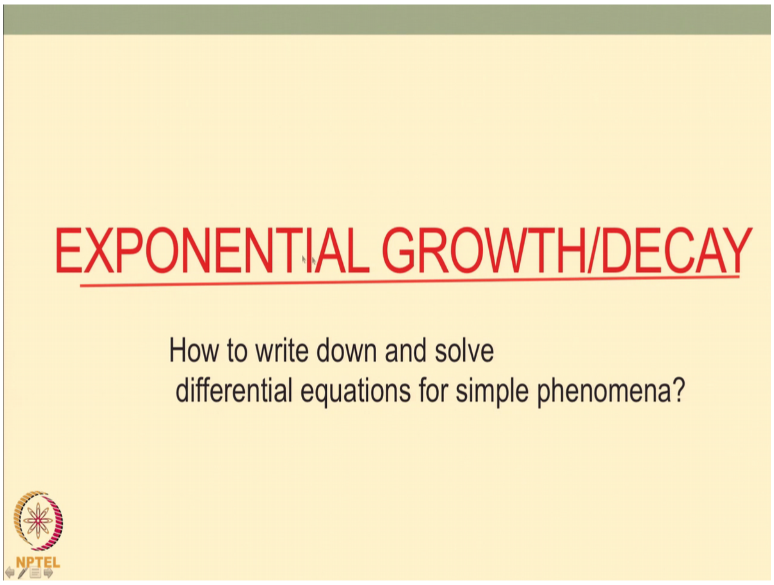

So, this is the title: exponential growth and decay, how to write down and solve differential equations for simple phenomena. So, we will take some simple phenomena of exponential growth and decay, and we will learn how to write down differential equations for this; that is equations involving derivatives and how do we solve them integrate them to get some sensible answers. So, that is the thing that we would learn in this lecture. We know examples of growth and we always we know a lot about the growth curve.

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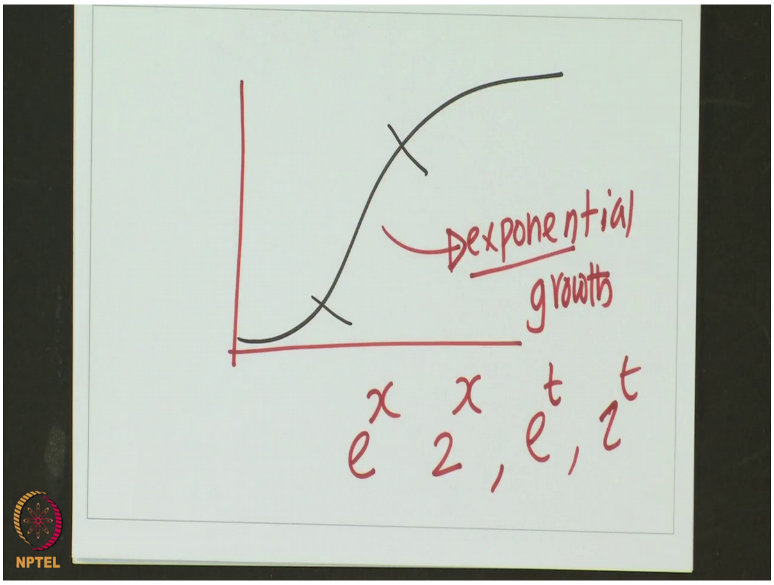

When we see growth curve we know that; we know something like this. In this there is some part which is the exponential growth, right. So, this people call this exponential growth. Why do we call it exponential growth? Think of we will think about it, but the word exponential would mean there is this function something exponentiated either e power x or 2 power x; something exponentiate or e power t 2 power t this function of time. So, you would have some function which is in the exponent, right

So, this is why this is exponential growth. You will try and understand this part; there are this has many parts there is this saturation, there is this initial part we would not worry about it; we would worry about this fast growing part this exponential part and we will think about it today. So, the classic example that we know is the bacterial growth. What is bacterial growth?

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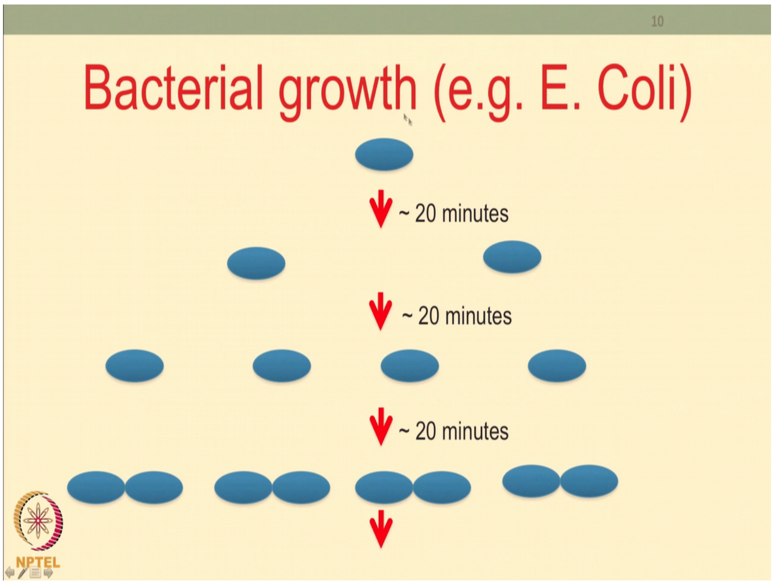

Bacterial growth we have e coli for example, we have 1 cell in about 20 minutes it would divide and become 2 cells, and in another 20 minutes each of them would divide and become 4 cells. So, this would become 2 and this would become another 2 total 4 cells. Again, each of them would divide into 2, this would become 2 this would become 2, this would become 2, this would become 2; total 8 cells, so each of them dividing doubling, which we would typically call as bacterial growth.

