Introduction to Biostatistics

Prof. Shamik Sen
Department of Bioscience and Bioengineering
Indian Institute of Technology, Bombay

Lecture – 25
Normal distribution Part-II and Exponential distribution

Hello and welcome to today's lecture. In last class we had discussed the normal distributions towards the end of last lecture. So, will begin by briefly recapping what we are discussed. So, for a normal distribution your probability density function f x has the expression 1 by sigma root 2 pi exponential x minus mu whole square by 2 sigma square. Where x can have any values between minus and plus infinity.

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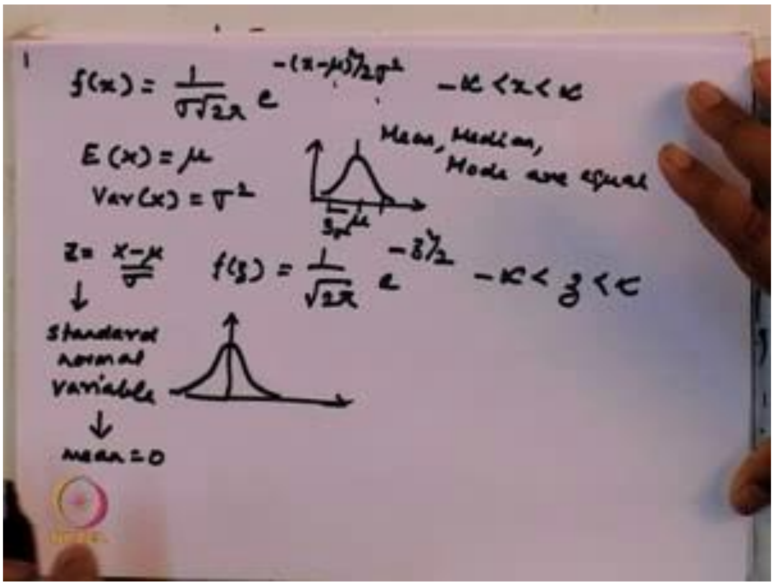

So, we had derived that for. So, as you see that there are 2 parameters in this normal distribution probability density function this is mu and sigma. And we had derived that for a normal random variable expectation of x will turn out to be mu and the standard or variance of x will come out to be sigma square. Also for a normal distribution because it is symmetric your mean, median and mode give you the same value are equal.

Now from this random variable you can convert it into a standard normal variable which is typically given by this expression z is equal to x minus mu by sigma. If you do this, then you get if instead of f of x you will have f of z which turns out to be 1 by root 2 pi e to the power minus z square by 2. And in this case the normal distribution, this the regular normal distribution has parameters mu mean mu. So, this is your value mu and variance sigma square. So, roughly 3 x, 3 3 sigma will mu plus minus 3 sigma will contain all your data. For this for the standard normal variable, this is written as the standard normal variable. So, this has mean of 0 and variance of 1.

So, this will be centered at the origin, and be symmetric. So, the cumulative distribution function is given by phi of x, which is given by 1 by root 2 pi minus infinity to x into the power minus y square by 2 dy.

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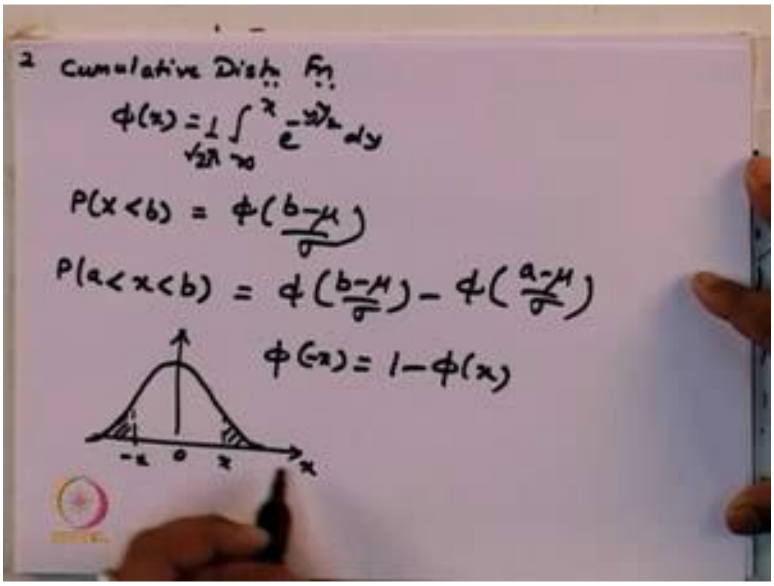

So, if you have any x, probability of let us say x which is greater less than b can we convert it to a standard normal variable is equal to phi of b minus mu by sigma. So, if x has mean of mu and variance of sigma square, then this p of x less than b is nothing but phi of b minus mu by sigma. Similarly, probability of a less than x less than b will give you phi of b minus mu by sigma minus phi of a minus mu by sigma.

