Optimal Control Formulation Using Calculus of Variations

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Module No. # 03

Lecture No. # 08

Summary of Variational Problems in Multiple Dimensions with Constraints

Hello everybody, we will continue our lecture series.

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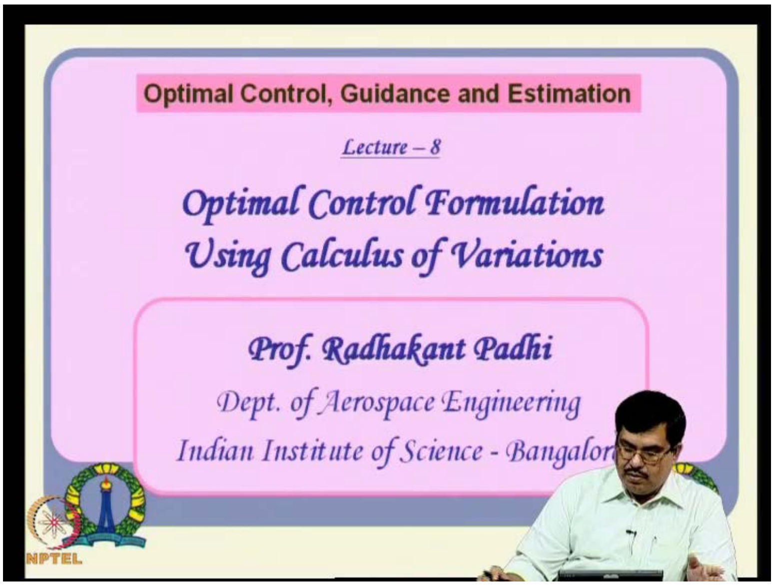

At last lecture, we saw this the concept of calculus of variation extended that to constant optimization problems, as well as well as we discussed about an example problem, which motivated that this concept of variation of calculus can be used for optimal control very well directly rather. So, now, we want to generalize this concept and then talk about how you formalize this optimal control concepts coming from calculus of variations, ok. So, we will try to use these equations again, but now it will be in the fully prime work of optimal control theory again, I mean we do not want to talk in a disguised way and thing like that anymore, we will define optimal control problem and try to attempt to kind of solve this problem as it is actually (()).

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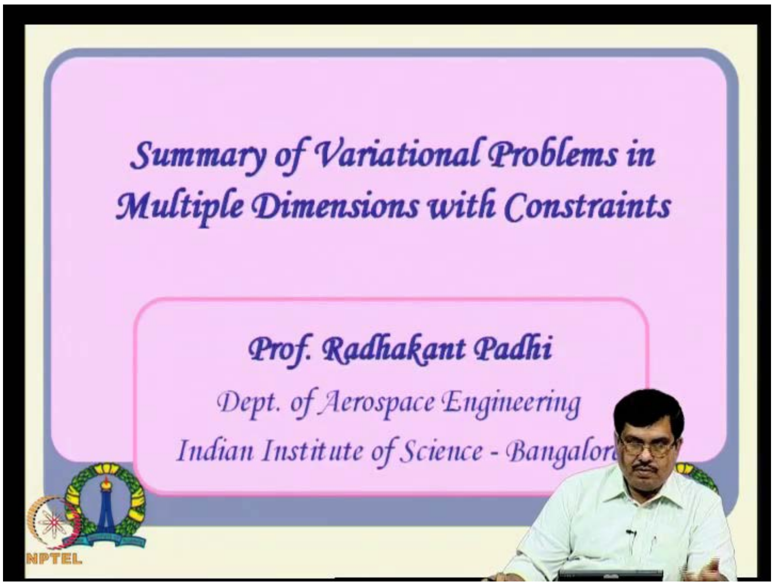

Alright, so what is the summary of various solve problem and multiple dimension with constrains?

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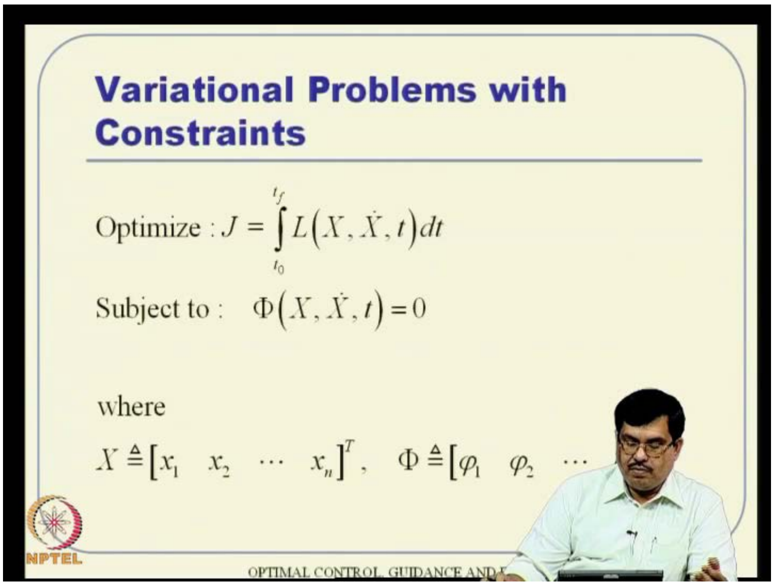

So, that we that is what we are we are were looking for, that is what we discussed in the just in the last class, that we are we were interested in the optimizing this kind of a cross function along with this this general equality constraints sort of thing, where x up in to end dimension of the phi up to the till the dimension.

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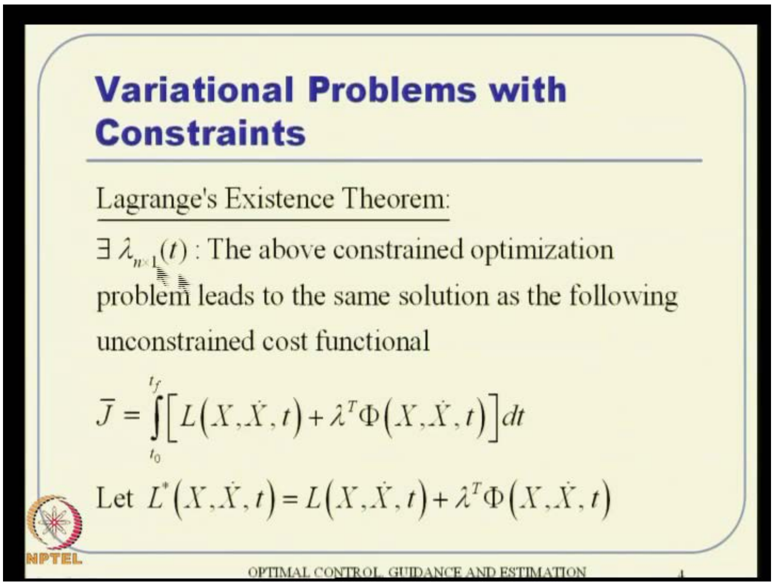

And then, we talked about this Lagrange's Existence theorem and things like that and ultimately, let us to this definition of this l star, again this l star happen to be a functional number also, so this this l star can be defined like this.

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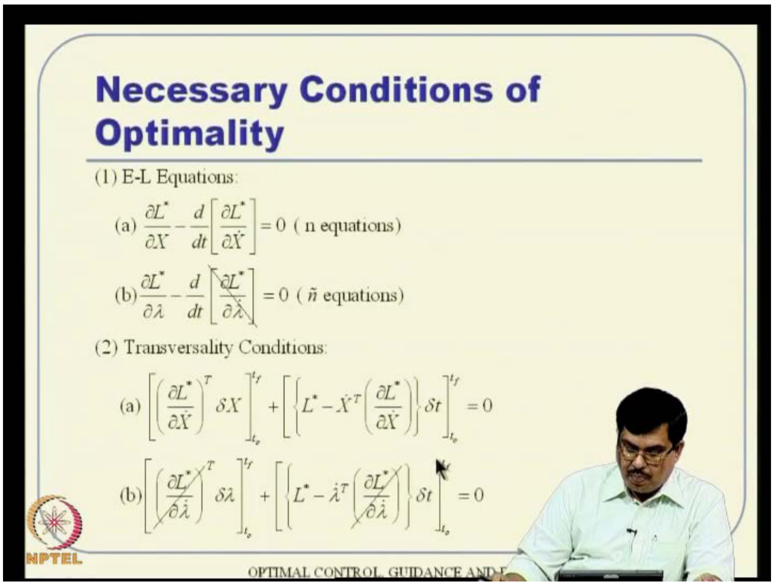

And then, we have this EL equation and thus EL equations happen to be something like this actually, ok. So, we have n equations coming from there and until the equations coming from there, transversality conditions were like this, we discussed everything in the in the previous lecture, some of you is who do not recall probably can revise.

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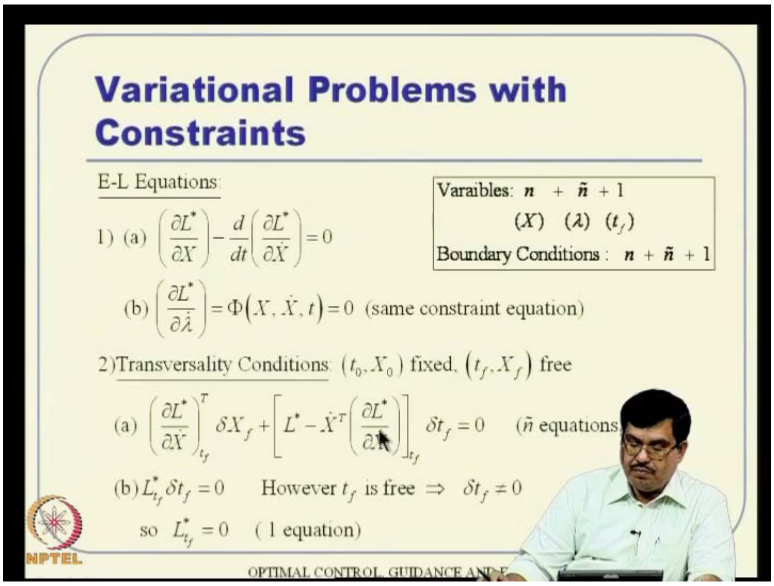

That actually and the ultimately lend up with these these kind of e l equations with with transversality conditions, ok. So, this transversaility condition depending on whether t f is fixed or t f is free we have these two classes of different things and we discussed about that in the last class. Ultimately the point is, key point is we have e l equations, lagrange's equation and there are transversality condition or boundary conditions like that actually.

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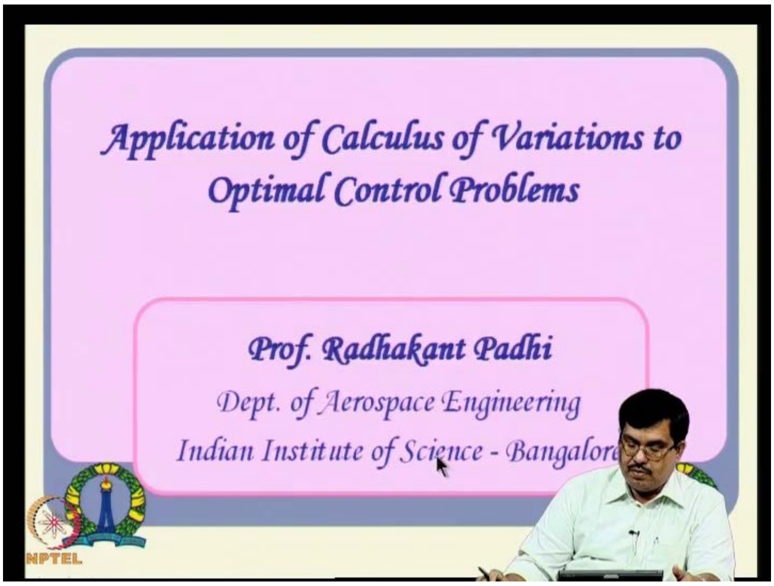

Alright, so now how do we use these concepts for these applications of these deals of optimal control problem rather ah.

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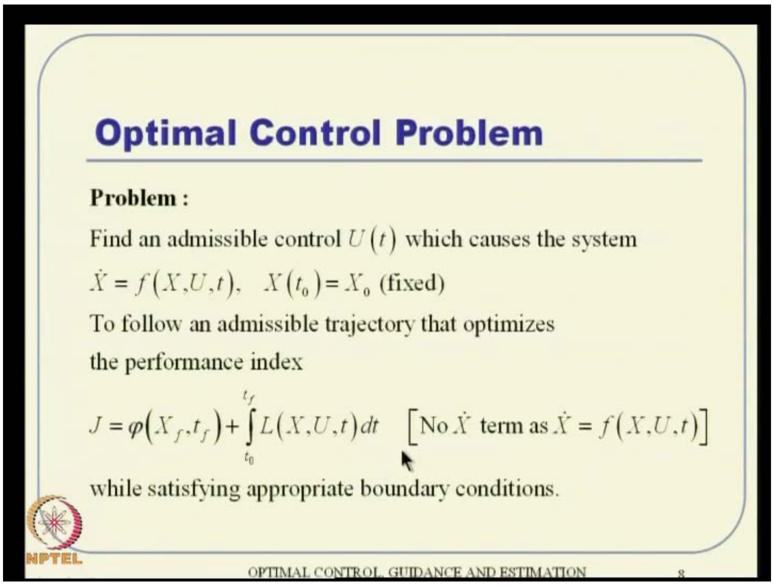

So, that is the formal definition of optimal control that also we discussed in the last class, a little bit towards end of the last class, but all I mean to revise it again. What you are looking for is to find and admissible control u t would causes the system this x star equal to f of X U t, where initial condition is fixed to follow an admissible trajectory that optimizes the performance index on the on the way while satisfying the appropriate boundary conditions. So, we have the p h of a good optimal control problem definition, you to you need to have a state equations system dynamics, you need to have a initial conditions or other final conditions, initial conditions together that means appropriate boundary conditions. And as a designer you should always try to construct a cross function that you want to minimize or maximize either way, which will lead to the object basically, ok.

So, there were the three component, that is always essential for a for a good optimal control problem formulation actually, alright. And also remember, now if you look at this thing, this this constant equation contain an x dot term, x x dot t equal to 0, where as this constraint equation what you are looking, what you are talking here, I mean this cross function, let me go back to cross function also there, cross function contain a x dot term also here. Now, this cross function, let me go back to cos function also there, cos function contain an extra term also here, now this cos function what we are looking for here typically does not contain an x dot most of the cases, ok.