So, this is a simple phenomena that we know. And now the question is how do we describe these phenomena mathematically? What is the method to describe this mathematically? So, let us see what we are observing, let us understand this is what we are observing and let us think about it a little bit. So let us rethink, let us see what we are observing.

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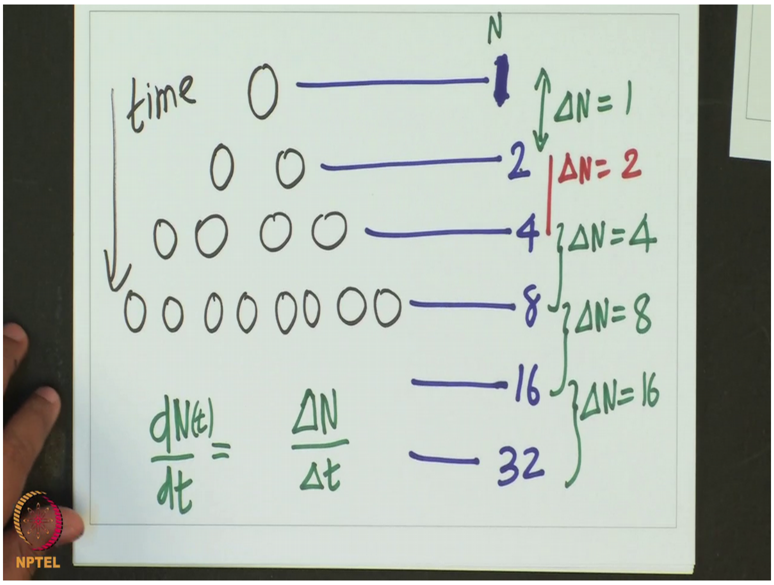

So what we are observing is; we have 1 cell, this was becoming 2 and then this 2 was becoming 4 and then 4 was becoming 8. So, this is what we have and so on and so forth, 8 would become 16 and so on and so forth. So, what did we have? We had here, 2 cells and we had here sorry we had here 1 cell and 2 here, 4 here, 8 here, and the next level it will be 16, and the next level it will be 32. We know that if I divide this is the bacterial growth phenomena that we know.

And if you want to write an equation for this, there is something changing here; what is changing the number of bacteria this is doubling here, number of bacteria. So, the number of bacteria changing with time, so this is time; so this is time. So, what we are observing here is number of bacteria changing with time. How do we write mathematically that the number of bacteria is changing with time? It is obviously, it is dN by dt. DN by dt is the number; N is the number of bacteria, t is time; dN by dt would mean how does this number N which is a function of t N is a function of t how would this N change with time. So, I urge all of you to plot think about plotting this N as a function of t and we are interested in how would this N change what is this. This is the question that we have.

Now, let us think about it, let us look at it what we have. So, dN is the change in N as the time changes by delta t. So this would also; this is equivalent of delta N by delta t; this is equivalent of delta N by delta t for small t. So, we know that from calculus dN by dt is delta N by delta t, and when this is delta t is very small we can write it as dN by dt. Now what is delta N by delta t? What is delta? So, delta t is some time by which you would divide, and delta N let us think about this. So, what is delta N here? So, this is delta N. How much was the delta N here? Here the delta N is 1. What is delta N here? Here the delta N is 2; it became 2 to 4. What is delta N here? Here the delta N 4 became 8; so delta N is 4. What is delta N here? Here the delta N 8 became 16, 16 minus 8 delta N is 8. What is delta N here? Delta N is 16.

So what are we seeing? We are seeing delta N and this is N. So, this is N, N becomes 1, 2, 4, 8, 16, 32, and delta N is 1 because 1 divides to 2. When there was 2 both will divide, so the delta N is 2 here and delta N is 4 here. And the fundamental idea is that if you have more number of bacteria if N is very large all of them can divide. If there is 100 bacteria all 100 would divide, if there is only 1 bacteria only 1 can divide.

So therefore, the delta N would be proportional to the N itself. This is obvious that delta N would be proportional to N itself.

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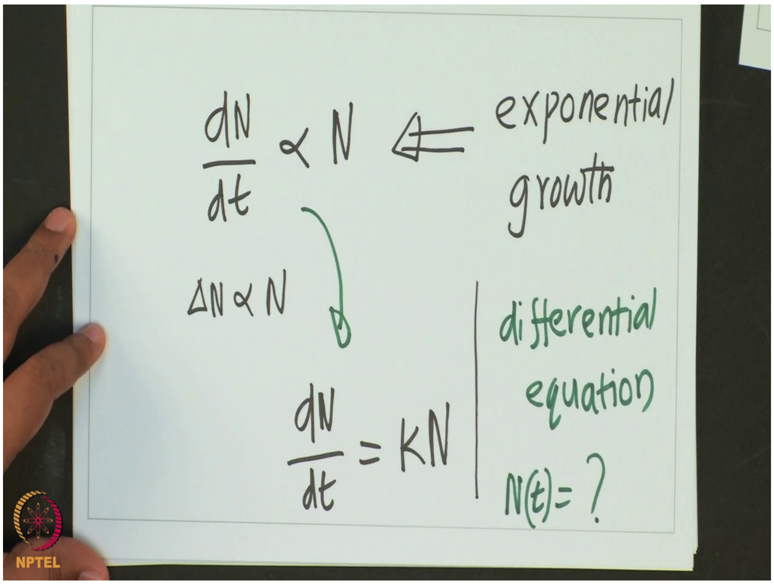

So, this fact if we understand this that the dN by dt or delta N by delta t would be proportional to the number itself. So always in exponential growth, whenever we would call exponential growth this is some phenomena always will happen. So, this is a fundamental of exponential growth; whenever an exponential growth happens the change the delta N would be proportional to N itself. The dN by dt would be proportional to N itself, which is obvious because the more number you have they all can divide at the same time, right. If there is 100 (Refer Time: 09:29) if there is 1000 bacteria all 1000bacteria can divide almost simultaneously that is roughly within if it an (Refer Time: 09:37) roughly within 20 minutes all 1000would divide. If there is only 10 within 20 minutes only 10 would divide.

