Introduction to Biostatistics
Prof. Shamik Sen
Department of Bioscience and Bioengineering
Indian Institute of Technology, Bombay

Lecture - 22
Binomial random variables and Moment generating function

Hello and welcome to today's lecture. So, in last class we had discussed about expectation variance and covariance and see how seen how you can use various functions or transformations to calculate expectation of g x or expectation of summation x i and same for the variance and covariance.

So, today we will introduce one more concept to calculate expectation and variance of a population, and that is using moment generating functions.

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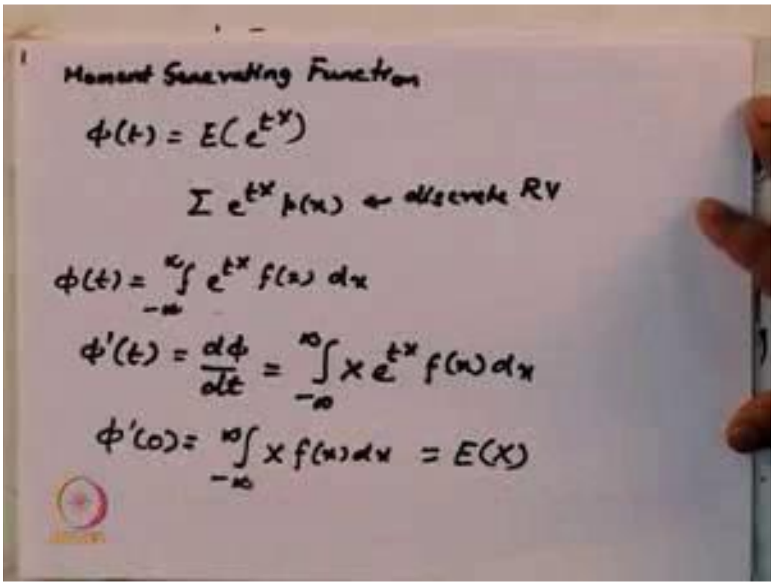

So, what are moment generating functions? So, a moment generating function phi of t is defined as expectation of e to the power t of x. So, this would be summation e to the power t of x into p of x for discrete R v or so phi of t.

Now, let us take the case of the continuous random variable and derive. So, if I do phi prime of t which is d phi of d t. So, then I can write this equation x e to the power t x, f of x d x why? Because this entire function is a function of x except for this e to the power t x, so when you take a derivative with respect to t, only need to take a derivative of this component e to the power t x. So, what do we get as a consequence of that. So, can write phi prime of 0, phi prime of 0 is equal to minus infinity to infinity, x f of x d x which is nothing, but expectation of x.

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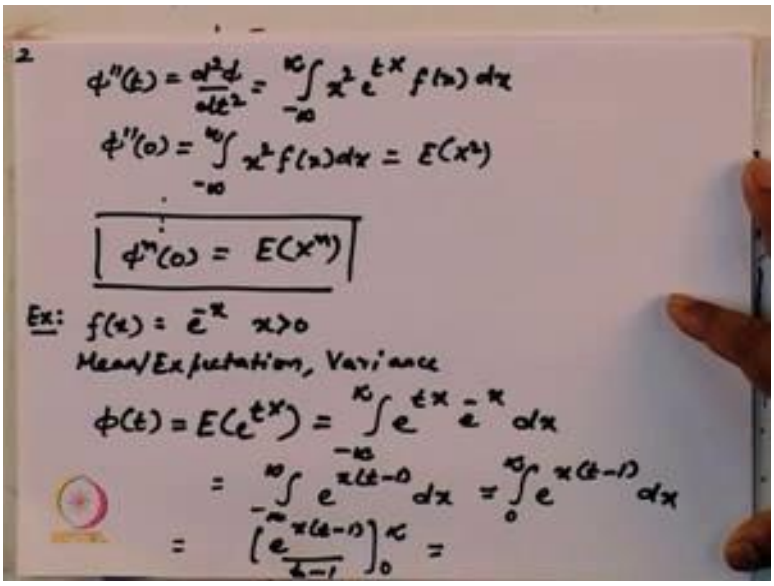