One more thing because your distribution is symmetric, so let us say this is x and this is minus x is your origin. So, phi of minus x which is this area will be equal to this. So, you can write phi of minus x as nothing, but 1 minus phi of x. You can see it visualize it from this plot, phi of minus x is this area phi of x is this entire area and one is the entire area under the curve. So, 1 minus phi of x will give you this area which is equal to this area. So, this is another useful expression remember.

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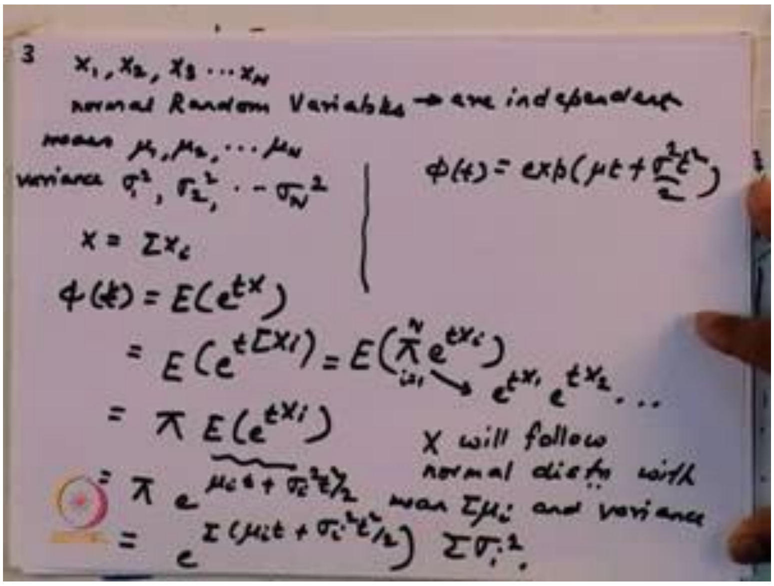

Or there is another interesting idea. So, if you have random variables x 1 x 2 x 3 dot dot dot x n, which are all normal random variables, with means of mu 1 mu 2 mu n and sigma 1 square and is variance is sigma 1 square sigma 2 square dot dot sigma n square, then for let us say if you have a variable x which you define as summation of x i. So, the moment generating function for this x will be phi of t is exponential of e to the power t x. Now let us assume also that these random variables are independent.

So your moment generating functions becomes 5 of t equal to e to the power t x. In last class we had derived that the moment generating function is gives you the expression exponential of mu t plus sigma square t square by 2. So, this is what single variable. So, for this case where x is summation x i, I can write as e to the power t summation of x i. And e to the power t summation of x i can be written as e to the power expectation of pi of e to the power t x i where pi is pi of x i I is equal to 1 to n. So, pi is nothing but the product that is, so, this is e to the power t x 1 e to the power t x 2 dot dot dot. So, I can take this product out and I can write down as phi of expectation of e to the power t x i. And for this I know that phi of t is has gives you this expression.

So, this is nothing, but pi of e to the power mu i t plus sigma i square t square by 2, and you have a product. So, eventually you will get this phi of t as e to the power summation of mu i t plus sigma i square t square by 2. This tells you that for the random variable which is defined as the sum of these individual random variables which are normal distributions, normal random variables the mean. So, x will follow a normal distribution with mean mu. So, x will follow normal distribution with mean summation of mu i and variance summation of sigma i square.