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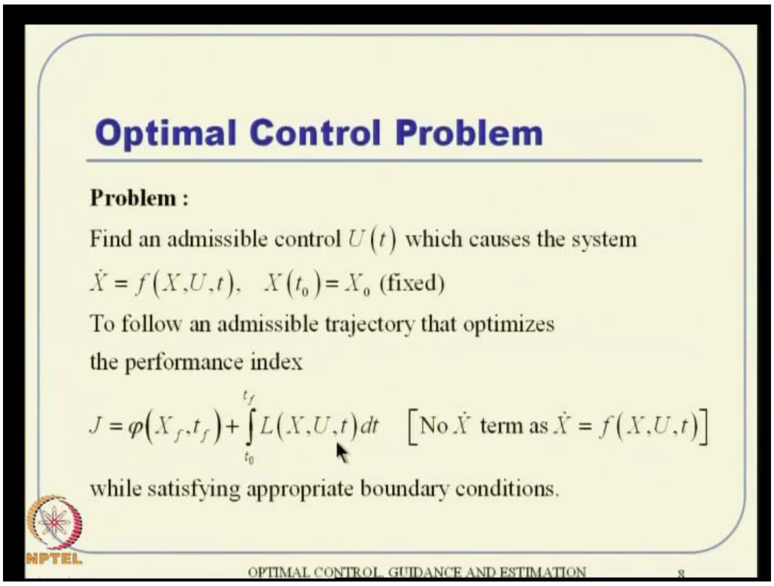

But x dot is also required, because x dot consists part of this state equation and by the way it does not mean that you do not really need x dot all the time, I mean you can have a prolonged x dot is also there, then in especially this l q r clause, linear quadratic regulator clause when we discuss that point at time, will see that the clause of problem for which you really need a x dot cross functional, in the cross functional also. But, in general, most of the time you do not need it, we it is sufficient to define in the form of x and u, because once x and u is this then x dot is constant anyway like this, you do not have a freedom of, you do not have a further freedom basically what I mean.

Once you get x and u and x initial condition is defined and then this x dot represent to be equal to that, so you do not have too much freedom for minimizing of maximizing that that quantity basically. So, that is the reason why we are not putting it here most of the time, alright, so this is the problem definition there how do you going do that, ok.

Now, the problem here is this this particular term, I mean if you say this this equation what we discussed in the calculus of of variation setting it did not contain any term outside the integral. What suddenly we are talking at term which is outside the integral actually, this term, this is outside the integral, how do you handle that, alright.

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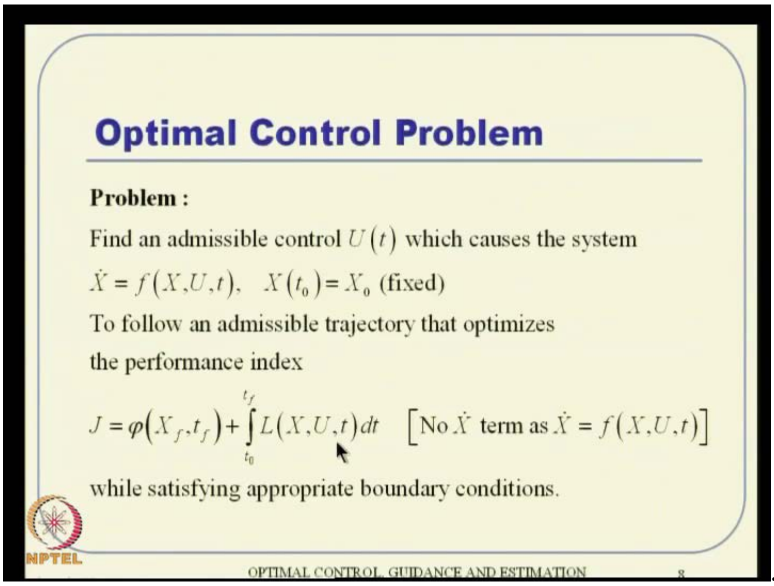

One way to handle that is something like this, let us first consider this integral, we will consider the integral t naught to t f d by of this quantity and this happens to be like that, I mean this function evaluated at final value minus evaluated in initial value. That means, this function what we are looking for, this phi of t f x, f small par f t f x f, that what we are looking for, this can be interpreted as something this one plus integral of this quantity, ok

But once you evaluate any function with a number, this particularly this as a number basically, so that means, that is irrelevant for optimization really, you are talking about this one plus something actually, which is which is variable. Anything that is has flexibility we can do something for minimizing and maximizing, anything that is number we can do anything too much on that, so that means, what I mean is this phi f x naught t naught is a constant value. So, instead of trying to optimize this j, it is equivalent to optimize j 1 ignoring this fellow.

So, we will ignore that and then tell now this is a integral of that and that is the original of thing is also integral of that, there is a term inside the integral, there is a term outside, this is gone anyway, again this is integral term, where combine these two, this what you are looking for here, now is compatible to what you what you seen before actually, right; this does not have anything outside. So, this is what you are telling this is the t that we are following actually in a way.

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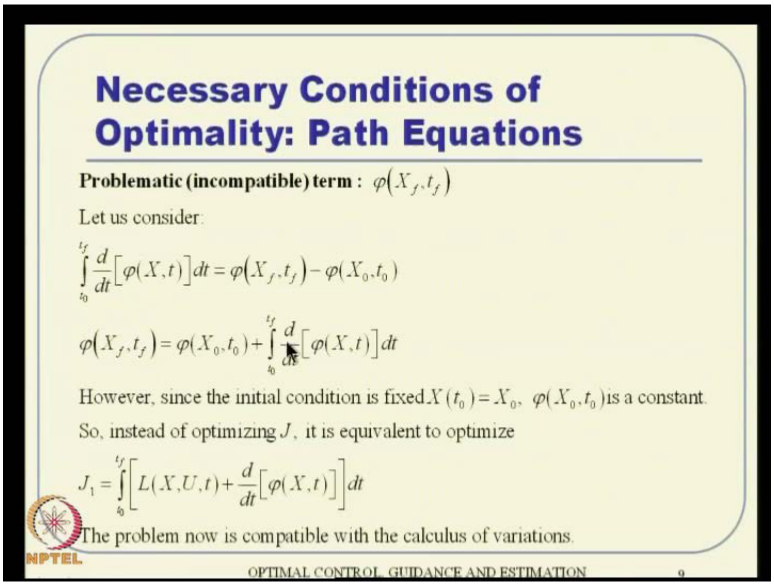

So, instead of trying to optimize the cross function j directly, which will which is also possible, I mean, I will talk to you in few slides to down the line actually, that is on the way of doing things, but we will consider in one to one equivalent of what we already know and how do we want to put it in that frame work actually, then we we do this algebra and then look at this, is just a constant value, does not matter. So, we will ignore that and then construct this j 1 and then take both the terms inside the integral and then it happens to be very comfortable to what you already know basically. So, now we can use this calculus of various ideas that we already know.

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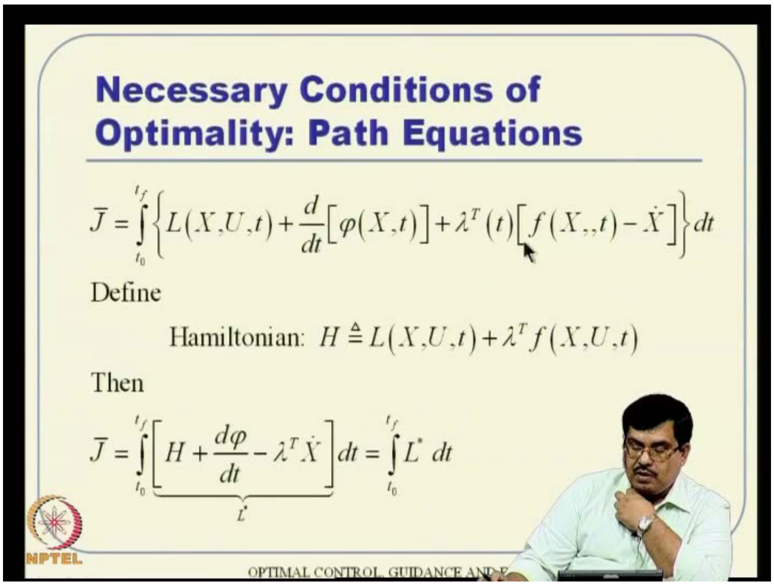

What is that? Then, obviously we will construct a j bar first and j bar is nothing but whatever we already have inside the integral plus lambda transpose, this constraint equation, f minus x dot is equal to 0, we have this constraint equation. So, f minus x dot will be equal to 0 and that constraint equation should form here, inside the integral itself, ok.

Now, the further algebra talks something like this, whatever, where ever you do not see any derivative term, you define that something like Hamiltonian, that this definition comes from continuous instantaneous and all that, that one, that history and all that is not here, that is not here, that essentially come from his ideas basically. Any way before that people where all talking about e l equation and going directly there, but this this particular concept of simplifying things and all that was (()). We will see some of this other contribution is constraint optimal control lectures later actually.

Anyway, come right this Hamiltonian is defined as something like this, all the terms without derivative term. Then I can apply simply j bar, something like this l plus lambda transpose anything, but h, now I will drop the arrangements for simplifying simplicity of algebra, I know that which one can be easy function of what, with that assumption I will drop this arrangement and then tell this is first h and then the next term is d phi by d t and the next term is this this quantity lambda transverse x dot, ok.

So, this is my quantity, so this is nothing but l star actually, so J bar is nothing but integration t 0 to t f l star td where l star is nothing but, that now it is very very compatible to what we know before actually. But what is l star and what are Hamiltonians and all, level l Hamiltonians is a function of h come l lambda and l star is also function of h c 1 lambda.

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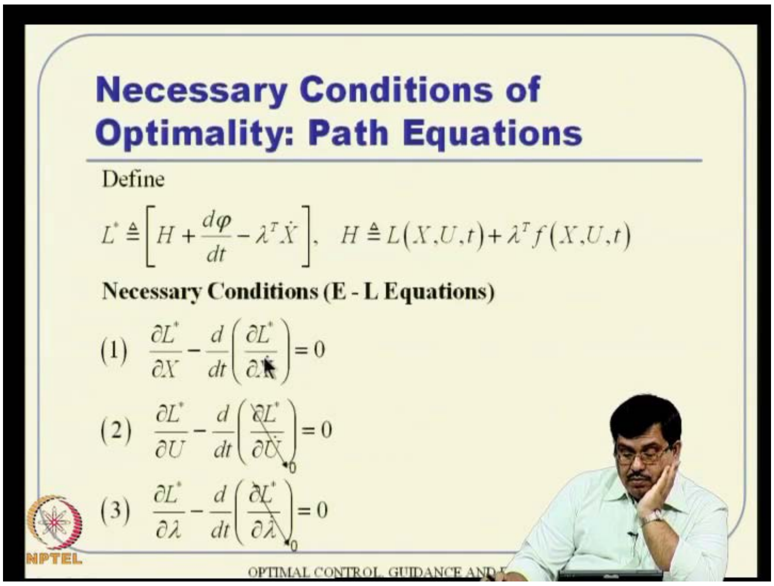

So, we have to apply the EL equations thrice once with respect to x, once with respect to u, once with respect to lambda actually. Alright, so we will do that and this l star just to summarize l star is defined like this, where h is defined like that, h is l plus lambda transverse, where l star is nothing but h plus d phi by d t, means lambda transverse t x dot. Now, what is the necessary condition? We have to apply this condition thrice.

And again this this expression nowhere contain u dot and lambda dot, again this to quantifier goes to 0.

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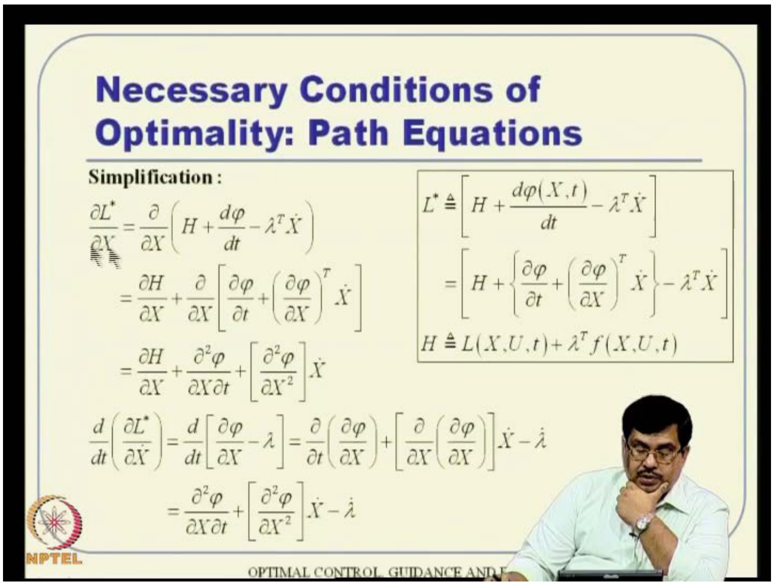

But, we are not done yet, this algebra we have to get actually, because what is del l star by del x, del l star by del x, so that by definition is del by del x l star, an l star is nothing but this quantity, right. So, we substitute this quantity here, whatever is l star is fine, here and then we carry out this algebra. So, first thing talks about del h by del x plus del by del x of this quantity on the where you have here and d phi is nothing but del phi by del t plus phi is a function of x del of t, so del phi by del t plus del phi by del x transverse time of x dot actually.

So, these two quantities come from this quantity, really of this quantity does not contain any x or I mean lambda, so the derivative of that, with that, specially with respect to that x is zero as equal to to left all to be right there. This is del h by del x plus this quantity del square phi by del x del t plus this quantity del by del square phi by del x square into x dot, this is double derivative of the thing actually, alright.

Now, what about the other quantity, this is one quantity, what about the other quantity del l star by del x, this is the first quantity that we discussed, where is the second quantity? d by d minus one of del l star by del x. So, that is why d by dt of del l star by del x dot. Now, we have to go to this l star sort of thing and (()) d by d t of that, d by d t of that is nothing but d by d t of del phi by del x minus lambda basically, ok.

Alright, because very it is easy to see that from l star definition something like this, so you can think of this del phi by del t again I have explained that here and we talk about d phi by d t, this d phi by d t is nothing but del phi by del t plus this term that is why you get an additional x dot t actually, ok. So, now, we need to talk about del l star by del x dot, now we are actually looking for this expression, h is not a function of x dot, so any personal derivative with respect to that is not there, but it will come from these two expressions, what is these two, one is del phi by del x, another one is lambda, that will come here and d by d t that d t, so del d by d t of that is nothing but again del by del t of that plus del by del x of that time x dot.