So, I can write phi prime double prime of t is equal to x square. So, then if I put substitute t equal to 0, I get phi prime double prime of 0 is equal to minus infinity to infinity x square, f x, d x; what is this this is nothing, but expectation of x square. So, if I continue like this, I will get n trivet of 0 is E of x to the power n. So, using these moment generating functions we can calculate expectation or variance of a population in the generic case. So, let us this is a calling let us solve an example to testing.

So, imagine you have a probability density function f x. So, this is an example which is given by e to the power minus x for x greater than 0. I want to calculate the mean or expectation and variance of this population. So, what do I do? I comprise I start with phi of t this expectation of e to the power t of x equal to. So, this I can simplify.

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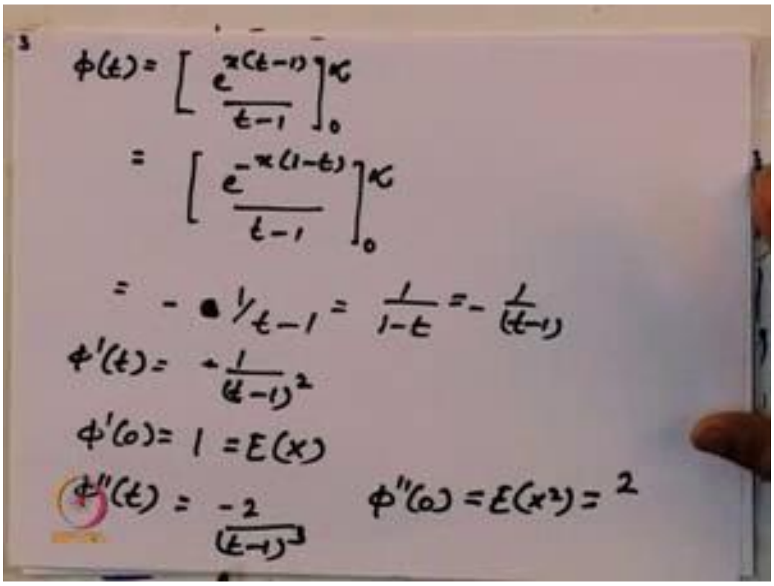

So, let me do it in the next page. So, I have got phi of t, I can write this minus of x equal to 0 I have 1 by t minus 1 equal to 1 by 1 minus t. So, phi of t comes 1 by 1 minus of t, what is phi prime of t is equal to. So, I can write minus 1 by t minus 1. So, phi prime of t is nothing, but t minus 1 whole square. So, phi prime of 0 will give you 1 is equal to expectation of x. So, I can accordingly find phi double prime of t, which is equal to minus 2 by t minus 1 whole cubed. So, phi double prime of 0 equal to minus is equal to 2, this will return me a value of 2.

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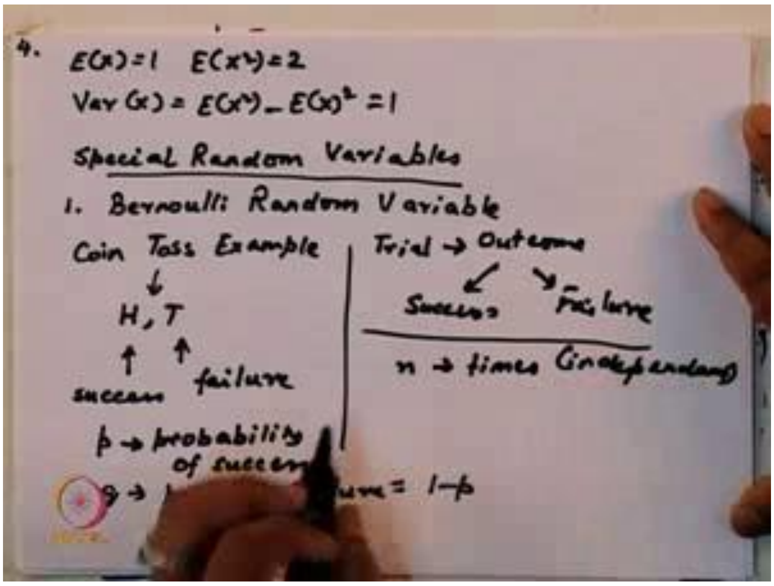