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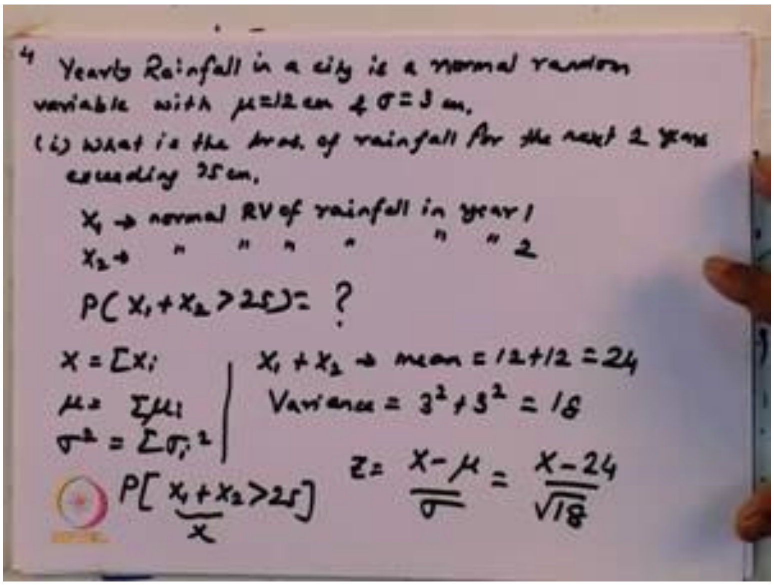

So let us consider an example where this concept might be very useful. So, imagine you have rainfall the yearly rainfall, in a city is a normal random variable with mu equal to 12 centimeters and with sigma equal to 3 centimeters. And you are asked the question what is the probability of rainfall for the next 2 years exceeding 25 centimeters. What you see here is that since the rainfall in a given year is a normal random variable and there is no correlation between the rainfall in a given year and the rainfall in the next year. So, you can assume that the rainfall in each year is a standard normal variable and these variables are independent of each other.

So, I can write x 1 is the normal random variable of rainfall in year 1 and x 2 similarly the normal r v of rainfall in year 2. So, what we have been asked to calculate is the probability of x 1 plus x 2 greater than 25, is what we have been asked to calculate. So, this is what we have been asked to calculate. So, now, in working the proof the theorem, that we just proved that for a variable x which is defined as summation x i your mu where x is our normal random variables, your mu is summation mu y and sigma square is summation sigma i square. So, if we use this particular idea then what we should have is the variable x 1 plus x 2 should have mean equal to 12 plus 12 that is equal to 24 and variance equal to 3 square plus 3 square equal to 18.

So, we want to calculate probability of x 1 plus x 2 greater than 25. So, I can convert this into a standard normal variable. So, let us say this is x. So, I can define x is equal to x minus mu by sigma, which is equal to x minus mu is 24, because this random variable has a mean of 24. So, sigma square is 18, this root of 18 is the sigma. So, what we can then do.

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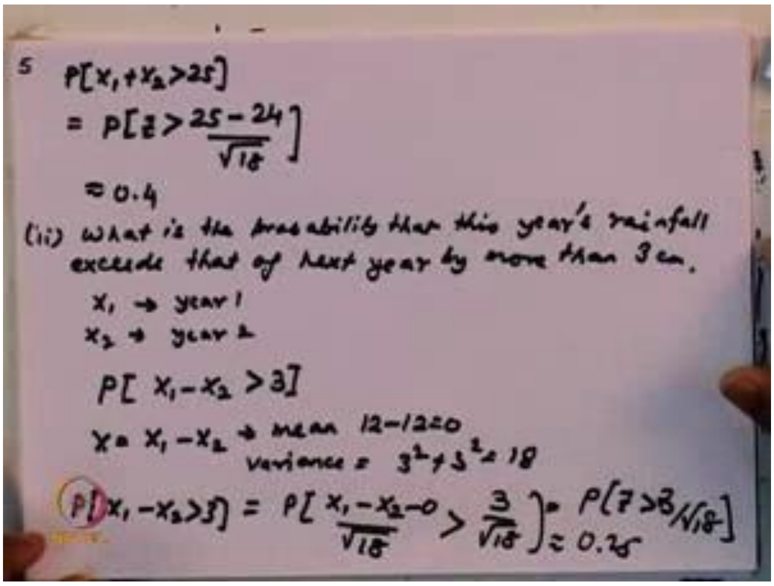

So, probability of x 1 plus x 2 greater than 25 is nothing but probability of z greater than 25 minus 24 by root of 80. So, this is you know roughly if you look up the tables the normal distribution tables, you will see this probability come out to be 0.4. We can ask another question, where what is the probability that this year’s rainfall exceeds that of next year by more than 3 centimeters. So, again if x 1 is for the normal random variable for year 1, and x 2 is for year 2, then what we have been asked to determine is probability of x 1 minus x 2 greater than 3.

So, once again we can use the previous you know theory and say that. So, the random variable x equal to x 1 minus x 2 will have a mean of 12 minus 12 which is equal to 0. And a variance of 3 square plus 3 square is equal to 18. So, thus probability of x 1 minus x 2 greater than 3, is nothing but probability of x 1 minus x 2 minus 0 is the mean by root of 18 greater than 3 by root of 18, is probability of z greater than 3 by root of 18, and this probability if you look up the tables it comes out to be approximately 25 percent.