So the more number we have, the more dN the change would be the delta N the change would be proportional to the number itself; the more the number more would divide therefore the increase would be more also. So, once we have this, this would lead to an I can convert this to an equation. So, this thing I could write it as an equation which is dN by dt and proportionality I could write as equal to with a proportionality constant times N. So, dN by dt is equal to kN. These an equation directly coming from the simple observation that the more number you have the more would be the change. And since delta N is proportional to N d within a given time of delta t dN by dt or delta it is proportional to N, therefore dN by dt would be k times N.

Once we have this, so this are the two points we should note.

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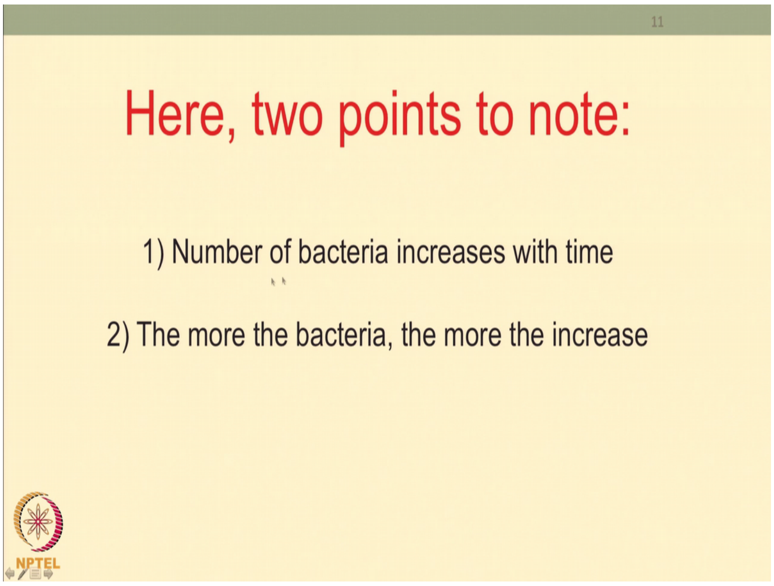

So, number of bacteria increases with time and the more the bacteria the more would be the increase. So, these are the two points we want to understand.

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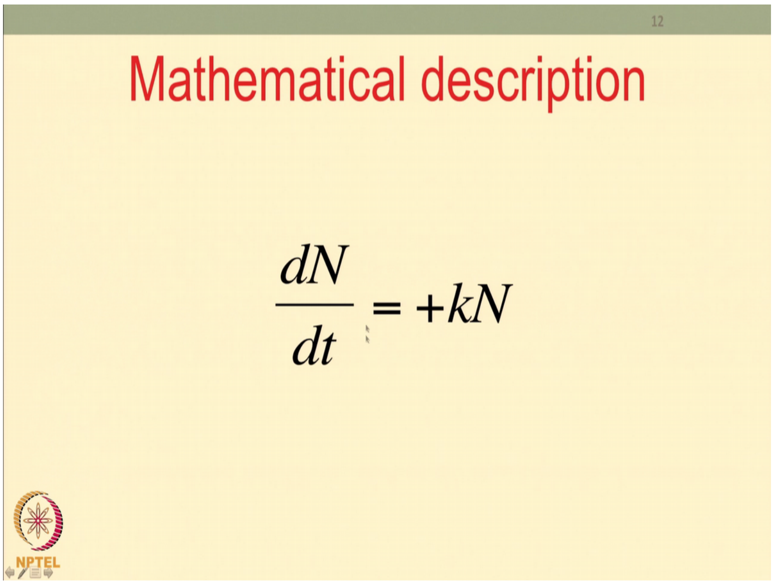

And that we can mathematically describe this thing as dN by dt is some constant times N and this is a positive sign here which would essentially mean this is increasing and we would discuss the case of negative sign here which would mean the decreasing which is not the case currently. Here we would have we have this equation, and this is a differential equation. So, this is a simple differential equation, ordinary differential equation dN by dt is equal to kN.

Now, the question is; how do we solve this equation and get N. What we want we want N as a function of t; how do we get this. Known this phenomena how do we get N of t. And this we can use the integration ideas that we learned so far. So, let us take this. So, what we have is dN by dt.