So, I can then calculate. So, I have calculated E of x equal to 1 E of x square equal to 2. So, variance of x is E of x square minus E x whole square is nothing, but 1. So, this is how you can see that you can really make use of moment generating functions to calculate expectation and variance and so on and so forth. So, that brings our discussion on expectation and variance to a close, and we begin by discussing some special random variables. So, today we begin by discussing the Bernoulli. So, let us take a Coin Toss example. So, here so I your outcomes are either head or tail, you can call your head as a success and tail as a failure. So, for an unbiased coin p is the probability of success p is defined as a probability of success, and q is probability of failure is equal to 1 minus p.

So, this is a single trial now imagine. So, this is just one particular example. So, if your coin is unbiased then p and q may not be half if your coin is unbiased then p and q will be equal to half, but if it is biased then p and q may not be half. So, in general so in each trial you call the outcome is either a success or a failure, depending on how you define an event as a success or a failure.

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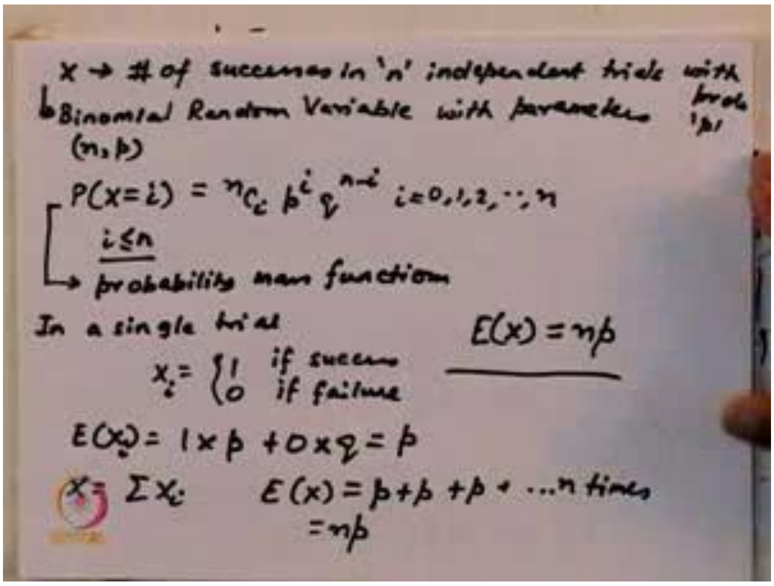

So, now imagine you keep repeating the same trial n times independent. So, each the outcome of one trial does not influence the outcome of the next trial so on and so forth. So, the. So, for this random variable we call x, a binomial random variable with parameters. So, x measures the number of successes in 'n' independent trials with probability p with parameters. So, x is called a binomial random variable with parameters n comma p.

So, in this case we can have compute what is the probability of x is equal to i? That is you want to have i successful events from n trials. So, that i events i is of course, less or equal to n, because n is the total number of trials. So, probability of x equal to I can be written as n c i p to the power i q to the power n minus i and with i can be anywhere between 0,1, 2 up to n.