Remember, whenever you talk, there are two things remain here, one is the total derivative, something like d phi by d t, other one is the personal derivative del phi by del t. So, whenever there is d phi by d t then is the first with the personal derivative with respect to this variable and then the account for change in the derivative as well basically, so that is how the algebra will proceed actually, sorry. So, this is this is the one what we are looking for here, d by d t of that turns out to be like this actually, because of that del phi by del by del t of that one plus del by del x of that into x dot minus this d by d t of lambda is nothing but lambda dot.

Because of this, look at this expression, whenever you have this del l by del x do not worry about it actually, any way this is the type of expression that I was talking about. So, do by algebra yourself, then you then the quantity will emerge actually and ultimately you will end up you in something like this. So, what is the EL equation, EL equation tells that this term minus this term x is equal to 0, then we talk this term minus this term, obviously these two will be cancelled of del square phi by del x del t, is in these both the places will get cancelled out.

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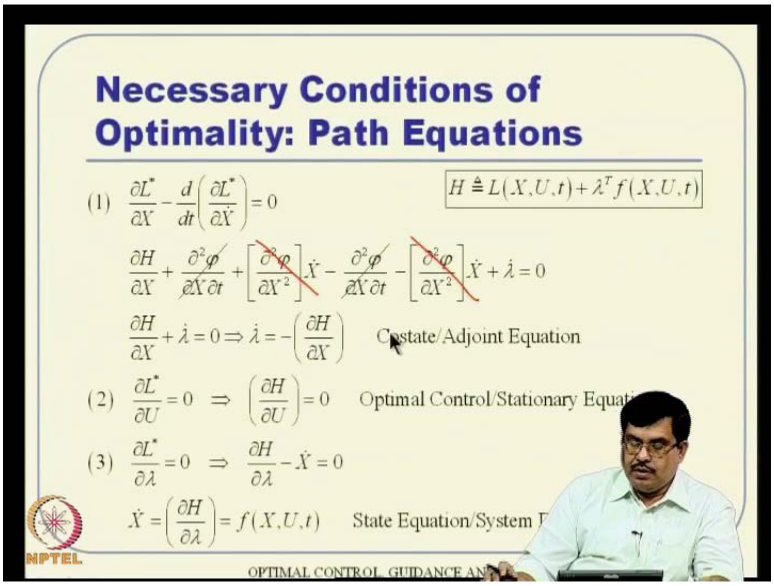

And then you will be left out with other term, so it is here anyway, so one term will cancelled out, the other term will will tell us that del h by del x from here, the other term will also cancelled out, this two terms, this is how this actually, ok.

So, then we are left out with nothing by del h by del x plus lambda dot is equal to 0, that means, lambda dot is equal to nothing but minus del h by del x and this particular equation is called costate equation or adjoint equation really. Then coming to the second part of the equation del l star by del u equal to 0, that is easy, that t will nowhere else u appears, u appears only in the definition of Hamiltonian. So, what you are talking is del h by del u is equal to 0 and then by del l star by del lambda will be nothing but the boundary I mean boundary, sorry this constraint equation sort of thing.

So, it will give us this thing like that del h by del lambda minus x dot equal to zero; that means, x dot is nothing but del h by del lambda and when you talk about h will like that, so what is del h by del lambda, it is nothing but f. So, x dot equal to f of x u t that is nothing but here state equation, so ultimately what you are looking for is what you one landing up a, where you landed up is something like costae equation and optimal control equation now is also called as stationary equation. So, it is just as algebraic equation sort of thing and then this is a state equation actually, so this equation has to be solved together basically that way.

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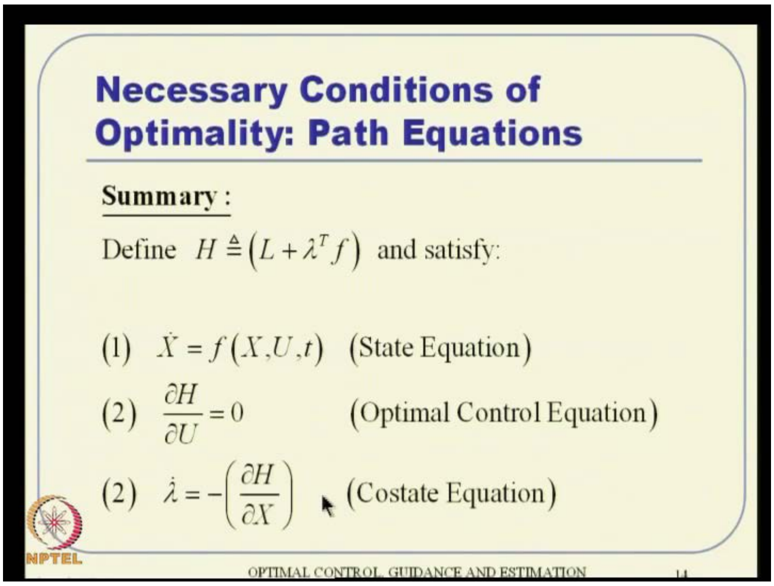

So, this what a summary. Summary is you have to define lamination which nothing but l plus lambda transpose f and then carry out this algebra, well one thing is x dot equal to f of x u t state equation and then del h by del t is equal to zero, optimal control equation, once you solve this equation at every point of time u will be a function of x and lambda alternatively. Hamiltanion is a function of x and lambda, everything, so if you solve this equation for U then we got that optimal control, for solving that you remember U will be function of x and lambda, both and for that you need these two equation together basically, ok.

So, one part of the equation is state equation, other part is costate equation, in regarding that I will talk one or two slides latterly, now what about boundary conditions, ok.

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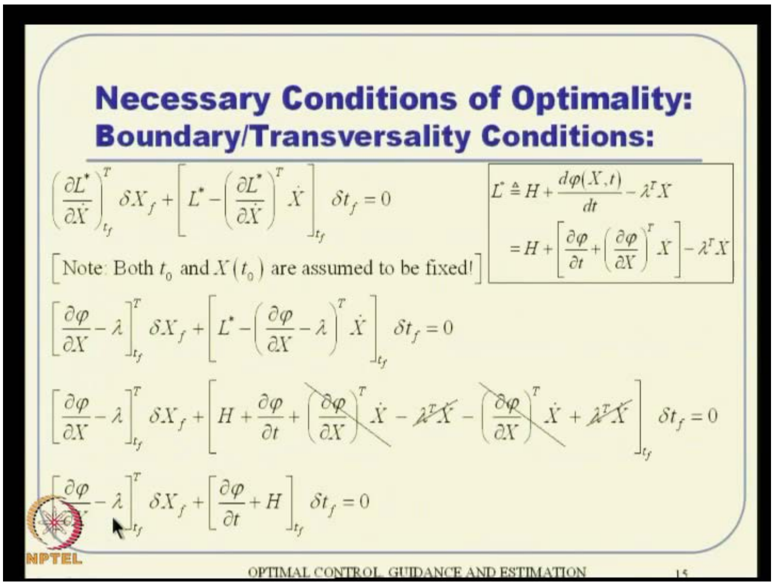

Boundary conditions are again, go back to that that transverse facility condition and all, the general transversality happens to be like that, but L star you remember something like this, this is that when the d phi by d t is nothing but del phi by del t plus del phi by del x into x dot sort of thing. So, you substitute this L star, whatever you see L star is something like this and then we get variety of conditions this way and that way and all, you can derive anything actually. For example, when both t naught and x of t not are assumed to be fixed, then this delta t not and delta x not, though those were 0 by default, we will, I mean we all that we have is delta x f by delta t f, that is the variation actually.

Alright, now what you do is you substitute these expression del l star by del x dot something like that, you are going to carry out these algebra will be more, cancelled out some of these other terms and land up some of these equations like this actually. So, what I mean is when ever this t naught and x of t naught are assumed to be fixed, which is many of the terms which is true, then this is a boundary, I mean transversality conditions, ultimately this will be place of something like that in the form of hamiltonian actually.

And in addition to that many times people tell a case at sometimes delta x f is 0, that means, x f is fixed, then this only these one you left out, delta t f is free, that means hamiltonian at the final time is nothing but minus of del phi by del t. Similarly when t f is fixed, what x f is free, then this is 0, but this is free, then this cannot be here, lambda f equal to del phi by del x actually.

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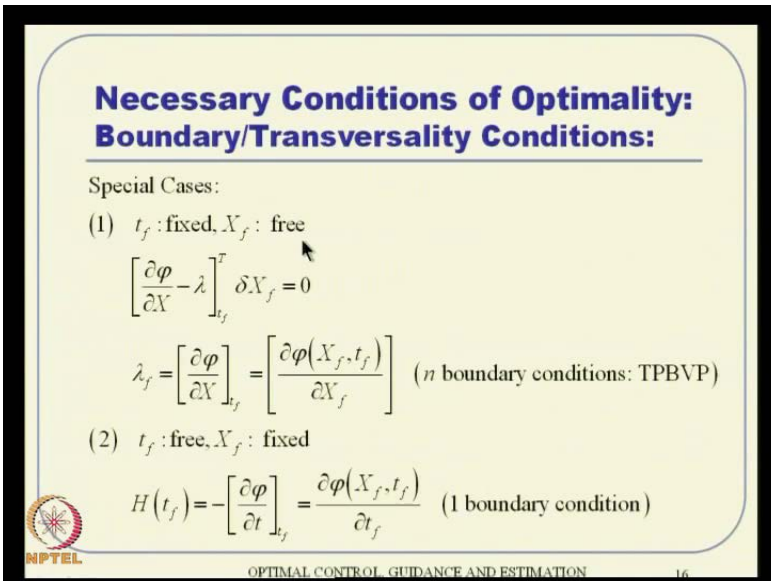

These is what is tried here, when t of f is fixed x of f is free, you land up with this equation, this what I just now described, so that means lambda f will turn of to be del phi by del x f. Actually, when on the other end when t f is free, but x f is fixed, then h of t f will be minus del phi by del t value to the t f and when both are free you have both these necessary anyway. If you x f is free and t f is also free then both these condition will come into the picture and also remember how many condition are there, those kind of phi is a scalar, x is a vector, del phi by del x happen to be n, so that means you are getting n boundary condition from here.

So, how many differential equation you have here, it is also again n, H is is a scalar, x is a vector, so lambda dot is an n dimension is x actually, so that means lambda is also n dimensional in a way you can give that picture. So, sincerely what is going on here, we have problem, optimal control problem and we have a once of differential equation and once of algebric equation, once of boundary conditions and things like that and all of them to be satisfied for meaning full solutions basically, ok.

Now, what about an alternate approach? We do not have to carry out all these algebra and things like that, but in a simplistic sense let as see whether we can have the same solution or not actually.

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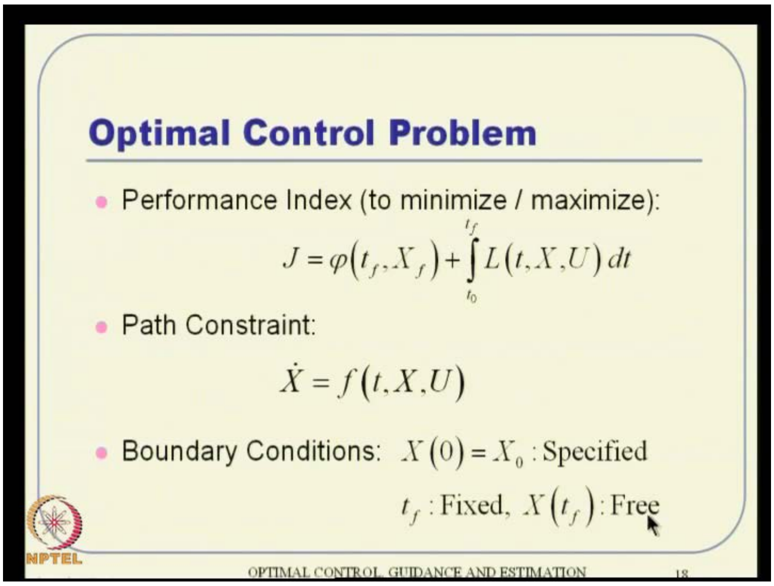

So that the performance index to be minimized, happens to be like this, the same performance index that we started with, path constraints nothing but the state equation and again the boundary condition are there, x of t j, X of 0 is the initial condition which is specified and t f is fixed and x of t f is free actually.

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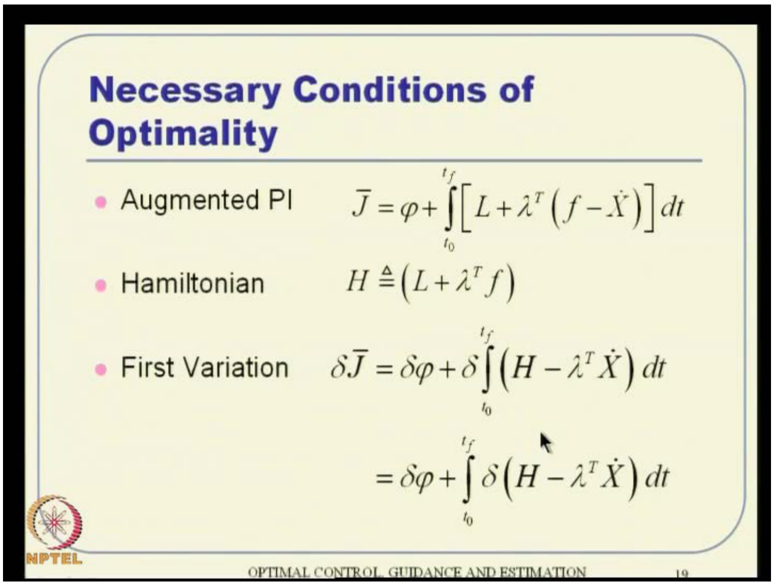

Anyway so this is what it is, then the necessary condition of of optimal value t what you do, will not worry about the calculus of various of e l equation and all we will go back little further and little more before that, then we will concept augmented performance index, j bar is nothing but that. So, L plus lambda transpose f minus X dot, so f minus anywhere, f minus x dot f minus f minus x along the path. So, will take it inside the integral, which is path dependent co structurally, where is something laminoarized here, put is as j bar or something like this, then directly define lamainarization like that, Hamiltonian is same Hamiltonian what we discussed L plus lambda transpose f.