So this is how you can make use of random variables of multiple random variables which are independent of each other. And find out the probability associated with the sum or and or subtraction of the 2 random variables so on and so forth. So, that brings our discussion of random variables or normal random variables to a close, I would like to discuss one more random variable which is called an exponential random variable.

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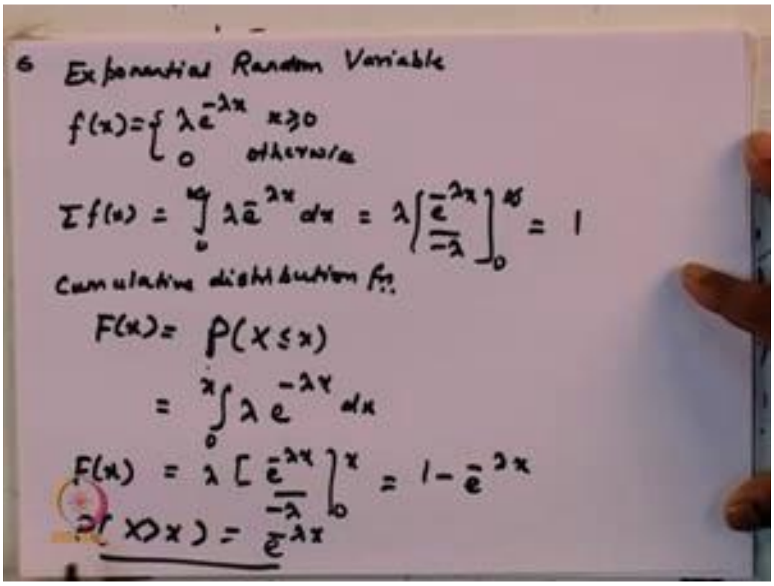

For an exponential random variable, so, your f x is defined as lambda into the power minus lambda x when x is greater equal to 0, and equal to 0 otherwise.

So, I can see whether, summation of f x would mean integral from 0 to infinity lambda e to the power minus lambda x by d x. I can take lambda out and it is minus lambda x by minus lambda 0 and infinity which is equal to 1. So, this fulfills summation of probability is all probabilities is 1. I can determine the cumulative distribution function f of x is nothing, but is probability x less equal to x is equal to 0 to x lambda rho, minus lambda x d x which turns out to be lambda 1 minus e to the power minus lambda x. So, your cumulative distribution functions is this which means that probability of x greater than x, will be equal to e to the power minus lambda x. So, we will make use of this probability slightly later.

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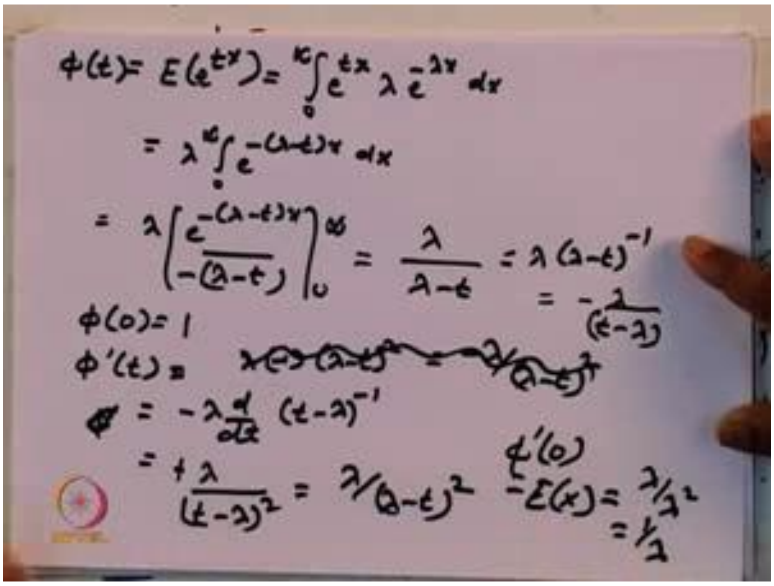

But for this exponential random variable we can calculate the moment generating function. So, the moment generating function will be exponential e to the t x is equal to 0 to infinity, e to the power t x into lambda e to the power minus lambda x, d x equal to, I can take lambda out 0 to infinity, e to the power minus lambda minus t into x d x equal to, so, this should give you a value of e to the power minus lambda minus t x by lambda minus t, with a minus sign this is from 0 to infinity. So, this should give you a value of lambda by lambda minus t. So, phi of t is this, which would mean that phi of 0, t is equal to 0 is equal to 1, 5 prime of t phi prime of t is, so, this is lambda minus 1 is equal to 1 square equal to minus lambda by lambda minus t whole square. If I write it as phi prime of t, let us see phi 0 is equal to 1 this can be written.