So, this since x is a discrete variable, this is my probability mass function. So, I can then calculate each of the probabilities; now in a single event in a single trial let us say if I say x equal to 1 if success, and 0 if failure. So, what is the expected value of x? Expected value of x is 1 into probability of success which is p, plus 0 into probability of probability of failure which is q is simply equal to p which means. So, this is for a single trial right. So, for x, so, we get x of i. So, x is nothing, but x is summation of x of i. So, then expectation of x is going to give me p plus p plus p n times; that means, equal to n p. So, expectation of a binomial random variable with parameters n and p is n p.

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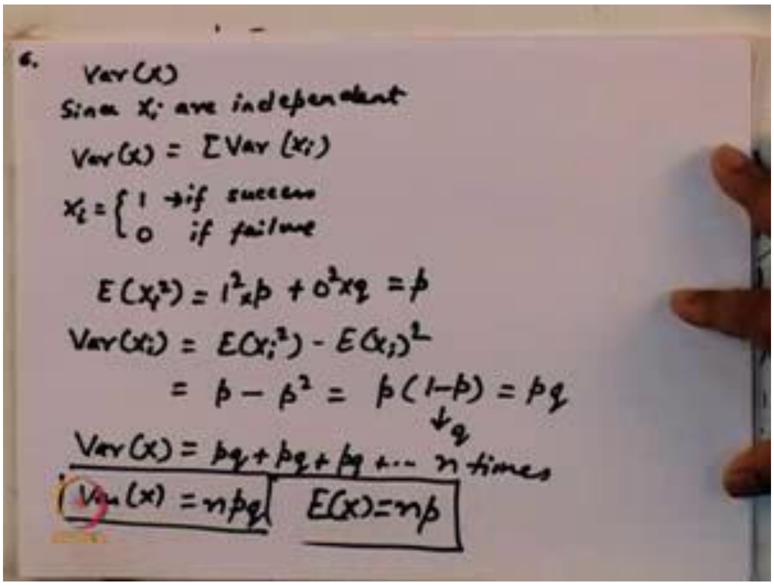

So, let us calculate the variance of x; now since x i are independent. So, I can write variance of x is equal to summation of variance of x i. So, again we have x i 1 and 0 right if success and if failure. So, E of x i square is equal to 1 square into p, plus 0 square into q is equal to p. So, your variance of x i is going to be E of x i square minus E of x i whole square this is p this is p square. So, equal to I can write p into 1minus p, and 1 minus q is nothing, but q. So, this is p q, variance of x then becomes p q, plus p q, plus p q, n times. So, for a binomial random variable with parameters n and p we have the variance as n p q and the x and the mean as n times p.

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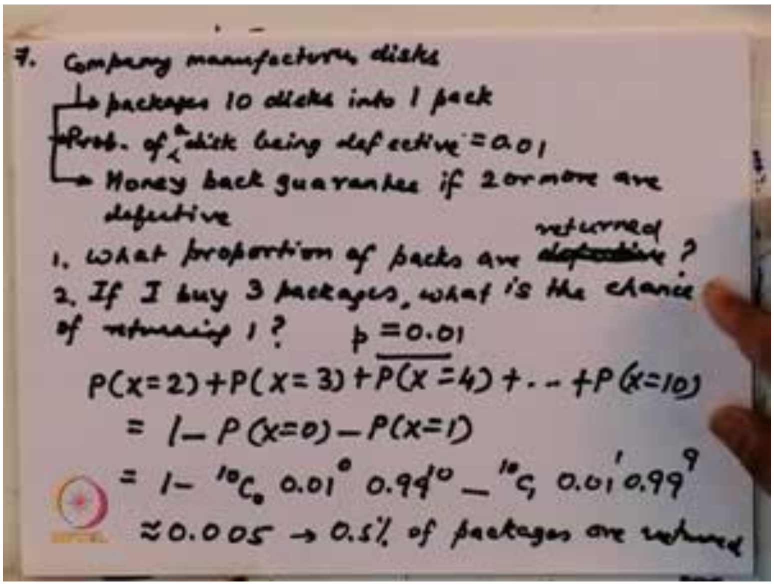

Let us solve a sample example. So, consider a company which manufactures disks. So, it packages 10 disks into one package, one pack. So, you are probability of a disk being defective is equal to 0.01. Now this company it has provides a money back guarantee, if 2 or more are defective. So, the question is first question is, what proportion of packages are defective. Second is so if I buy 3 packages, what is the chance returning 1. So, what you see you have pack size 10 disks in a pack, the probability of disk a disk this is of probability of a single disk being defective is 0.1.