And now we are interested net replying el equation, but we will consider first variation of j bar directly, when we do first variation of j bar is nothing but first variation of phi plus variation of j bar is nothing but first variation of this integral. And remember, the fundamental theorem of calculation of variation that we discussed at time that a variation of integral is integral of variation actually.

And then take this variation inside the integral, so then we talk about this this algebra, the variation of these terms can be expanded and then you can write variation of this is nothing but variation of Hamiltonian plus variation of these two multiplication variation quantities.

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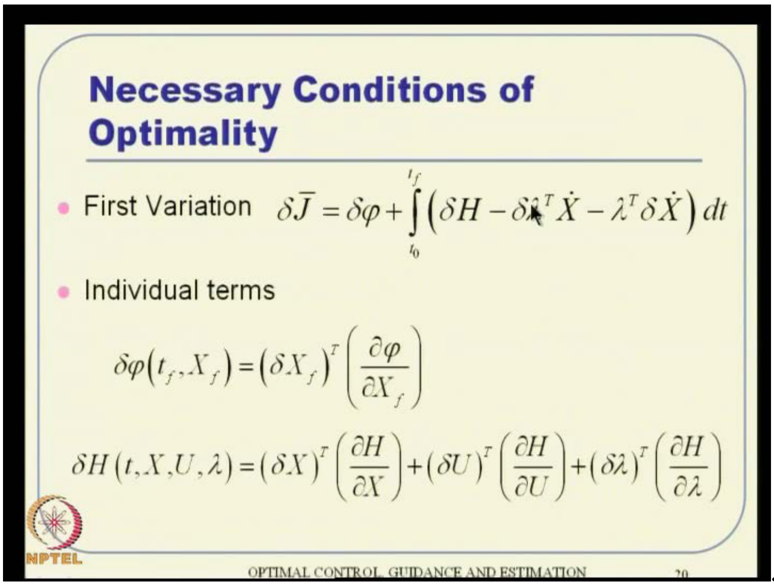

That means variation of this and the variation of that. Sincerely the first variation happens to be like this and now look of the individual term del phi is nothing but, del phi what is del phi? del phi can happen through variations of x f actually. So, del phi nothing but del x of transverse times del phi by del x f, I can write that way. Similarly del H I can write it that way, del h is nothing but, remember x is a function of x v number, all the three quantities actually. So, then variation of h can happen through variation of x through variation of u, through variation of lambda as well and this then essentially contain these three quantities. What about the next? Next is that, that one will consider with integral term, so this with integral.

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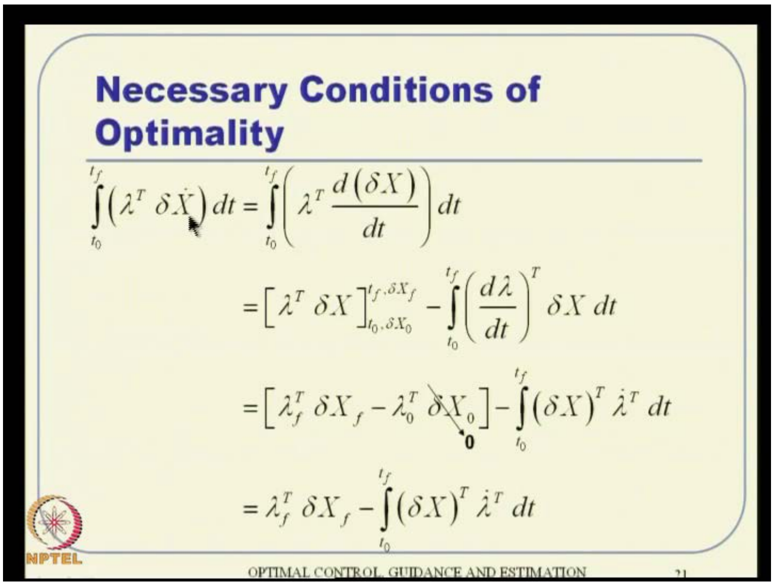

This is nothing but, by definition delta x dot nothing but d y by d t of delta x actually, this by definition of x dot. Or you can exit another calculus of the variation of result, that variation of derivative is nothing but derivative of variation, we also discussed this actually. So, variation of derivative is nothing but derivative of variation actually, then it is ready for applying this this person this integration by points. Then this is nothing but these first one then the integration of second with respect to time is nothing but delta x minus derivation of first one into integral of the second one.

So, this is what it is, so ultimately it leads to this this x to s on of lambda, is transverse to x f, this is lambda f transpose delta x. Well, this experimented evaluated at t f by delta x and minus this expression evaluated at t naught delta x naught, this is how it is. Minus of this, this quantity now can be interchanged and because it is something like x transpose y and x x is a vector, y is a vector of same dimension, then x transpose y is nothing but y transpose x, ultimately it is nothing but x 1 y 1 plus x 2 y 2 plus x 3 y 3 like that, we can exchange it and tell x transpose y is nothing but y transpose x actually.

So, that is what is being going on here, (()) just for I mean for further simplicity we require this delta x team of the left hand side to be compatible with all this expression that we have been doing. Somebody can do other way around, which is also true, this del phi by del x f transposed time delta x f like that actually then you do not need this algebra.

Anyway we have been doing in the left hand side the variations first, so we will be compatible with that actually, just to do that it is put that way (()). And one second, there is small mistake, again here this transpose is not required, yes, this transpose is erased, because it is x transpose y is nothing but y transpose x actually. So, this transpose y is nothing but y transpose x actually, so that is lambda dot the transpose and all we get obviously the multiplication is to x square that that also need to be evaluated anyway. So, this how it is, so this this integral it happens to be something like this, ok.

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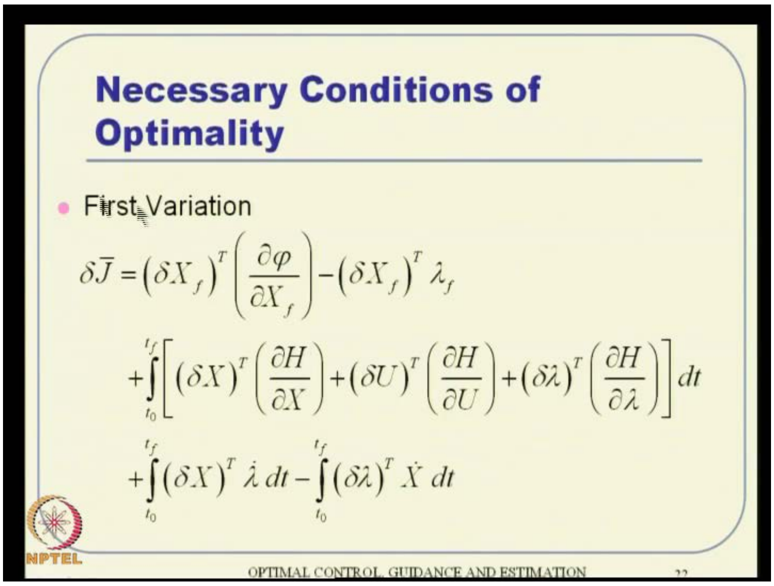

We will put them together and tell this is the first variation, all the quantities that we know, this is the first variation of phi, let it be there and then first variation of all those other quantities lambda, I mean Hamiltonian with these term and then the other quantities are put like this actually. You know this term will go to this term, outside the integral basically, that is how it is (()).

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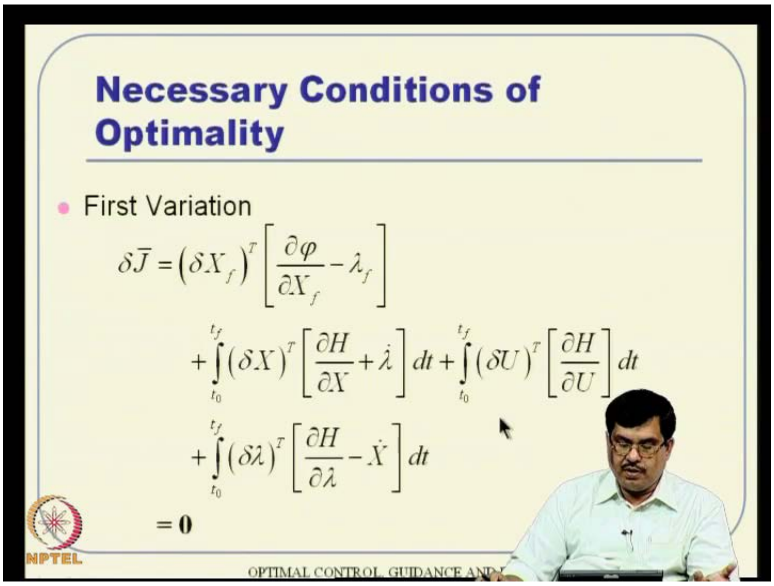

So, what you are looking for? The first variation if you collect all the terms and try to put it together, the first variation del f bar represent to be something like this, delta x f transpose of this plus integral of all these and then plus integral of that actually. So, again this leads to, that we have to say that the first variation represents to be 0 and it happen to be zero for all possible variations of this, this x, I mean x u lambda delta x f everything. Actually I mean the coefficient needs to be 0, this coefficient needs to be 0, this (()) this is the fundamental theorem that we discussed in calculation class also.

For all variation that one function is not 0, then the coefficient has to be 0 in the interval and all that, actually that that theorem you can you can revise also if it is necessary. So, what do you is all the coefficient happens to be 0, so the coefficient means whatever this results from here we will see that in this actually. So, what result from here, this particular equation, x dot is nothing but del h by del lambda and what is Hamiltonian, Hamiltonian is l plus lambda transpose f. So, when you talk about del h by del lambda then it is nothing but f actually, so that is what you are doing, you are doing here.

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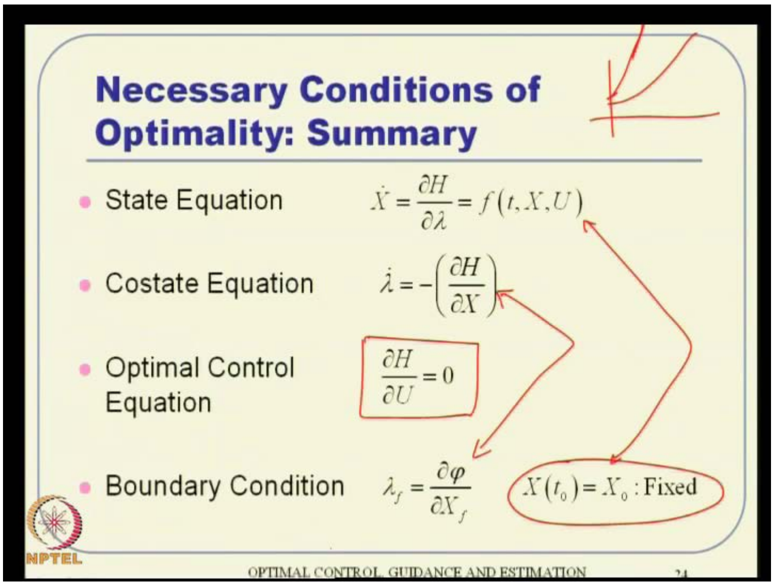

So, x dot equal to del H by del lambda from here and that is nothing but f, so that means x dot equal to f actually and this nothing but state equation. Similarly, if you talk about this equation, lambda dot equal to minus del h by del X, what we derive just before. Now, if you take this coefficient is nothing but del h by del u equal to 0, what we already derived actually. Similarly, the boundary condition this will come from here, lambda f equal to delta phi by del x f, we would not consider other generic boundary condition and all, we have consider where t f is fixed actually, so that is how it is lambda f actually, alright.

So, this is what the summary is, we have this this bunch of equation that is to be satisfied ultimately by that the first state equation, the second is costate equation. And remember the dimension of costate is same as x basically and then third is optimal control equation, del H by del v equal to 0 and the fourth is boundary condition and whatever associated with boundary conditions actually.

And remember if any of these equation is not satisfied then it is a non optimal solution, it not really optimal solution at all. Actually for an optimal control to be, for control to be an optimal control all these equations needs to be satisfied simultaneous actually for the entire duration t naught to t f.

Alright, now this is the problem what you call as a kind of dualarism and things like that, because for solving an n dimension problem we actually look for a differential equations

2 n dimensions face actually. This is n dimension coming from here actually, but the thing is this boundary condition, I mean this initial differential equation is is available along with this boundary condition and this differential equation is available along with that boundary condition. So, boundary conditions are available, it is a matter of only a counting for that actually probably, ok.

Alright, so what you have, what you can see here that is what I am telling here; let me see that. This these two conditions what you are looking for, these two are compatible and these two conditions are also compatible (()), alright and this equation has to be solved for a kind of optimal control. If we solve this algebraic equation u will be a function of x and lambda at any point of time, to get that x and lambda we have to use this differential equation set along with their boundary condition, you have an idea of what is x and what is lambda at that point of time actually.

Alright, some of these comments are given here, so the state state and costate equation are dynamic equations and optimal control is a stationary equation obviously. And the most problematic or other painful part is something there, we know that differential equation and boundary conditions are available that is not a problem, but you simply cannot take any numerical integration algebra like algorithm, like (()) and thing like that and then start propagating, it will not be possible.

Because the boundary conditions are not given at the same time, that is very very critical observation actually. The boundaries are available at the same time for all the variables then you could have simply integrated it either forward or backward that is not a problem. And the point here is you have 2 n differential equation, but n differential equation conditions, initial to its n, I mean for boundary condition sense, for n differential equations you have initial conditions, condition are given at t naught, for rest of the n the differential equations the condition are given at t f for that, ok.

So, that is the why it leads to be difficult of something called two point boundary value problems; so for solving that we need this, this iterative solution procedures mainly because of that actually, ok. What is another observation? You can see this boundary condition the way it is given, the state equations initial condition is given; that means, it double of forward where the costate, the final boundary condition is given, so that means this equation double of backward actually.

Even you can integrate it backwards and remember, the eventually the what I am talking about its lot of numerical intensive algorithm, iterative solution and all and that is why this become computationally very complex actually. There is computational intensive and you read it like iterative numerical procedures, numerical procedures to solve it actually. And even if you solve it this iterative of procedures one vary into two things, I do not have the initial conditions for lambda but let me guess initial conditions for lambda and then the initial conditions for both x and lambda will be available, then I can integrate it further and let be equal to t f. I will see what my integrated value of lambda is and whether you satisfied this equation or not.