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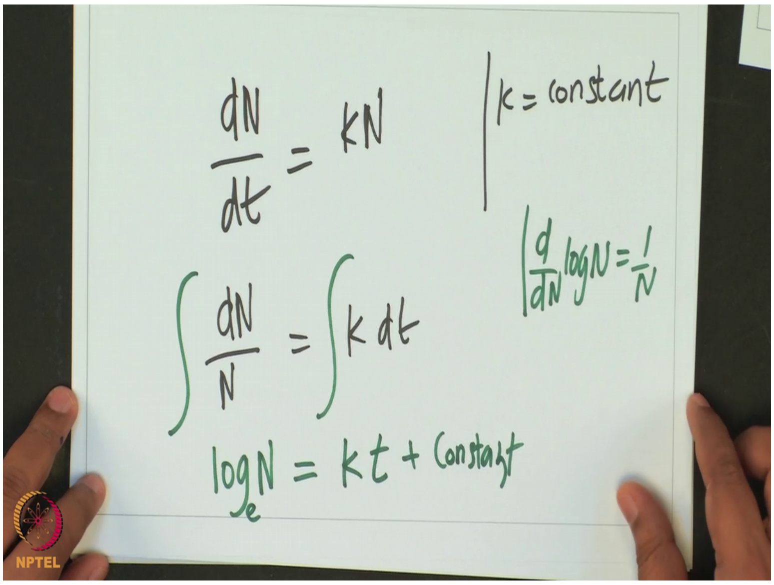

Is kN, I can take N to all one side and if I do that I would have dN by N is k dt where k is a constant; is a proportionality constant we put in it could be some number, and this number of course is related to the rate of growth in some way and we will see how does it actually depend.

So, if I have any equation like this I could rewrite this way and then I could integrate this. Now what we have? I have my function here is 1 over N dN. So, 1 over N is my function, 1 over N dN integral is k dt integral. So, what is 1 over N? And it turns out that derivative of log N d by dN log N is 1 over N. So, integral of 1 over N will be log N. So, this is integral of this is log N. And this would be log to the base e; it is log to the base e, right; N log to the base e which is equal to k is a constant integrator of dt is t plus another constant this is the answer that you would get if we integrated.

So, let us understand this. So, what we would get is log N to the; log N plus kt plus a constant. If I have this, I can right this as the way of getting N from log N is exponentiating. So, I can exponentiate what we have is the following; what we have is log N is equal to log to the base e.

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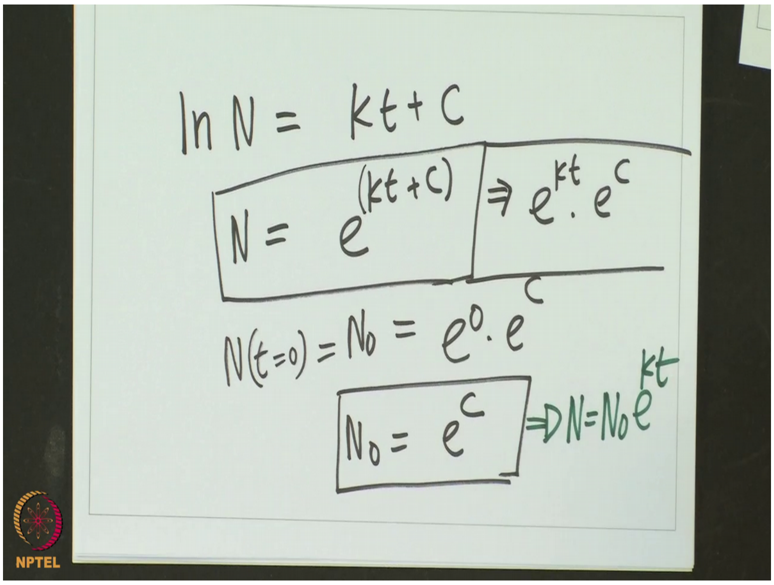

Log to the base e is also written as lon; lon N is equal to kt plus some constant let me write this as C. So, which is log to the base e. If I exponentiate this e power log N would be N. So, N is equal to e power kt plus c. I can exponentiate and I would get; so this is my answer to this integral N as a function of t.

So, if I know a time t I can get this N if I know this c and if I know this k; k is some constant related to the rate of division which is the rate of bacterial growth and you will see how it is related. But this is the integral of this equation. And this is why it is called an exponential growth. N increases exponentially with a time t. How do we get now c? If I take N at t equal to 0, if I call it N 0; N at t equal 0 which is e power 0 plus times e power c. So, e power kt plus c is e power 0 times e power c. So, this N 0 would turn out to be e power c.

So, this thing I could write. So, this is also equal to; just remember this is equal to e power kt times e power c; e power kt plus c this is an exponential e power a plus b is e power a times e power b. So, e power kt times e power c is this and e power c is N 0. So, this is nothing but N 0 times e power kt. So what we get is that, what this would imply is that N is N 0 e power kt. This is what we would get the answer from this.

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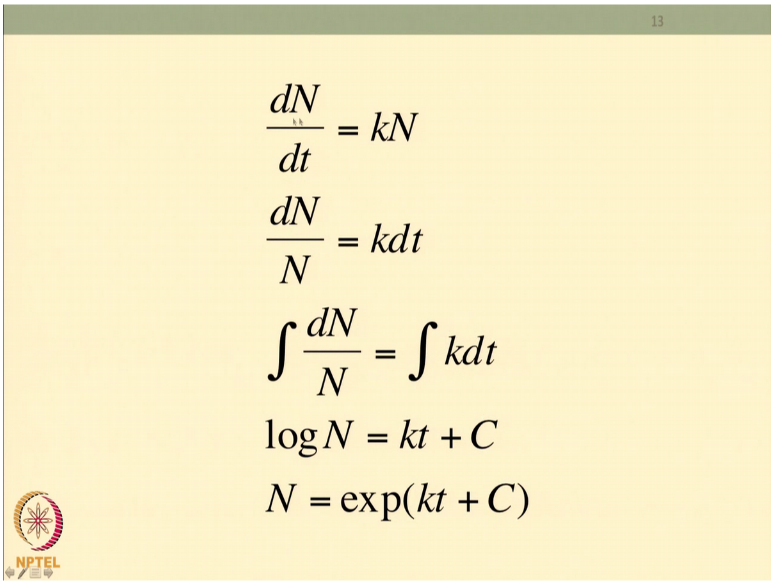

So, let us see what we have here. We had dN by dt is equal to kN; I take dt that way and N this way. So, dN by dt dN by N is kdt. So, I can rewrite this and the N by N is equal to kdt. I can integrate on both sides. So, this is integral over 1 over N; integral of 1 over N and integral over a constant time dt integral our 1 over N is log to the base en and kt plus a constant which I could write as N e power kt plus a constant.