So, when you ask. So, let us try to solve the first question first part of the question, what proportion of packs are defective. So, which means or what proportions of packs are returned is a question. So, now, it is showing this being a statement is, if there are 2 or more detect defective in a single pack then the company offered some money back guarantee, which means that your packages return.

So, in order to calculate the proportion of packages returned, which means I want to calculate probability of x equal to 2, plus probability of. So, if x is the random variable which is the number of packages is definitive. So, I have to compute probability of x equal to 3, plus probability of x equal to 4 plus so on so forth plus up to probability of x equal to 10 right.

So, the probability p in our case is equal to 0.01 this is for a single. So, this is what you have to calculate, but this is nothing, but p minus. So, if x is the number of defective pieces. So, this is what for all these cases you have to return for these cases you do not have to return. So, this is nothing but 1 minus probability of x is equal to 0, and minus probability of x equal to 1. So, this I can write as 1 minus. So, from 10 10 c is 0, 0.01 whole to the power 0.99 whole to the power 10, minus 10 C 1 to 0.01 whole to the power 1, 0.99 whole to the power 9. So, if you do the calculation you will see this will return you a value of roughly 0.005. So, this means that on an average 0.5 percent. So, 0.5 percent of packages are returned.

So, another thing you have to remember that this p is the probability of single disk being defective it is point 0.01, but this is the probability of the package being returned which is 0.005. So, in order to answer the second part of the problem that if I buy 3 packages what is the chance of returning 1. So, I have calculated this probability let us say probability of returning which is 0.005 this case. So, I can I am labelling it as p prime because p prime is separate than p.

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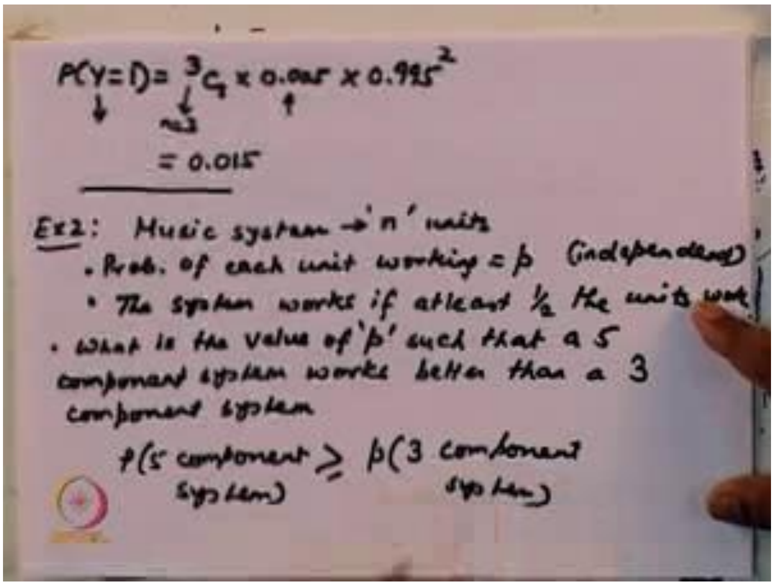

So, what is the probability that I have to return one package from 3 packages? So, that probability is probability of y, y is the random variable let the packages returned. So, we would need to calculate probability of y equal to 1. So, total number of packages is 3. So, 3 C 1 into 0.005 into 0.995, so 9 and 5 whole square, so this is how you do it, so n equal to 3, so n equal to 3 in this case at the package level one is defective you can have three different combinations, and then from here you have this is the probability of the package been defective which you have a package being returned which is 0.005, and then this is 1 minus 0.005. So, this will return your value of 0.015. So, from 3 you have to return 1 the probability is nearly 1.5 percentage.