If you do not satisfiy I have to come back and corrected it control equation, (()) try to satisfy in the next iteration basically. So, that kind of ideas are something for suiting method and all, will also talk about that in the next class actually, but the other ways of doing that (()). So, come up to the point, point here is we really required to solve two end differential equations and this process is called realization actually, that means you are at this lambda happens to operating in the dual place also and you got two dimensional differential equations to get of here.

So, is here is actually learning of with realization of a problem actually and this realization problem is difficult to because it is it leads to this two point what really problem. And one more comment on the way is this third equation happens to be stable most likely this costate equation will happen to be unstable actually.

Just now the idea that I told that initially just some lambda zero integrated further and all, it is not really very good, because what we are talking about for a stable system this equation is any one unstable. And anyone to unstable the differential equation numerical integration the error of the initial condition guess will be amplified quite (()) basically, ok.

That means, to a pictorial sense if you have some on stabled differential equation, if you start with these initial conditions opposite goes off that way, then if you start with little bit differential equation it will go that way. That means, separation between them happens to grow a quite a lot actually, that means an error value get amplified as well basically.

So, that is the problem of even numerical integration actually. And then the ideas of multiple suiting and all do not suited too much and do not propagate too much, you break the trajectory into parts and all that, that you break it again, at it is sound all difficulties, how to do you have see a continuity with this, all that those things will discuss in numerical method procedure class itself (()).

Any way all these things are given here and also any more after all these difficulties suppose if you are able to solve it ultimately you are solving it for a particular…, what you are doing here, if I am solving it, we are solving it for a given initial conditions only basically. That means, we are actually solving an open loop control strategy, the control is not enclosed (()) core loop sense basically.

So, we have the difficulties of a optimal control problem found as in general problem formulation is nice, but you will end up with this computational difficulty, is use in all that of this is this problem is called this curse of complexity actually. So, this is a what you were looking for is all (()) you were lending up with a situation, you have dealt with complex problem usually. So, instead of n dimensional problem we will end up with 2 n dimensional problems and then the boundary concern is split, then the nature of the differential equation is opposite. And then unless is always you do not get a control anyway, actually at least you have iterative procedure and think like that way. So, this is called curse of complexity (()), ok. Alright, so ultimately even if you do all that and finally it leads to an open loop control solution ideas actually (()).

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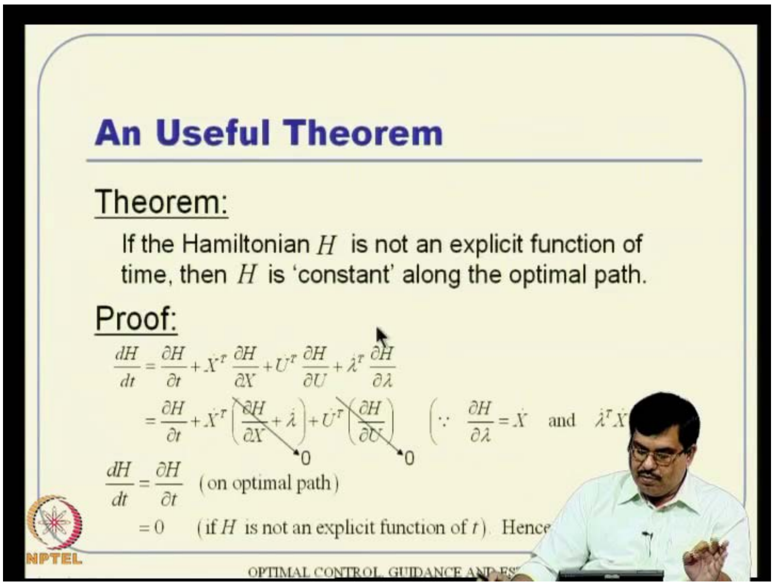

Alright, on the way there is a different comment here; one theorem tells that if the Hamiltonian h is not an explicit function of time H is not to be constant along the optimal path. Why so, because you have content d H by d t is nothing but del H by del t plus this variations coming through this X u and lambda, so X dot transpose times this plus u dot transpose times that lambda dot transpose time that and all that actually. Now, you combine this two quantities, this X dot transpose times del H by del X plus this lambda dot and all that actually, because del H by del X, del by del lambda is nothing but X dot. So, and you remember lambda dot times lambda dot transpose X dot is nothing but X dot times X dot transpose times lambda.

And lambda dot actually this algebra, what you were looking (()) del h by del lambda is is nothing but X dot and lambda dot transpose times is X dot is nothing but X dot transpose times lambda dot. So, if I do this algebra and substitute were here and I can observe it here, then I tell ok this is nothing but my costate equation and this nothing but my optimal control equation and I am walking on optimal path only way, that means all these equation are satisfied. Once you satisfied then d X by d t is nothing but del H by del t and that is equal to zero because by definition H is not an explicit function of time, so any partial derivative of H with respect to t 0. That means d H by d t happens to be 0, that means, what does it tells you it tells us that if H is not an explicit function of time then H has to remain constant along the optimal path, ok.

And typically for a regulative problem and all will see that H is not only constant but that constant happens to be 0 also basically, thus H will becomes 0 at t equals to infinity, but H is a constant anyway, so if it is 0 at 1 point of time then it has to be 0 everywhere actually. So, using that idea then it turns out that in general H is constant value for whatever problem is that and optimal path as long as it is an explicit function of time. And most of the time we do have this similar situation actually, this those state equation and all will not be explicit function of time constant parameters as well as especially time invariant systems and all we need to talk about that.

So, for these class of problems H as to be constant and that can be a verification to be numerical solution procedures also, once you get a solution you can plot H versus time and ever feeling whether it is constant or not, actually that can be verification tool (()).

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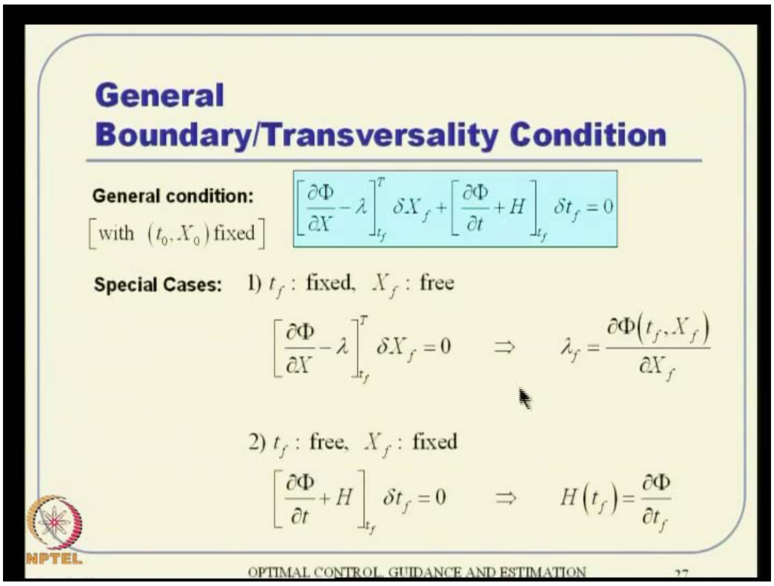

Anyway, so come up to the general boundary boundary condition or transversal condition and all that, so this is what it is. We can, this, I mean the similar ideas that we discuss in calculus of variations, we can you can talk here also. Start some something like this and then depending on various various cases case 1, case 2 case 3 all that you can you can derive the corresponding equations and all. Primarily, we were interested in these two cases most of the time, so in one case t f has to be fixed, other case t f is free and t f fixed and X f free, then you will you from here you can see that an t f is fixed, this is 0, you are up to something like this, that means lambda f is nothing but del phi by del x f.

And in certain case t f is free, but X f is fixed, that is kind of hish constant formulation we can think about with free final time, time doesn't matter, but ultimately the error has to be 0 that kind of situation, then this pair has to be 0 actually, that means h of t f is del phi by del t f rather here. So, this is how you would use this conditions as a one which is necessary in a particular formulation actually, (()) now with all those in ideas, (()) all these in mind, especially this conditions and all, will discuss this some example problems again to get our ideas clear.

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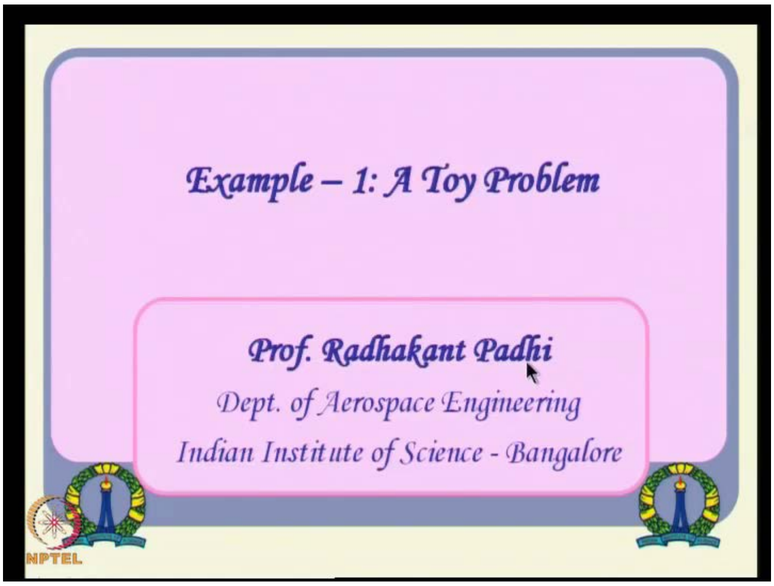

First, we start with a something called a toy problem, not very challenging, but it gives us some clarity about now to do you use this, this ideas and all.

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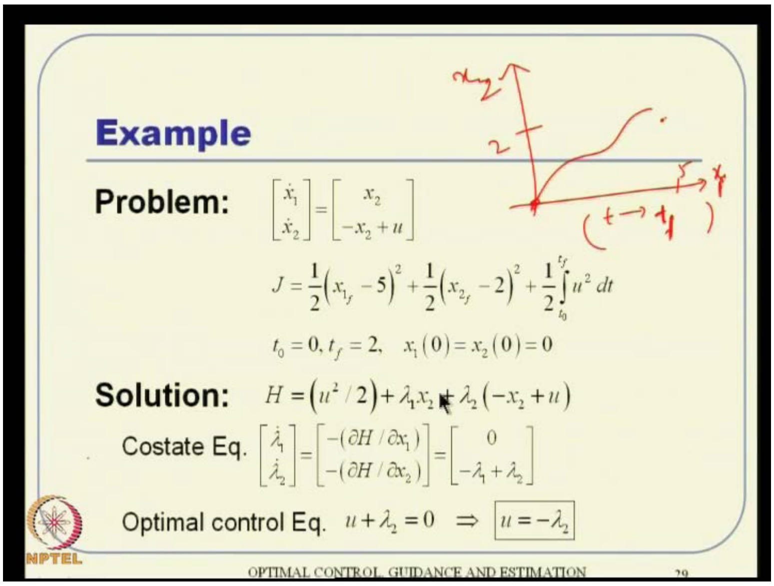

So, let we start with this, let us start with this. So, this X1 dot is X2 and X2 dot is minus X2 plus u. I think your problem cannot be simpler, then that is actually a liners, second order linear equation actually, linear system dynamic sense. And J is nothing but this kind of thing, so what we are looking for is at t equal to t f and t f is nothing but two actually, that also valuable. At t equal to t f our state values X1 and X2 should remain close to 5 and 2, we are not interested in exactly meeting 5 and 2, as long as these terms are minimized, we are ok actually, that means you can start with this initial condition which is 00.

This you can start with the initial condition 00 and at t f if t if its t f t goes to t f and 1 our value has to be something like if you are plotting in terms of X1 and X2 let us say, ok. So, t equal to 0 is starts 00, when t goes to t f, let me write it properly, (()) when t goes to t f what happens here is that we are interested in taking this trajectory, I do not know how it will work, but ultimately this is there is a 52 value, this is pi let us say and this is 2 basically, ok.

So, this point it does not have to go there, but it has to remain close to that value. So, that is what we are looking for actually, it may not go there, proceed, but there is a point 52 and this is 2, go double often, stay somewhere close to that that point actually, that is the whole idea of that actually. If it refers to the exactly meeting we are looking actually, but in general its do not you need not bother about that what I mentioned. So, what I doing here, we have start with the Hamiltonian definition, Hamiltonian is L plus lambda transpose f, so that means L is nothing but this half u square plus lambda transpose f, f is f 1, f 2 towards this. So, lambda 1 times f 1, which is X2 plus lambda 2 times f 2, which is minus X2 plus u, so that is a definition Hamiltonian.

Now, costate equation tells us that lambda dot is is minus del h by del x, that means lambda 1 dot is minus del h by del X1 and lambda 2 dot is nothing but minus del h by del X2. So, lambda 1 dot del H by del X1, it does not contain any X1 term, so that is 0 and lambda 2 dot is del H by del X2, that means which is lambda 1, X 2 is here, which is minus lambda 2. Again take minus sign of that that means minus lambda 1 plus lambda 2, that is how you get I mean costate equation. Remember, state equation already given, no need to derive it, this is there, now costate equation is already there by this algebra and optimal control tells us that del H by del u has to be equal to 0. So, del h by del u one term is u and second term is lambda 2, that means u is nothing but minus lambda 2, as simple as that.

As long as we get lambda 2, we are done actually, that means once lambda 2 value is available, u is nothing but minus of that, is very simple, it looks actually depures very simple this one a type. So, let us see how most effect is needed to get this lambda 2, anyway so u is nothing but minus lambda 2, so I can substitute it here, if I want to minus X2, minus lambda 2, I can write actually, ok.