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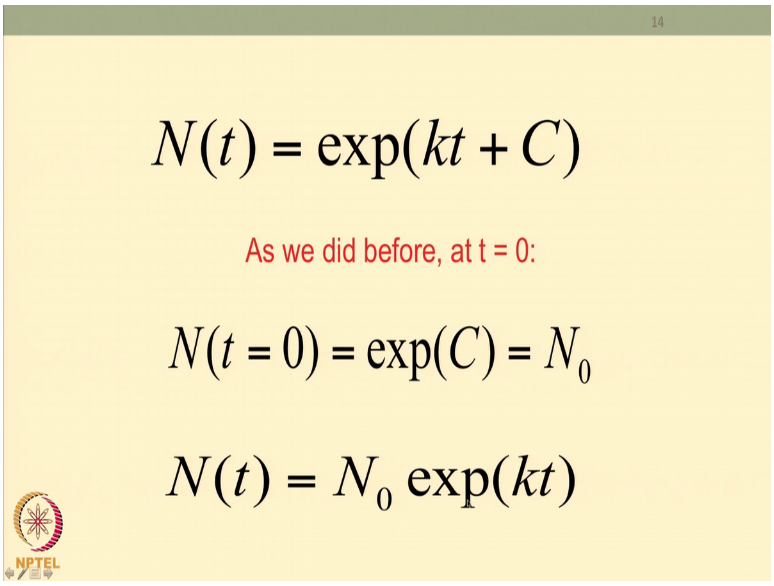

So, which is the exponential function that we have; N of t is e power kt plus c. So, if I substitute t is equal to 0 I would see that N 0 is e power c and I could write this N is equal to N 0 e power kt. So, this is what we have at the end of the day: N is equal to N 0 e power kt.

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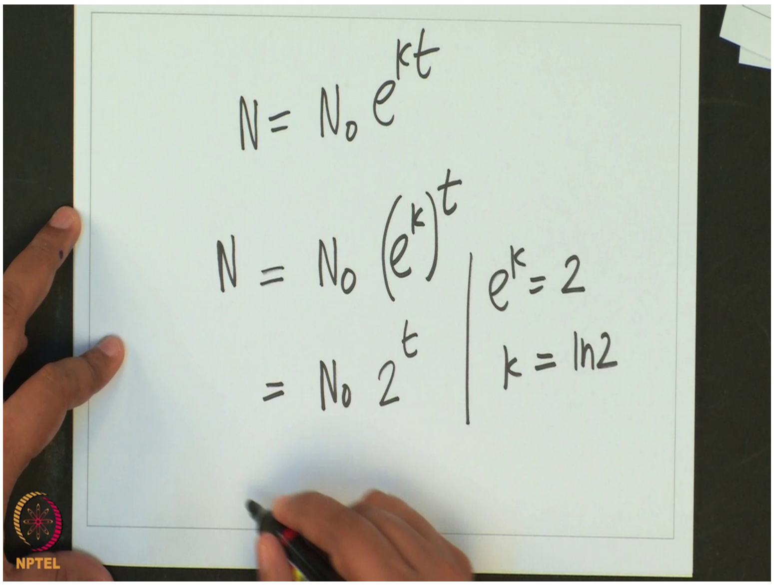

What we have is N is equal to N 0 e power kt. This is an exponential example of an exponential growth. But very often we would get to power something right; we would get to power something. So now, let us think of this. If I want to write this as; I can write N is equal to N is N 0 e power k power t. And if my e power k is 2, I can write it as N 0 2 power t if I want write this as either I could write it as 2 power kt or I could write as 2 power t. If I write it as 2 power t what would this mean, I would mean e power k is equal to 2 which would mean k is equal to log 2.

So, if k is lon two or natural log 2, if I have k is equal to natural log 2 that this is my rate. This would mean that N is equal to e power 2. Very often you would also want to write 2 power N. So, you would have seen very often that; very often you would heard about bacterial growth has 2 power some generation time.

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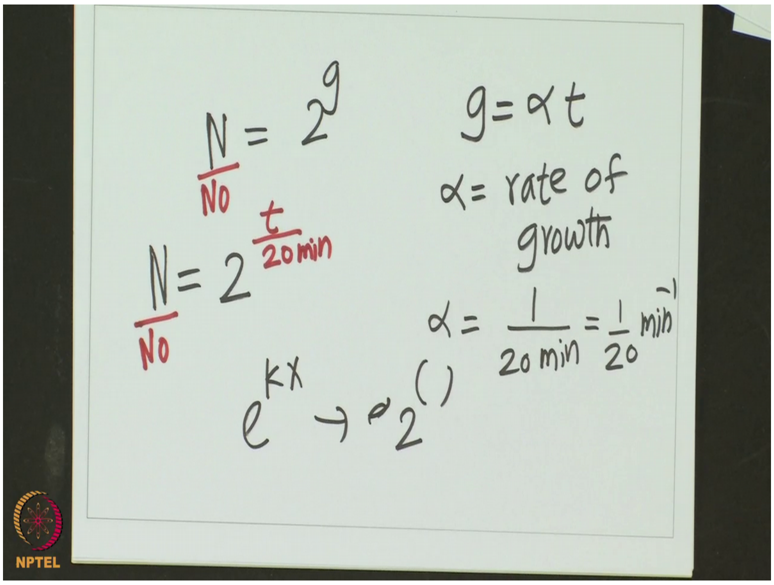

So, what you would have heard is N is equal to some 2 power g; the 0 generation this is 1 second generation to how do we convert this to the exponential thing that we learned now. One is this g which is the generation I could write as alpha t. What is alpha? Alpha is rate of growth. This alpha g I could write as alpha t.