Let us take out another example. So, imagine you have a music system, which comprises of 'n' units. So, the probability of each unit working is equal to p and this is independent. So, the probability of one unit working is independent of probability of another unit working; this is one, other clauses the system works if at least half the units work. So, the system will work if at least half of the units work, the question is what is the value of p such that a 5 component system works better than a 3 component system. So, you have n units right and we say. So, there is a clause that the system works if at least half of the units work. So, for a 5 component system, so we want to find out the probability, that probability of 5 component system is greater or equal to probability of 3 component system, this is 8 and so each unit operates with the probability of p right.

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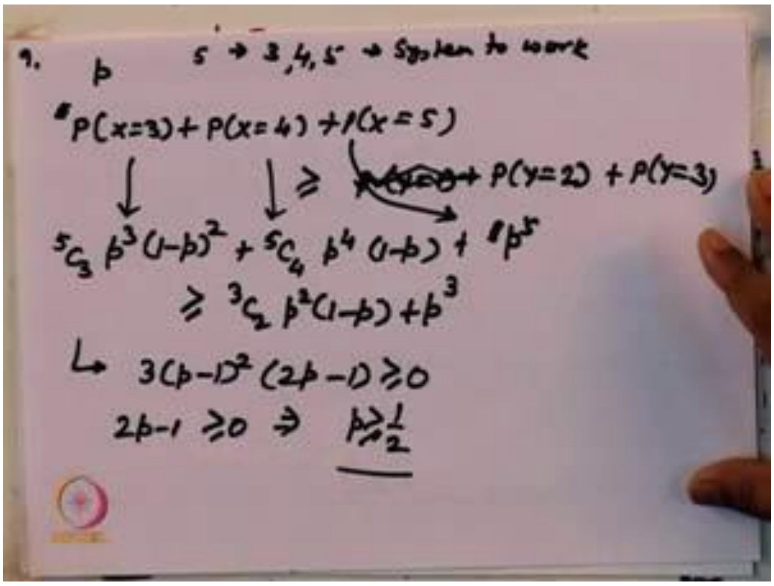

So, in a 5 component system, so for the 5 of component system to work it must at least have either. So, from 5 you must have at least 3 components which are working 4 components working or 5 components working for the system to work.

So, what is the probability? This probability is probability of x equal to 3, plus probability of x equal to 4, plus probability of x equal to 5; and we have this is greater than probability of y equal to 1 plus probability of y equal to 2. So, y equal to sorry sorry at least half is working probability of y equal to 3. So, for this I can write it as 5 C 3 square this is this component which is this component right. So, for a 3 component system to work any 3 of the 5 can be chosen in 5 C 3 different ways, the order does not matter is just the combination. So, a 5 C 3 p cubed is the probability of success of each component, and 1 minus p is a probability of failure of the 2 other components.

So, this has to be greater than 3 C 2. So, you can simplify this particular equation you see you have at least piece p cubed terms common across the board. So, you can rewrite this equation as 3 into p minus 1 whole square into 2 p minus 1 greater or equal to 0, so p greater equal to 1is of course, not possible this implies that 2 p minus 1 grater equal to 0 implying p greater equal to half.