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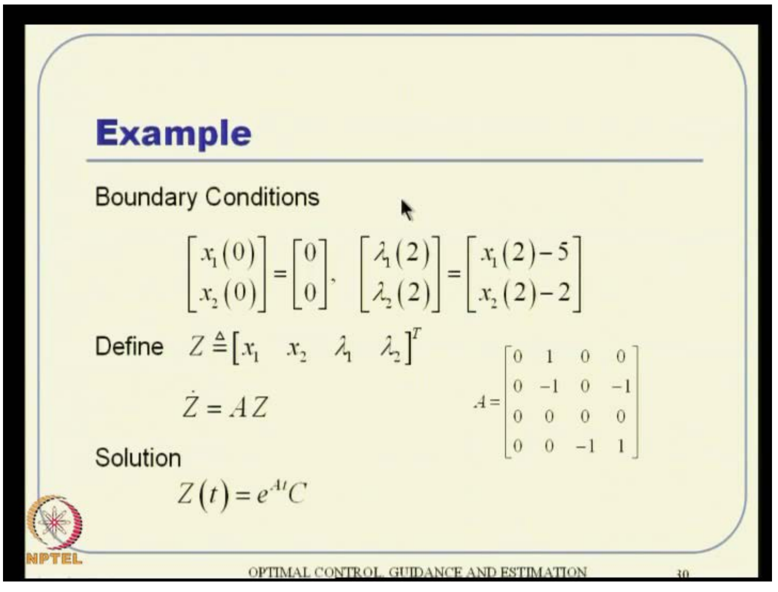

Anyway boundary conditions are like this x1 of 0 and x2 of 0 or 00 and lambda 1 of 2, because you remember the x f is free anyway, right, it is only close to that value, what it is in truly seeking is free. So, we can talk about using corresponding boundary condition will tells us lambda f is nothing but del 5 f del x f. So, lambda f, which is lambda 1 of 2 and lambda 2 of 2, it del 5 by del x f and what is 5 here, 5 is this term. So, the lambda 1 is nothing but lambda 1 f nothing but del 5 by del x f, that means del 5e divided by del x1 f, this is first term, this is all it is.

Similarly, lambda 2 of f t f is lambda 2 of anything, that actually… Now, how we solve this equations that more very important, now the differential equation, boundary conditions everything is available and our main aim is to solve for lambda 2, but you cannot do that as in isolation, use are lambda 2 dot what you see here is also function of lambda 1. So, lambda one dot has to be accounted for this equation, for persionalating this equation its its happen the lambda 1 dot is on zero and lambda 1 is constant value.

So, that it kind of (()) some simplicity actually and you will not go through those, it what we will consider, these some like at, I mean Z vector, Z it contains x and lambda to gather that, z contain x 1 and x2 lambda 1 and lambda 2 together. Now, if we look at the differential equation this is a linear equation and this is also a linear equation, so I can put them together and right in the form of Z dot equal to Z, where a a happens to be like that actually, ok.

Alright, so x 1 dot is x2, this is what will get if you get that, x1 dot is 0 times x1 plus 1 time x2 plus 0 times lambda 1 plus 0 times lambda 2 that is how we get it. And similarly we have all over things use if it end end up this this elementary features, so when you help this linear differential equation of this form, Z dot is equal to Z, the solution we know Z of t is nothing but Z to the power C.

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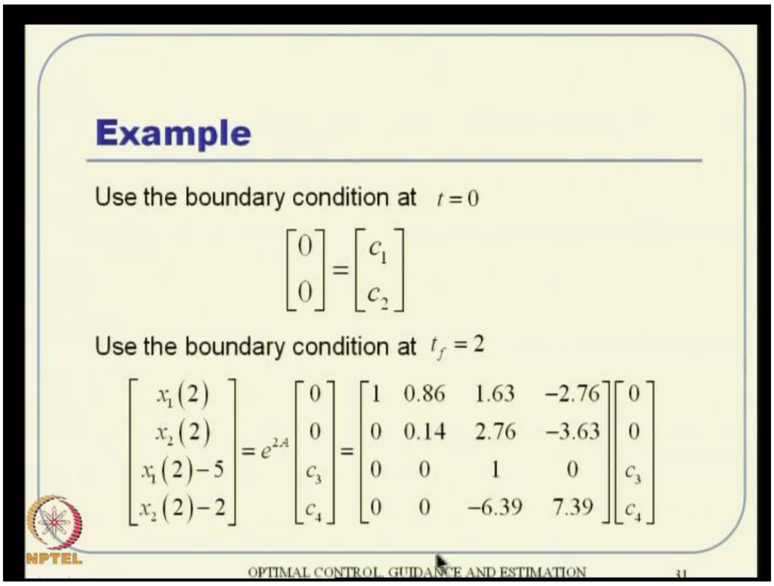

So, now we use the boundary condition at t equal to 0, then t equal to 0 t to the power a plus identity, so that is what will do, then ultimately you get this this seriously. If the power a plus is I basically right, so will end of C, so C1 and C2 happens to be 00, the form, the from the very first conditions actually, this is 00 sort of thing, right. So, you have this condition appearing, it will give us e 1 and e 1 is 0 and what about the other thing, other thing has to be applied here actually. So, when you apply t f equal to 2 we are looking for x1 of t f, x2 of t f, then lambda 1 of t f and lambda 2 of t f, but lambda 1 of t f is this 1 and lambda 1 of t is that 1, we just derive this, ok.

So, this is what will put it here and that these equation here. Again now w to power lambda lambda plus and t f is 2, so if the power 2 essentially and a is nothing but that. So, you can evaluate if the power 2 lambda represents to be like this actually. Now, what I mean looking at we will looking at this equation, what we are looking at, we are looking at variables as x1 of 2 and x 2 of 2 and C3, C4, when this equation what we are having here is 4 equation with 4 unknown quantities, so x1 of 2, x2 of 2 and C3 and C4.

So, these four unknown equations and four equations we can solve it and while solving it we have to rearrange in this equation little bit and you to take this prevariables into one side in the vector and all that, so that will do and will end of with some equation like that.

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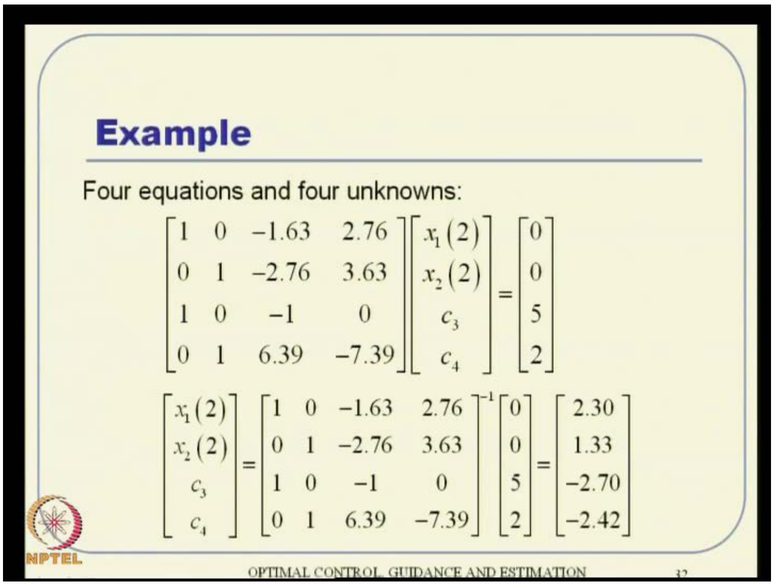

So, it something but a x equal be a form, you can solve it for that, now you get the values for x1 of 2, x2 of 2, C3 and C4 itself in the series of all that.

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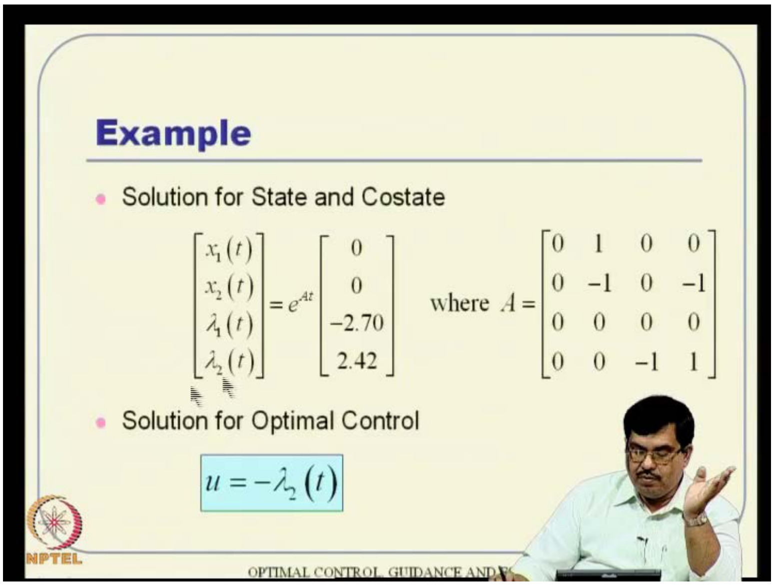

That is where we will get the solution ultimately, x1 of t, x2 of t, lambda 1 of t, lambda 2 of t, that what here, z vector. z vector contains x and lambda 2, get the way, so we got the lambda basically in the process, lambda 1 what you are looking for you got the control has to be u equal to minus lambda 2 of t, that is what our main priority ambition was actually, got it.

Remember, to get this value it is so much difficult, because even though we started with the very simple looking for that, we absorb minus lambda 2, to get the lambda 2, it is not that easier process actually, it goes through lot of algebra and then ultimately you will get it there.

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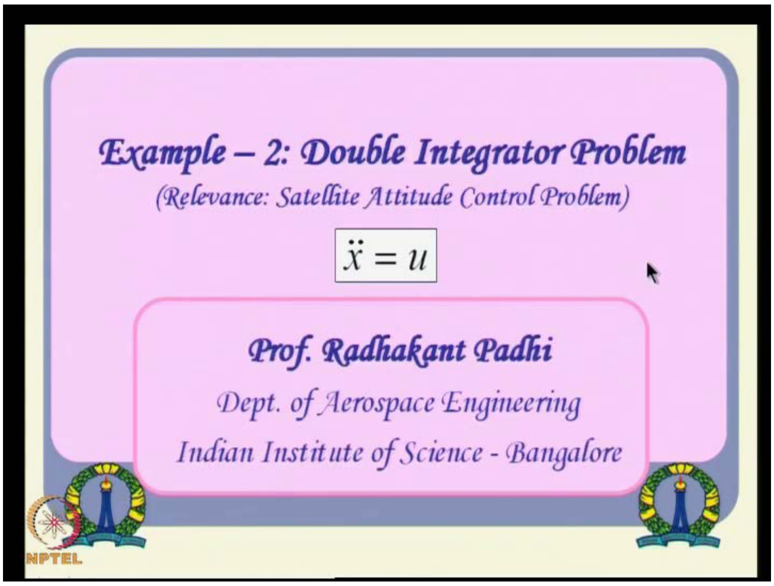

What about the second example? It will more complicated, if well, not that much of complicated either, but it is more relevant of particular application, which is like double integrator problem, x double dot u. The relevance of that you can think about satellite attitude control problem without this this dumping terms and all that actually, if this I w dot is a function of something if it all control torque in a different exist over there. So, this is double integrated problem are very popular primarily because of those kind of ideas, the the aero dynamic must they no draft, no drift, it just to tumbling of mass basically of all of things, that is how it is, our double integrated problems are quite relevant in those kind of equations. But in general, it is not there are more complications and all you should approximately it is is valid actually.

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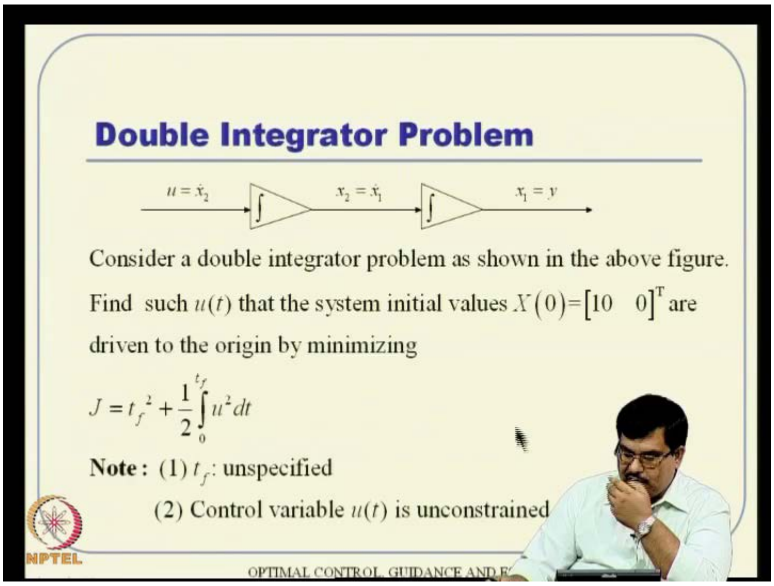

This problem that we are talking we want this double integrator system and we want to minimize this. Remember, final time is to minimized as well, final time is not fixed here, but if will unspecified, it has to be minimum, minimum time problems add or you think about, where it'll not only exactly minimum time, what we want is this t f to be minimum, along with the condition that control has to minimum as well. So, that is why this control dependency is also available, I mean also part of the coefficient. So, t f is unspecified and control variable u of t is unconstrained actually, that is what we are looking for, ok.

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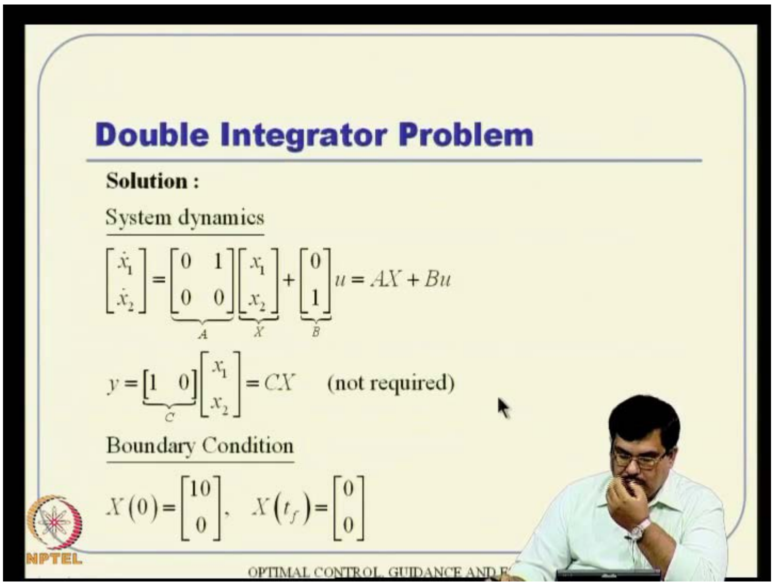

So, the double integrate problem happens to be like this, system dynamics is the linear form Ax plus Bu, y is applied again, it not required really, but somewhere you can think of Y is may x1 set of something actually. So, x dot is a x plus u, y is equal to CX and the boundary condition is X of 0 is 10 0, X of t f is 00, that it, this is what, you ten is the angle and 0 is the attitude, velocity you can think. So, than the moving, the tumbling mass is some some 10 degree deviation initially relates to it and you want to be need to 00 actually, ok, so that is what objective actually.