So, let us say alpha is equal to 1 over 20 minute that is 1 over 20 minute inverse. So, this is 1 hour 20 minute inverse, so 1 over 20 minute inverse. If alpha is this I could write N is equal to 2 power t which is time divided by 20 minutes. And this is, I actually should be N by N 0 I would say I could write even this is N by N 0 if I want, if I write this I would get a formula I would get a plot which would be again an exponential growth. And, if I start with N 0 is equal to 1 of course at t is equal to 0 you will have N is equal to 1. So, if I start with 1 bacteria which is my N 0 is 1.

So, I have 1 and I would get this as 2. And I could start with after 40 minutes this would be 2, so after 20 minutes will be 2 power 1, after 10 minutes it would be 2 power 2 and so on and so forth I could get this appropriately and plot this as a function of time. So, there are different ways of doing this, think about it. All of any exponential function e power k x can be also converted to 2 power something else as we saw and we can deal with this.

So very often when we do calculus, when we do calculus like the way we learned we are actually not interested with discrete numbers, but some kind of continuous fraction; the fraction of bacteria. So, we would want to have our density like the bacterial density. So, very often you would be interested in a continuous time. So, at any time you can measure a density. So, very often we would be interested to get the bacterial density which is e power kx; so rho of x or the concentration or density.

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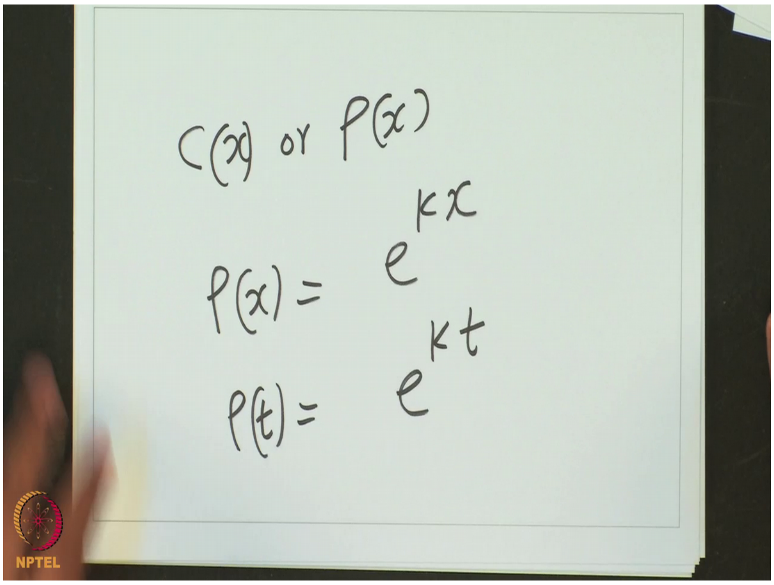

Let us say the bacterial density or a concentration if I wish or density rho of x. So, if we are interested in bacterial density you could write this as e power kx where or e power kt rho of t at any time the density is e power kt, and this k would have appropriate meaning and we could get this.

And now the opposite thing very often we see is a exponential dk. So, that is the death. So you have cell death; so same thing applies to decay or cell death also. So, let us think of an example which is cell death and we have large collection of cells.

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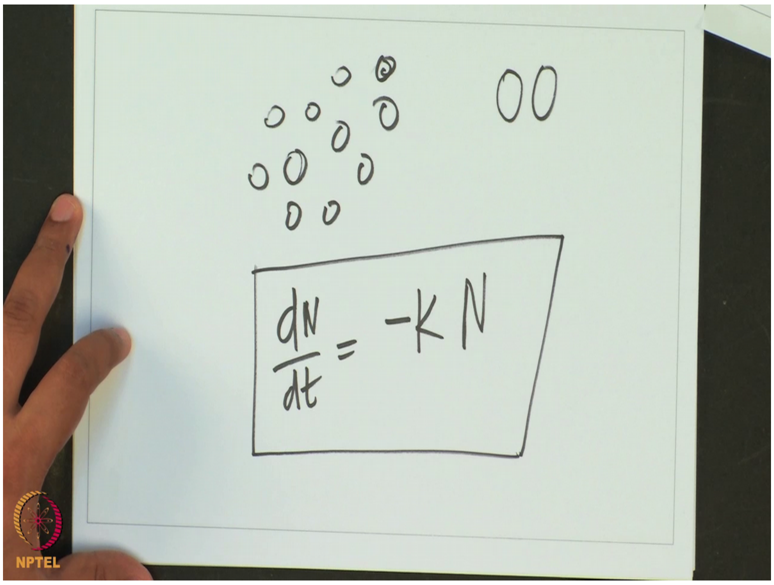

The more the cell they all can each of them can die with some rate k, so then we would write a similar equation dN by dt. How would this change? This would decrease minus k times N, because the decrease the dN by dt; the more the cell the more can die if there is only 2 cells only those two can die, but if you have a 100 cells all 100 can die or a good fraction of this can die. So, if I write similarly for death I would get similar equation with the minus sign.