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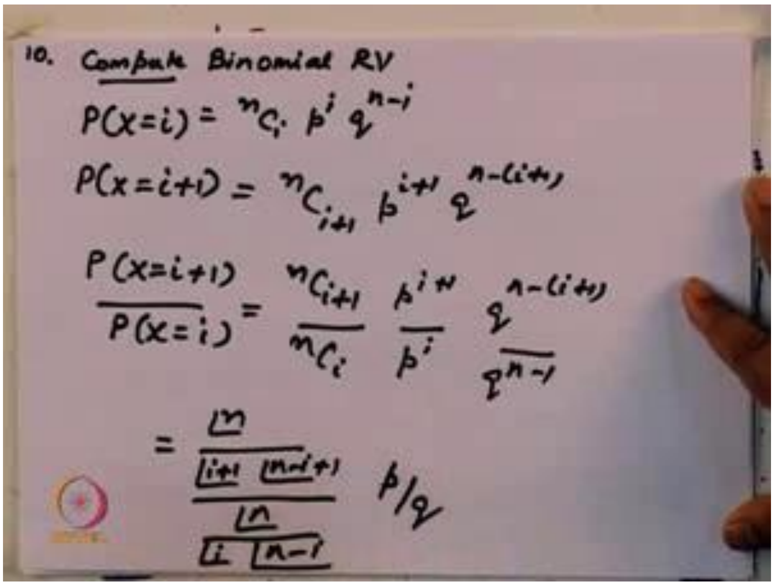

So, this is the answer your probability has to be greater or equal to half; for a 5 component system to be more successful than a 3 component system. So, one last thing I will introduce today. So, is how to use how to compute binomial random variable compute means in a computer? So, you know that probability of x equal to i is equal to n C i p to the power I, q to the power n minus I; and probability of x equal to i plus 1 is equal to n C i plus 1, p to the power i plus 1, q to the power n minus i plus 1. So, I can take the ratio i plus 1 by probability of x equal to i n C i plus 1 by n C i, p to the power i plus 1 by p to the power I, q to the power n minus i plus 1 by q to the power n minus i.

So, what is the n C i? N C i you get factorial n by factorial i plus 1, factorial n minus i plus 1, this is factorial n factorial i, factorial n minus i this is p by q. So, I can simplify the following page.

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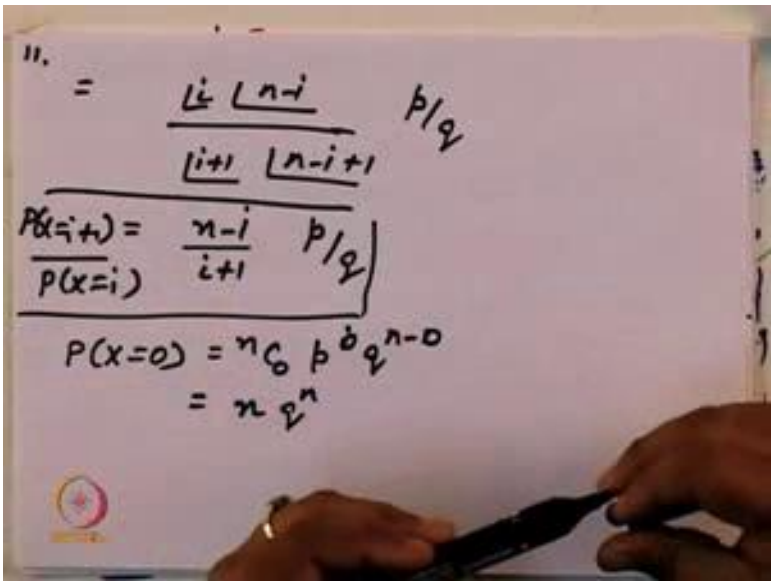

So, this becomes factorial i into p by q, i, i plus 1 into p by q. So, this is a recursive formula. So, if you put i equal to 0; so probability of x equal to 0, is n C 0, p to the power i, q to the power n minus i ok is simply n, q to the power n. So, then you can use this expression to calculate x equal to 1. So, you substitute i is equal to 0 and then accordingly you get. So, this is how you can use recursion to compute these binomial probabilities in a computer.

With that I stop here today and I will continue with Poisson random variable in the next class.

Thank you for your attention.