So, you find that minus lambda by t minus lambda t minus 2 minus. So, this can be written as phi prime of t can be written as minus lambda into d, d t of t minus lambda whole to the power minus 1 is equal to minus 1, minus plus lambda by t minus lambda whole square equal to lambda by lambda minus t whole square. So, phi prime of 0 which is nothing, but exponential of expectation of x, returns your value of lambda by lambda square equal to 1 by lambda. So, expectation of x gives 0 value of 1 by lambda.

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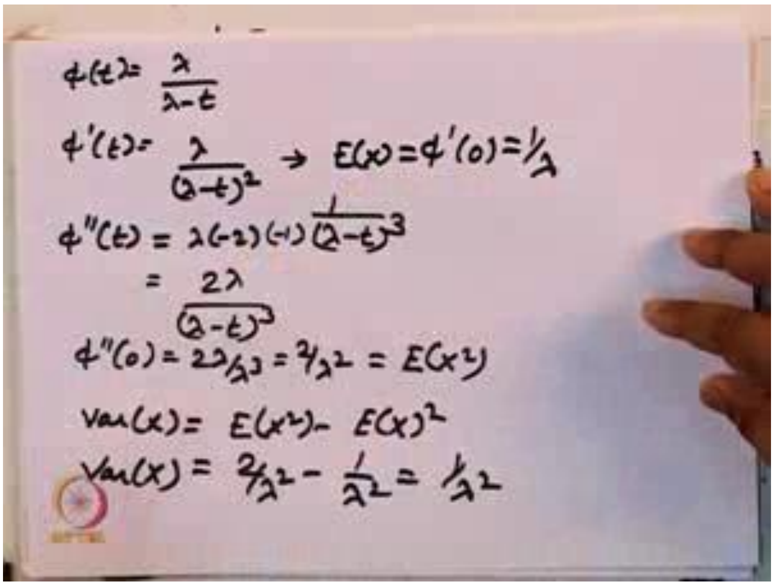

We can accordingly do phi double prime of t. So, phi of t is gives you a value of lambda by lambda minus t, phi prime of t is lambda by lambda minus t whole square, phi double prime of t will give you minus 2 minus 1 to lambda minus t, whole cubed is equal to 2 lambda by lambda minus t whole cubed. So, phi double prime of 0, will be 2 lambda by lambda cubed is equal to 2 by lambda square. So, variance of x will give you if a e of. So, this is E of x square E of x square minus E x whole square equal to 2, by lambda square minus 1 by lambda square is equal to 1 by lambda square. So, variance of x gives you 1 by lambda square an expectation of x. So, this gives you expectation of x equal to 5 prime of 0 is equal to 1 by lambda.

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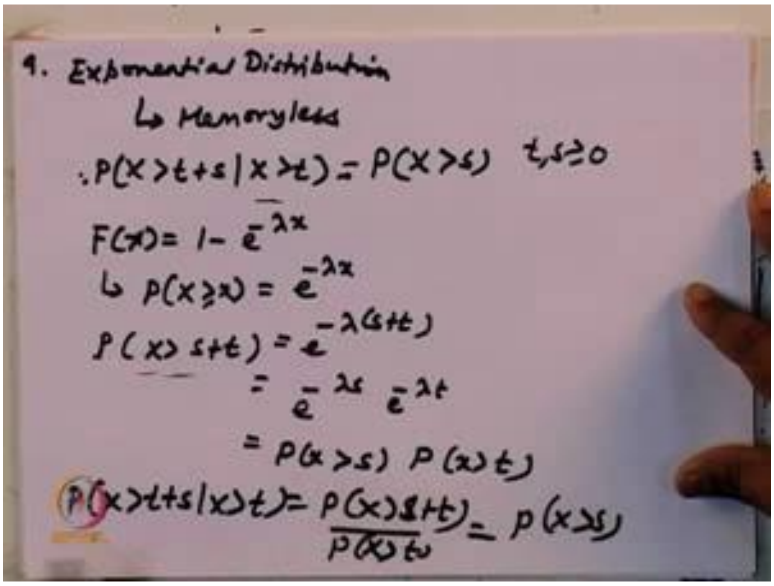

So, one very important property of exponential distributions this is like number 9. So, one very important property of exponential distributions is it is exponential distributions, has this property that this distribution is memory less. In other words, let us take a case where we are considering the lifetime of a battery of a car or lifetime of a disk or so on and so forth. So, we would typically expect that if we know that the battery was operating as of today it will probably depend. So, how long much longer it will last will depend on how much it was operating now; however, in the case of a memory less distribution. So, you can it probability of x is less the lifetime greater than t plus s given x is greater than t.