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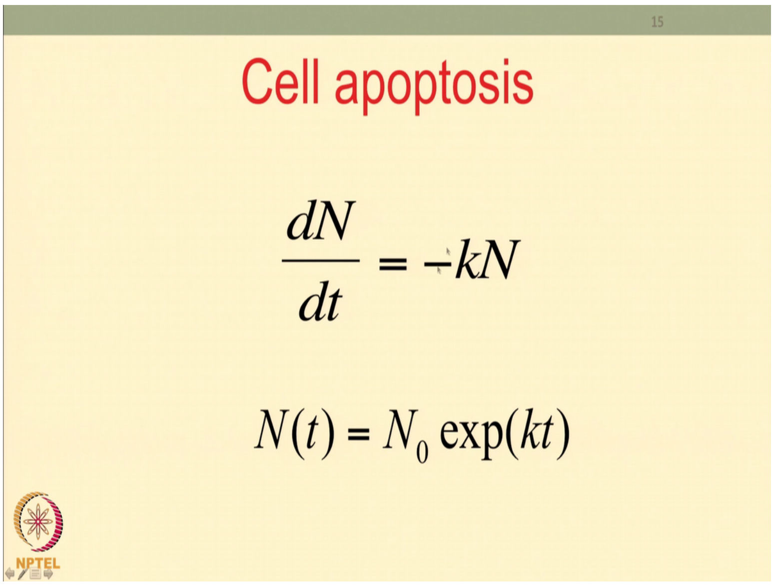

And if you have such equation you would get dN by dt is equal to minus kN; and this is a mistake here should be a minus sign here and you would the answer you would get similar way you could do this calculation.

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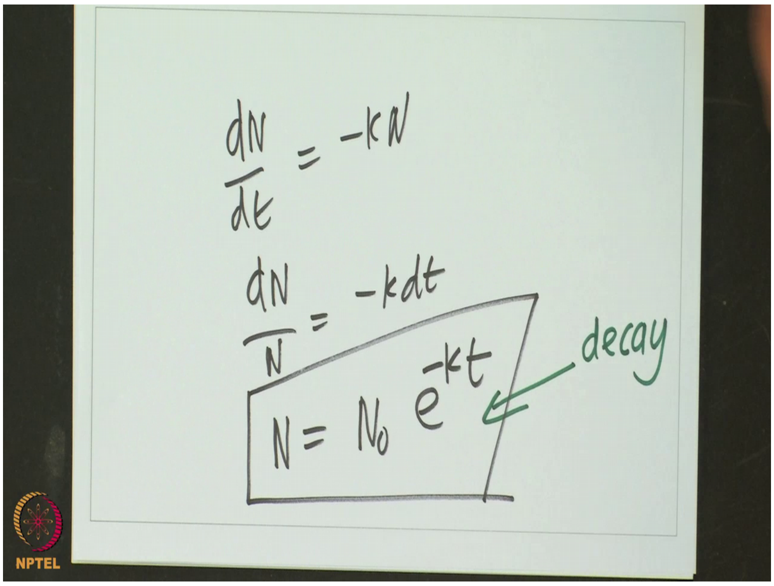

And show that if you have dn by dt is equal to minus kN, I could write dN is by N is equal to minus k dt. And you would get that this would is N is equal to N 0 e power minus kt. So, this is exponential decay. So, follow the similar mathematics, but the only thing is that here is a negative sign the N would decrease. So, this would N would decrease with time; N is equal to N 0 e power minus kt. So, this would exponential decrease, and therefore this is an exponential decay.

One very typical other example would be if you have bacteria; if you have set a proteins bound on the DNA.

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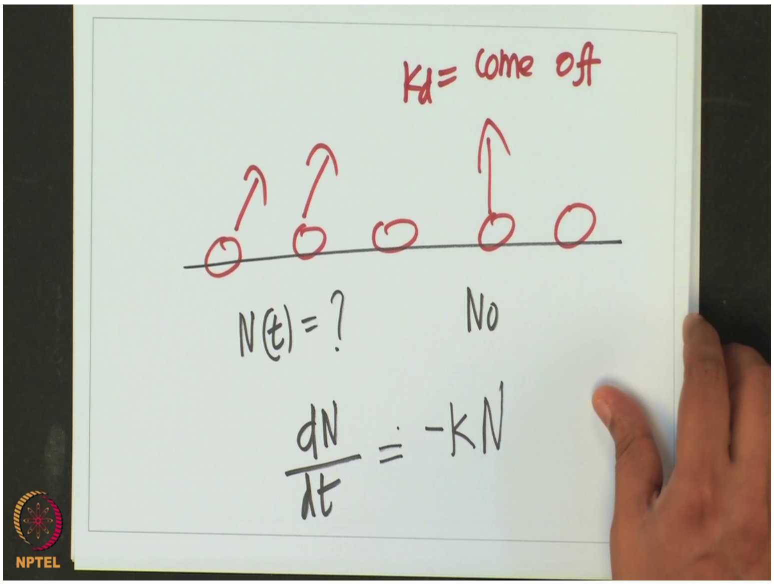

Imagine that you have a DNA or any filament and there are many proteins bound on it, so many proteins there are N proteins bound on it. And, they all going to dissociate from this with some come off with some rate kd. So, kd protein dissociate come off right, protein would come off from the DNA with some rate kd. If that happens what we want to calculate the number of proteins bound on the DNA at any time t; what is the number of proteins bound on this DNA anytime t. If I have N 0 number to start with and they all going to come off with some rate kd.

We would write a similar equation the number of bacteria bound N would change, where N is number of bacteria bound on this DNA that would change dN by dt and this would decrease and this decrease would be proportional to the number itself. If there is large number of them bound all of them can come off, if N is 0 nothing is bound, nothing can come off.