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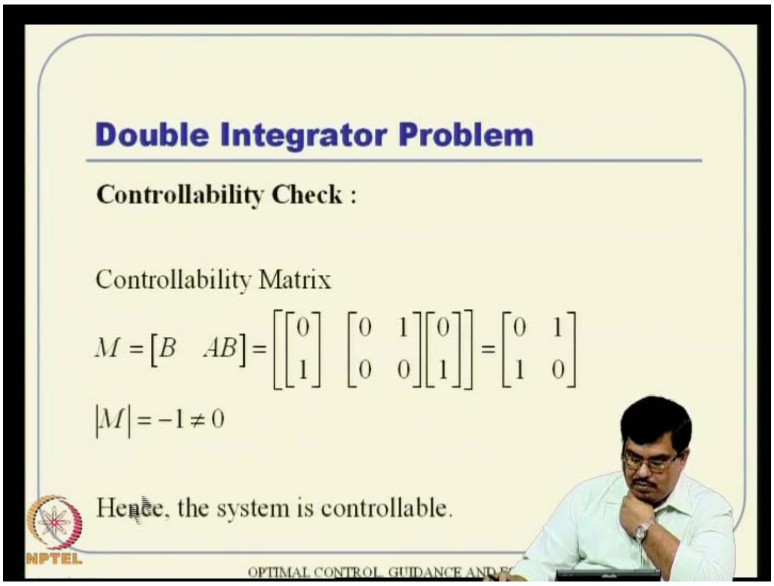

First thing, first we will control the controllabity check, you should not forget that the system will not control, you cannot do anything. So, the next wants to quickly check, quickly do a controllability check for any problem, so you do that and then turns out that determinant of m is minus one and more importantly it is not equal to 0, hence the system is controllable. Well, you can do that, you can proceed with the our optimal control equations and all. So, first thing is Hamiltonian, which is half of u square coming from this term, else that is half of u square plus lambda transpose f.

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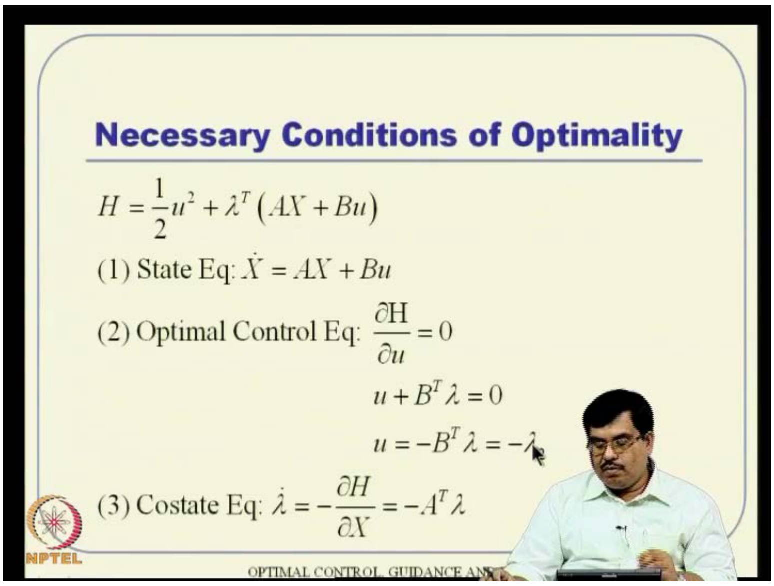

Lambda transpose f is nothing but AX plus Bu, lambda transpose f, lambda transpose f, f is but AX plus Bu. So, x of equal to a x plus b u that is what it is here, so it is like that. So, the state equation already we know, costate equation is lambda dot equals minus del H by del X we can see that. So, minus del H by del X from here is nothing but minus a transpose lambda, only this term. And optimal control del H by del u is equal to 0 and del H del u equal to 0 means del H by del u is nothing but u, I mean coming from here, u plus 1 more term B transpose lambda coming here. So, u is nothing but minus are in well minus r inverse in general, provided you have this this r metrics here.

Ok, but if the r is identity here, so we will not worry about that, talk about u equal to nothing but minus B transpose lambda and B is B is something like this, B metrics like this. So, if you substitute there and again we will end up with up with u equal to nothing but minus lambda 2, well then how difficulty is that to get the lambda 2 let us see that. So, the costate b u is to be satisfied simultaneously, actually.

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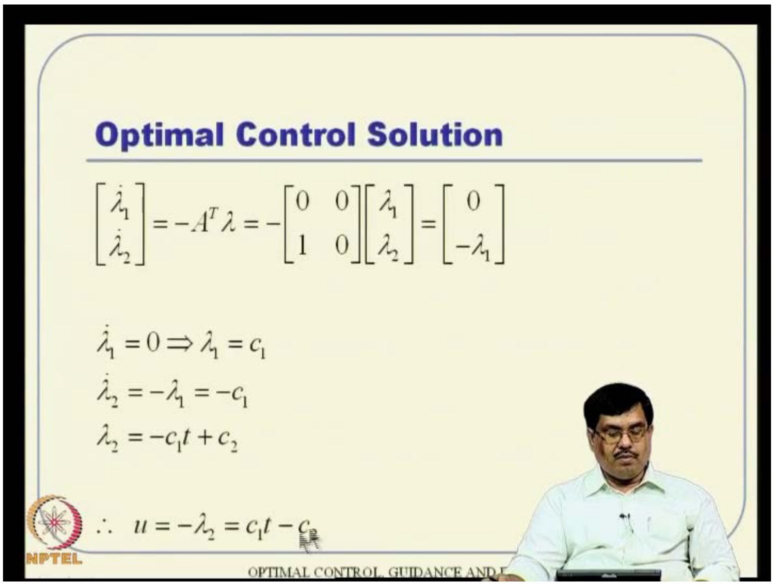

Now, lets us go to the optimal control solution thing, so lambda 1 dot, lambda 2 dot is minus a transpose lambda 2 dot is minus lambda 1, so that is minus C1, so lambda 2 represents minus C1 t plus C2 in this equation. So, u is nothing but minus lambda 2, so it is C1 t and C2, as long as we know C1 and C2 for then our control is available actually.

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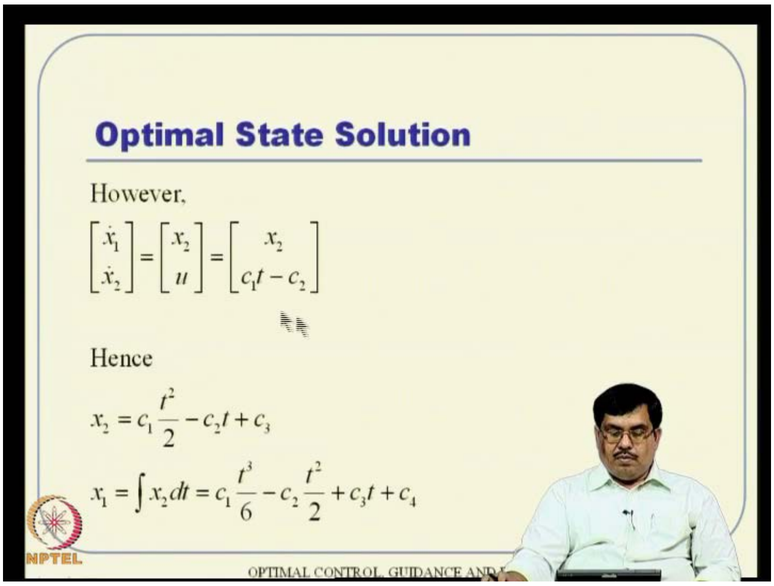

So, go back to the state equation now and tell x1 dot is x2, x 5 dot is u and u is nothing but these and hence a solution sense what happens to here, thus x2 dot is available explicitly in time variable now, that is nice nice actually. Then I can talk about x2 is integration of that is nothing but this kind of thing, C 1 t square by minus C2 times t plus C3, but x1 can be integrated again and I will end up with this. So, x can can be integrated again and I will end up with this integrating one more and hence I have it actually. So, x1 of 2 is like this and x2 of 2 is like that, ok.

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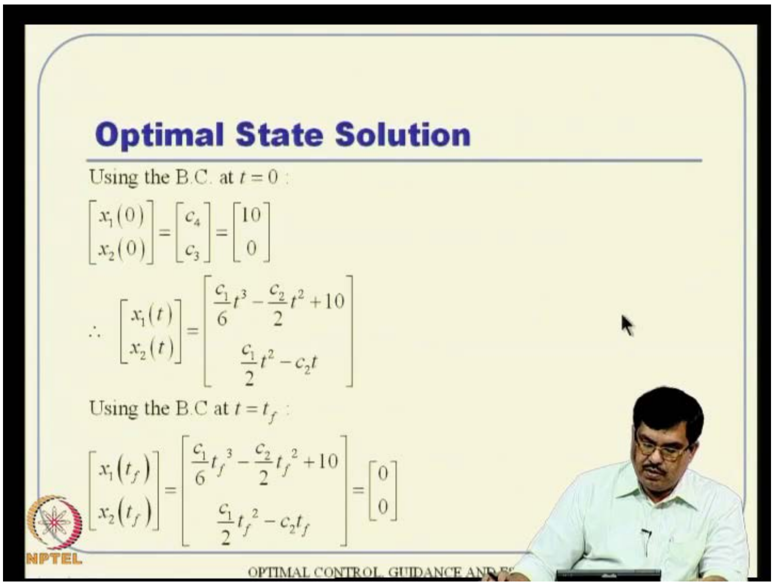

Now, what about the boundary conditions? So, using the we will use the boundary conditions at equal 0, so that turns out out to be like this tends 0 and hence we attempt to solve, I mean we got this C4 and C3, as you get that that is eliminated, we are left out with C1 and C2 . So, this is like this.

So, how do you get that actually, now we are to use this boundary conditions for t f also. So, you use t f and then this is what the result is, x of t f and x2 of t f is 00, that is what our aim is actually. Our aim is try it to 00, we put 00 here, so this is the equation. And what you what you observe here, the equation contains there variables C1, C2 and t f, but the number of equation of two actually, so we need one more equation.

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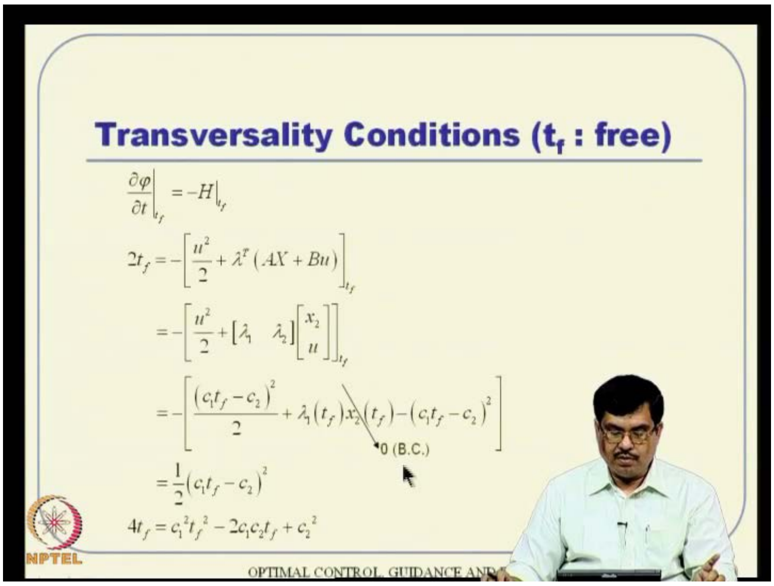

Now, the one more equation is transversality condition, because t f is free we can write it that way, del phi by del t is h of t f and you substitute all the necessary things you will end up with some constant like this, ok.

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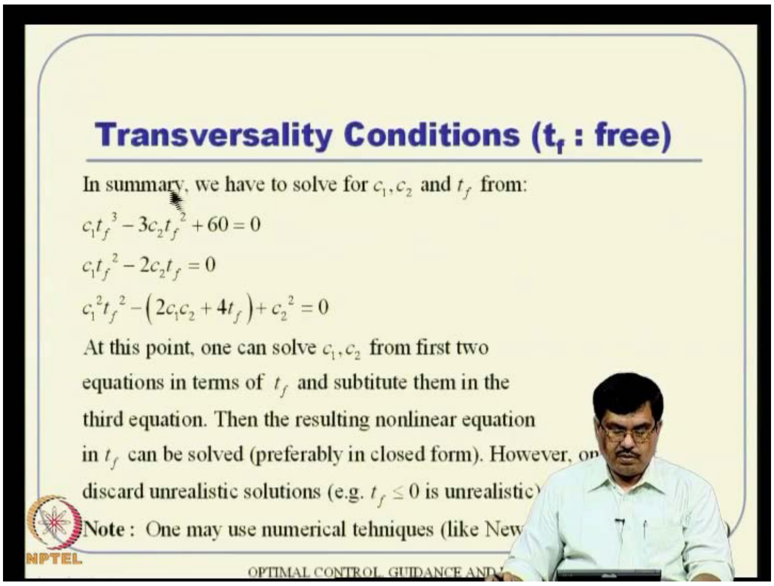

So, what you are looking for, what you are telling is we now have three equations and three variables, so there is chance to solve it. But, remember this is this is a set of equation that are non-linear, there are terms like t f square c multiplied with c a C1 and then C2 t f square, C1, C2 all sort of thing are there.

So, sincerely it is not that easy to solve it, but looking at this equation itself we can think first two equations are linear in C1 and C2 at least, I can consider there is an equation for C1 and C2 which is linear. So, it is possible for me to write it in the form of AX plus B u sort of things and then tried to eliminate C1 and C2 as a function of t f. Then I put that some that function in t f here, whatever I get C1 and C2 and all that and then I will end up with a huge polynomial expression for t f. Everything will be a function of t f actually, then I can find the roots of that and then get get my t f and once again my t f C1 and C2 will be available, ok.