In other words, it has lasted at least t amount of time what is the probability it will last for another x amount of time. So, this one would ideally expect to depend on this t, but this has the probability this is does not depend on t, and this is simply probability of x greater than x. So, this is the memory less. So, this is true for t comma s greater is equal to 0. So, this is the memory less property of an exponential distribution and we will see why this is.

So, we can see we have calculated, we know that if x is given by 1 minus e to the power minus lambda x, which would mean that probability of x greater than x is equal to e to the power minus lambda x. So, if probability, which means that probability of x, great or equal to x is e to the power lambda s. So, probability of x greater than s plus t would simply be equal to e to the power minus lambda of s plus t. And this I can break it down e to the power minus lambda s into e to the power minus lambda t, is equal to probability of x greater than s into probability of x greater than t.

Now if this is, then from this and this I can write down. So, probability of x greater than s plus t given x greater then t is simply probability of x greater than s plus t. So, you can simply write down this divided by this is. So, probability of x greater than t plus s given x greater than t, is equal to probability of x greater than s plus t s plus t, by probability of x greater than t. And this gives me the value probability of x greater than s. So, this proves that this is the memory less property of this particular distribution.

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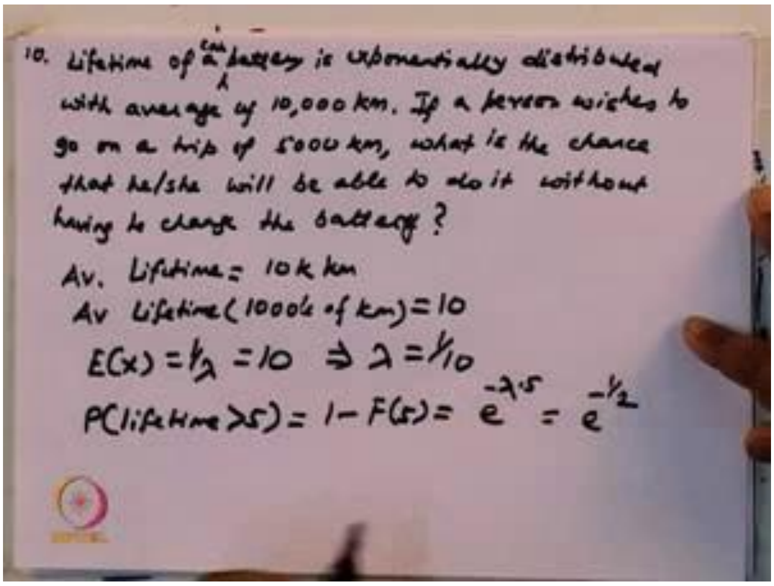

And let us take one particular example to discuss how it is useful. Let us say the lifetime of a battery for car battery is exponentially distributed, with average of 10,000 kilometers. If a person wishes to go on a trip of 5,000 kilometers what is the chance that she will be able to do it without having to change the battery. So, what you are given is the average lifetime. So, average lifetime is equal to, so, 10 kilometers. So, let us say I put the average lifetime in unit is of thousands of kilometers. So, my lifetime is 10 and I know the expectation. So, expectation of this exponential distribution is 1 by lambda and this is same as this 10 which would mean my lambda is 1 by 10. So, I want to calculate probability of lifetime greater than 5, which is nothing, but 1 minus f of 5 equal to e to the power. So, we know e to the power lambda into 5 is nothing, but e to the power minus lambda times 5 equal to e to the power minus half.

So, this is what you see that this distribution does not depend on to how long the battery lasted before, but it only depends on what is the average. And using that you can calculate the lifetime. With that we conclude our discussion of probability distributions. So, across multiple lectures we discussed about binomial and special random variables including the binomial random variable, the Poisson random variable, the uniform random variable, the normal distribution and the exponential distribution. So, from next lecture onward, we will start discussing about sampling distributions.

Thank you for your attention.