So, this coming of quantity has be proportional to N itself and if I solve this similarly you would get a similar equation you would get an exponential decay you would see that dN by dt is minus kN would have a solution.

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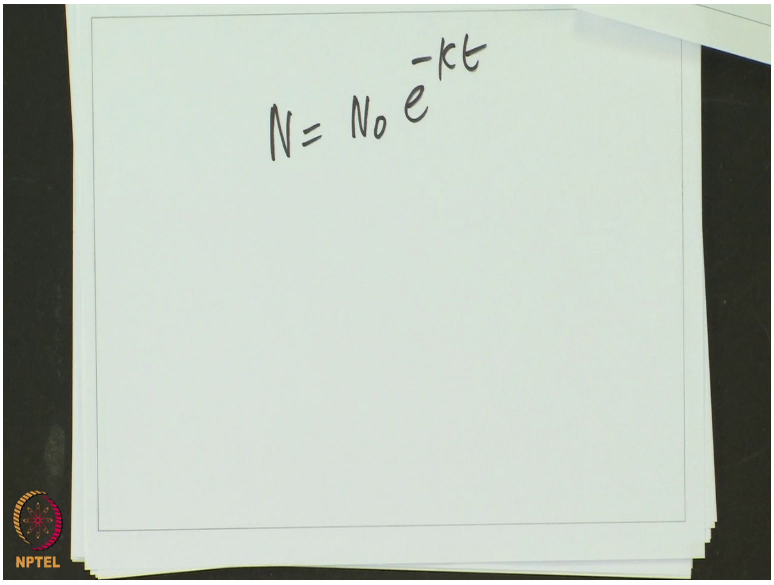

N is equal to N 0 e power minus kt. So, these two are examples of typical differential equations.

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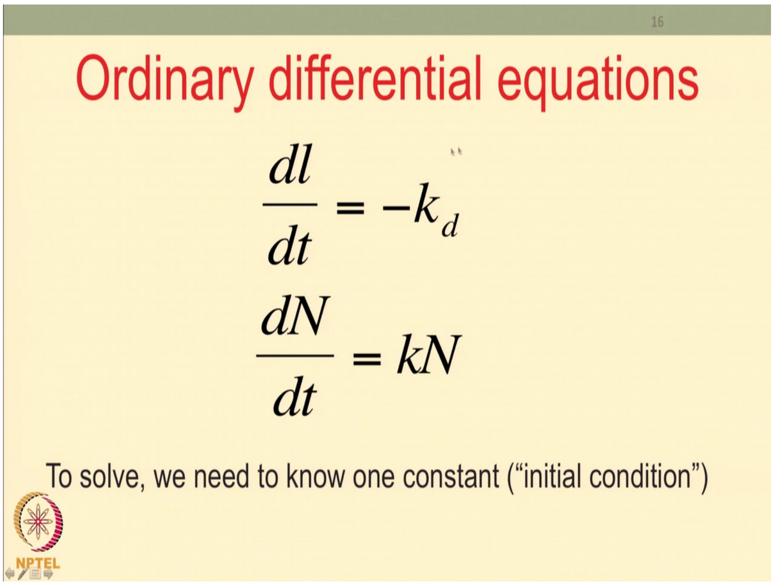

You would see dN by dt is equal to minus kd this is an ordinary differential equation; dN by dt is equal to kN here there was no N dl by dt is equal to minus kd something we learned for de-polymerization polymerization where there was no N here, here there is an N here. So, think about the difference between these two where we did polymerization wherever this polymerization on dependingly only at one end here there was an exponential growth or decay. Think about the difference between these two and we can solve them by knowing an initial condition.

Similar example apply for DNA replication and many examples in biology.

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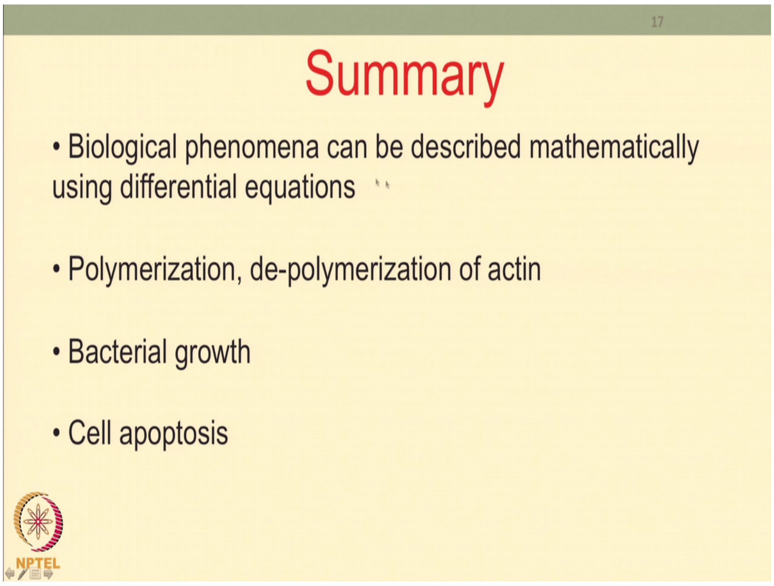

So, summarize biological phenomena can be described mathematically using differential equations. Polymerization de-polymerization of actin something we learned. Bacterial go growth, cell death, DNA replication; there are many examples where one can see note the change in number and write down an equation for it and solve it to get the answer. So, these are simple applications of calculus to think about a typical biological problem, and this describes how to think about it.

With this we will stop this lecture and continue in the next lecture. Bye.