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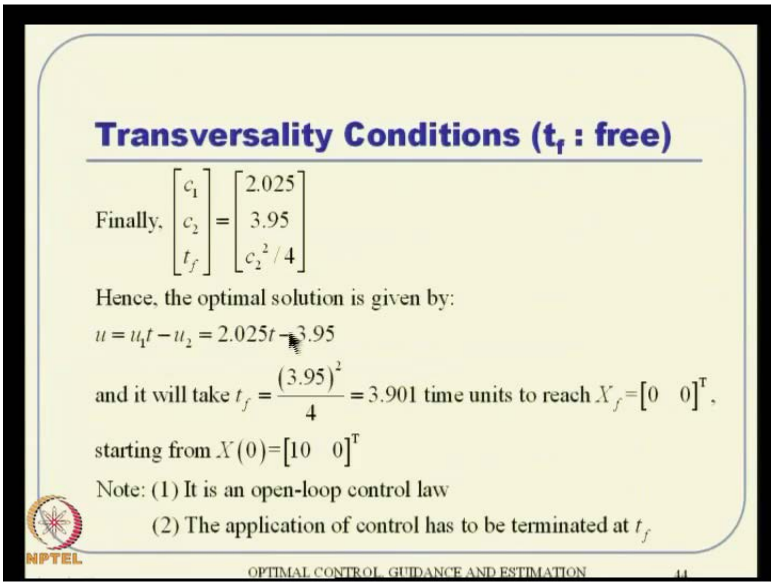

And ultimately otherwise somewhat you can always think of doing it numerical solutions way talking the help of Newton rap son technique, this also possible actually, alright. So, finally, we will end up with some some expressions like this if you do all that and hence the control solution will be something like C1 and C2 was our primary concern, ok.

Well, this is this small mistake again, this is now u 1, this is c 1 and this c 2 (()), so C1 of t minus C2 sort of thing is C1 like this and C2 like that and t f is also available, t f is nothing but that, so you can calculate that. So, you can also tell this will not only erase this tumbling, but it will also take that much of time to erase the domain actually.

So, essentially it is an open loop control law, because it is a function of purely a function of time and application of control has to be terminated at t f, if you keep on applying this beyond t f then, there is no gurantee at all actually, that should be a mechanism of stopping the control at t f actually, that is more important.

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Another class of problem before you wind up this this particular class, you can think that function of state variables prescribed at fixed prime of t f, we know that that kind of thing actually.

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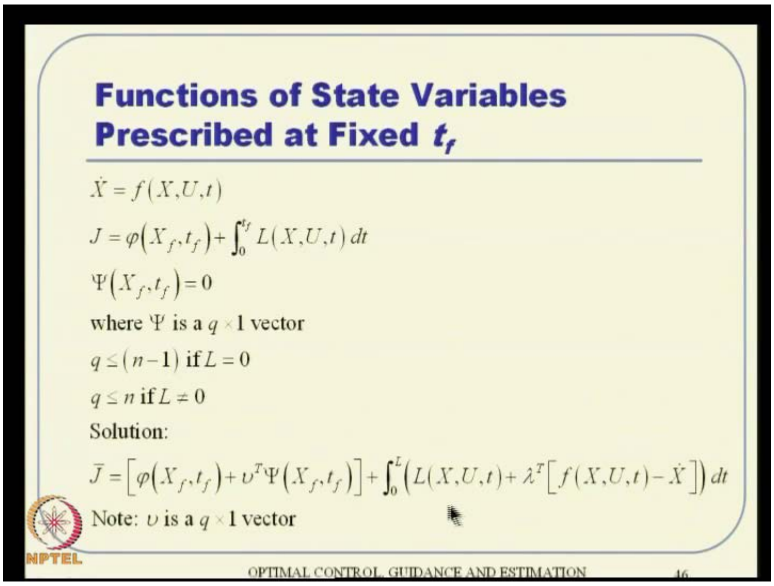

So, you have a concern something like this, X dot equal to like that, J has to minimized where this additional constant is coming into picture, this an algebraic constraint. The function of states at t f equal to 0 basically, how do you handle this?

This can be handled with this auxiliary condition sort of thing, where you you consider this further, I mean some sort of Lagrange variable, one Lagrange variable happening outside the integral, this this happens inside the integral, because of state equation. Because of this constraint equation this gets coupled with this condition, that you have it happens to outside the integral. Now, you can apply your regular optimal control, consider this as some sort of a phi function and what about you have here is something like al function actually.

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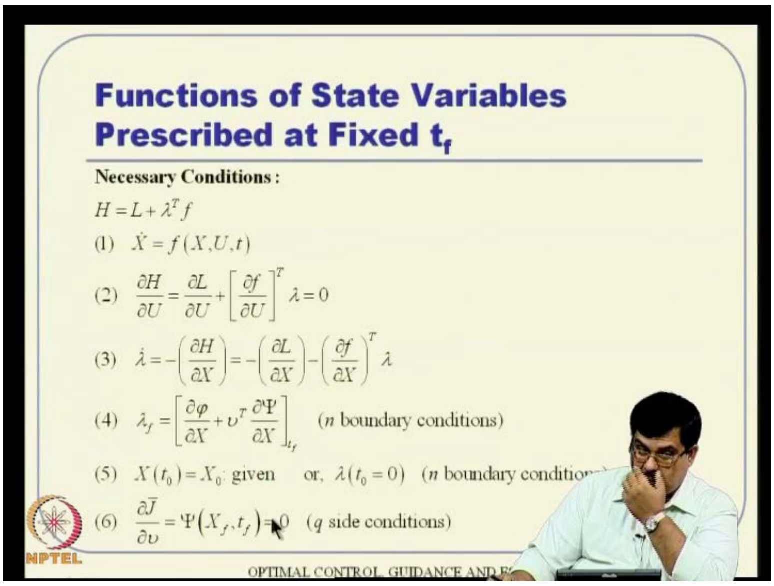

Do that this thing, what you are looking for, what you are looking here is very similar to the conditions what lambda f will not be only a function of del phi by del x, but it will also contain this, lambda f and we will also let (( )) consider this del j bar by del u is some sort of a side conditions actually they occur. So, you have to solve both lambda f as well as mu to get the answers actually.

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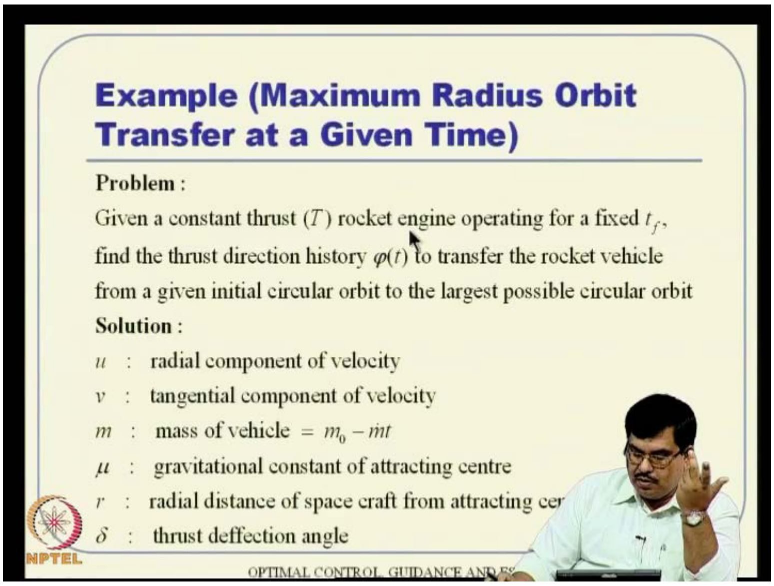

So, a quick example of that is emiteral transpose, where you talk about a rocket engine operating at some t f and things like that.

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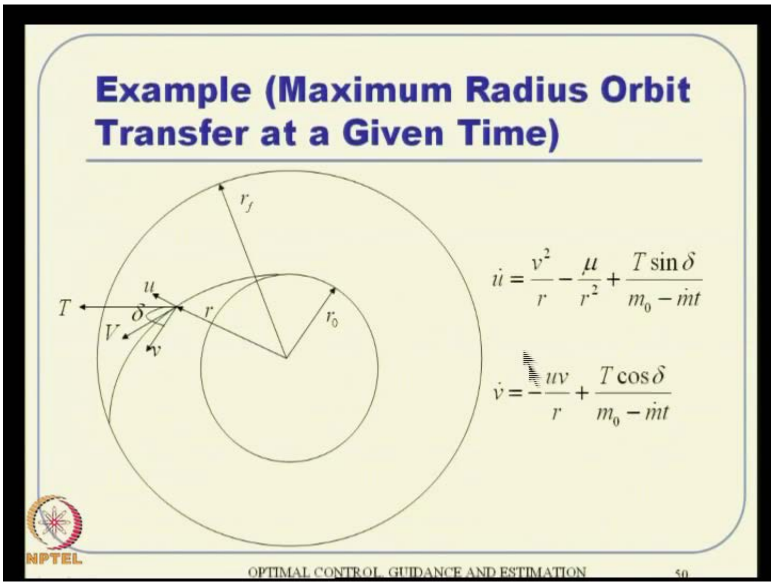

So, sincerely some satellite was there or some v2 was there and then it must be transport to a different orbit using the thrust mechanism; you can operate a new intermissive system if it starts here and our ultimate aim is to merge with this orbit actually. So, the dynamics are (( )) mean equations are like this.

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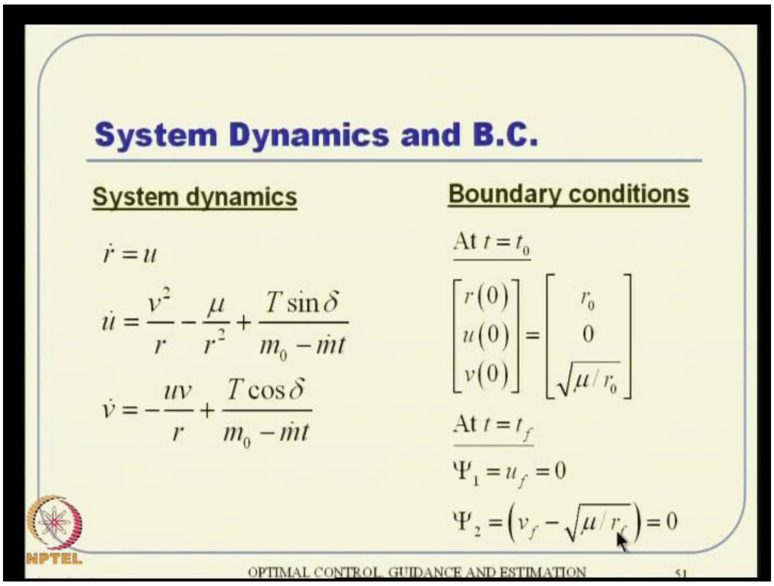

And then you can talk about system dynamics in these three and the boundary conditions finally. Initially the boundary conditions are the the orbit one condition and the finally, the boundary condition will be some sort of something like this, this u f has to be 0, ultimately this is of a perpendicular velocity has to be 0 and tangential velocity has to be maintained actually, that is your orbitary velocity sort of thing. So, these conditions come as some additional constraint equation outside the integral actually, so this equation if you do that, an algebra and Hamiltonian and all that we define and ultimately we will end up with this equation, this regional equation, this costate equation, optimal control and all that.

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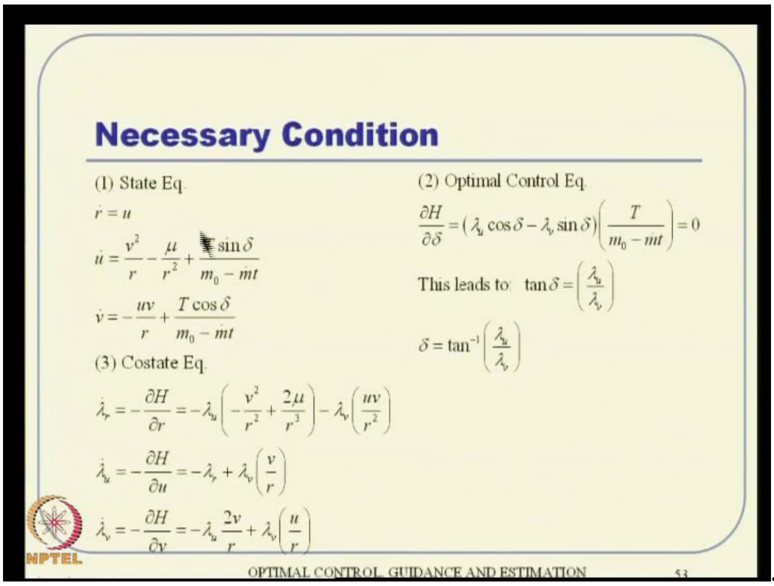

And obviously this equations cannot be solved using this (( )) radius in close form. So, essentially the point is here, the message here is to get these values, what you are looking for is control value like this, but to get these values you certainly knew numerical techniques actually. So, in general, particularly relevant problems will typically leave to numerical solutions as well, so we we have to gear up towards that. The boundary conditions are also split, so essentially what you are looking for is just that this state equation, costate equation, optimal control equation and then boundary conditions.

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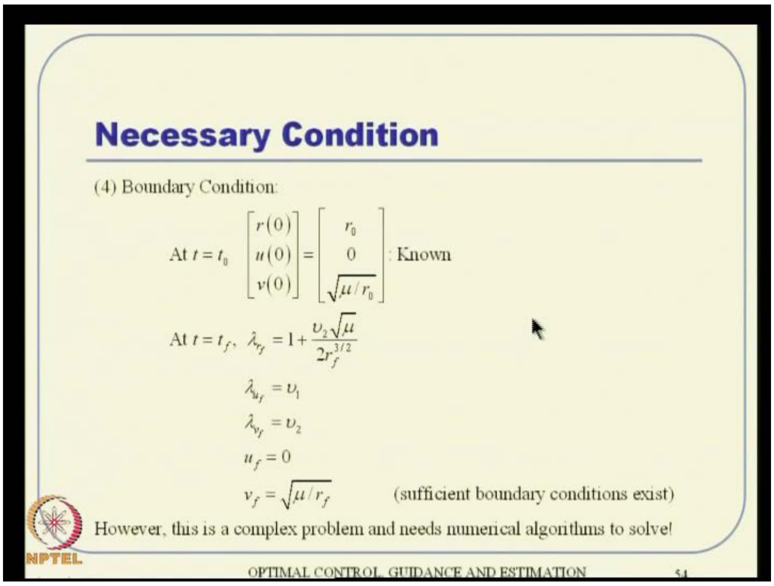

So, say little bit complex condition. Conditions are there, I mean the equations and conditions are available, just that it is not possible to solve between close forms and hence, we need numerical solutions actually, alright, that is the message that I want to give in this class. So, we will see this numerical numerical techniques and all in the next class actually; thank you.