Optimal Control Guidance and Estimation

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Module No. # 03

Lecture No. # 07

Review of Calculus of Variations – 11

Hello every one, we will continue our lecture series on this optimal control Guidance and estimation course.

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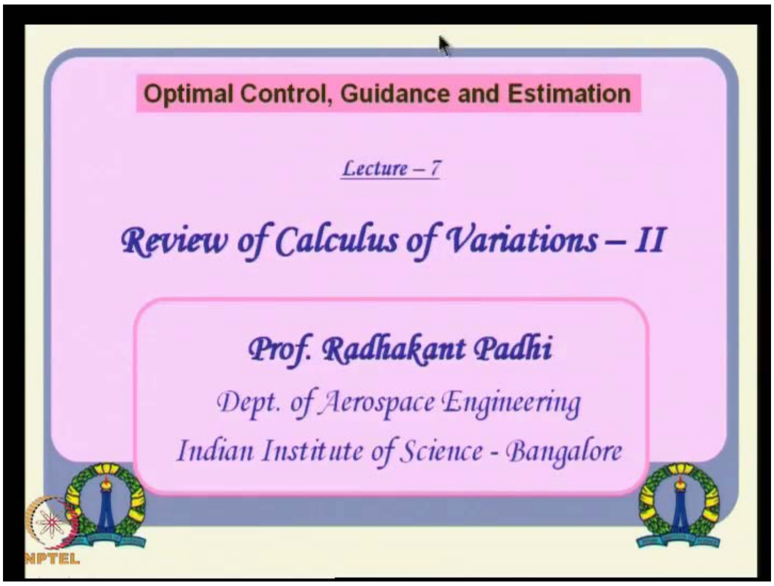

We have finished as six lectures already and we all to do already, get some over view of calculus of variations is scalar. I mean, scalar variables what I am going to do here, in this particular lecture is to review that briefly, what we discuss in the last class followed by its extension to vector. I mean vector consumption all that multi dimensional things. This is very similar to scalars anyway.

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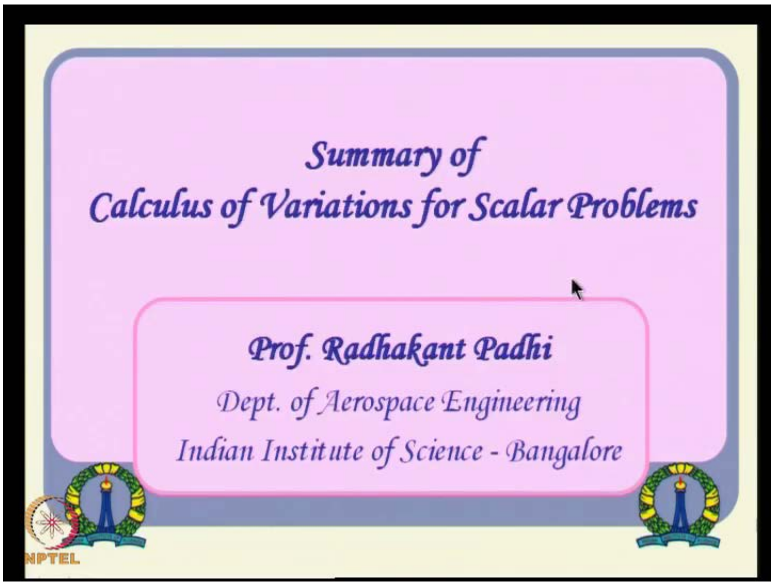

So, first thing is summary of what we discuss in the last lecture, the calculus of variations for scalar problems especially.

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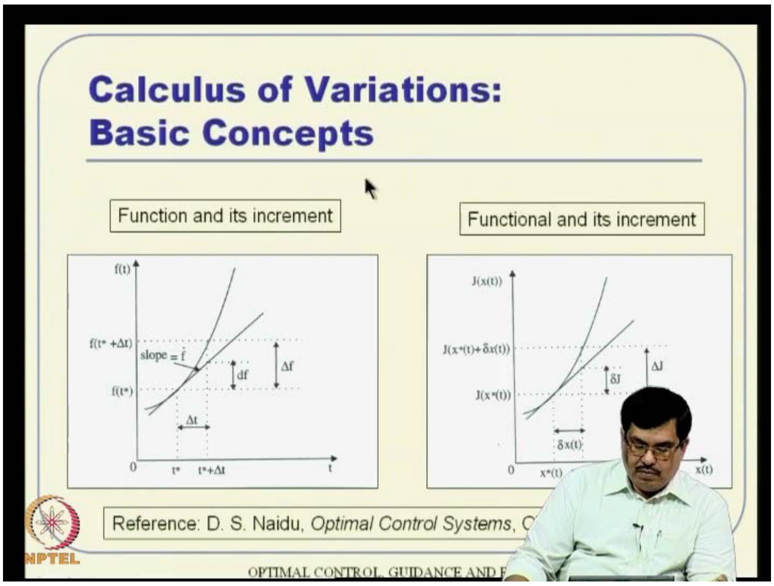

So, this is the concept that we discussed, we discussed something like on one side is a function and its increment and the other side is a functional and its increment. And both the appear to be quite a similar provided the x x is turns out to be like, in this particular case it is free variable time whereas, in this it is the x of t basically. So, it you can interpret is something like a function of functions sort of things and that is, that how?

Where this? The this delta x can be a variation with respect to time. Then if it varies with respect to time then, how this J varies? That is ultimately the implication basically. Then we discussed many things about an increment of a function is this delta f approximated is d f similarly, increment is delta J approximated is del J.

Slight notification change this is d f was it is delta J sort of thing. So, just to may say that this is a variations.

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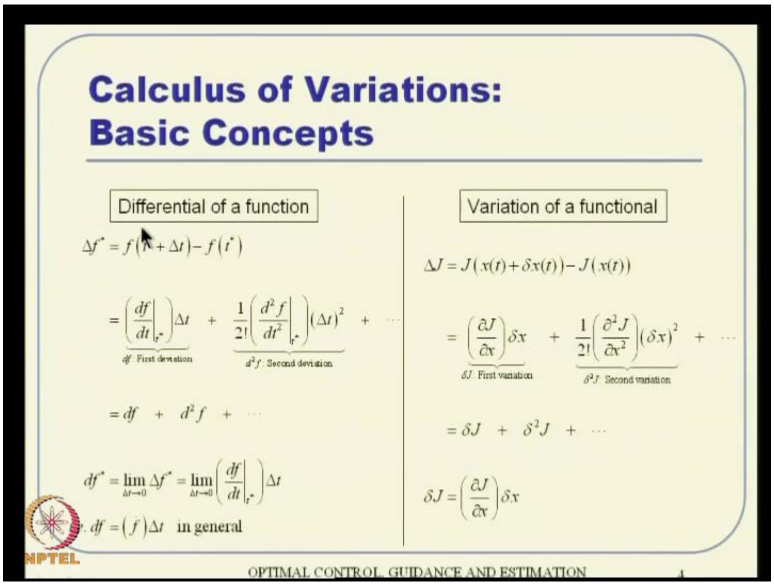

Then we discuss about differential of a function and variation of a functional, think like that and we carried out this algebra. And using this taylor series, we learned it up there kind of expression for delta f and then it turns out that, when delta t goes to zero and we learned of something like d f. So, in general you can tell d f is some thing like f dot delta t in general basically. So, very similar (()) concept you can do this way also here and you can take first variation, second variation thing like that, the first variation is can be computed this way. And as I told last class somebody does not tell where the (()) default variation means, first variations can be computed this way.

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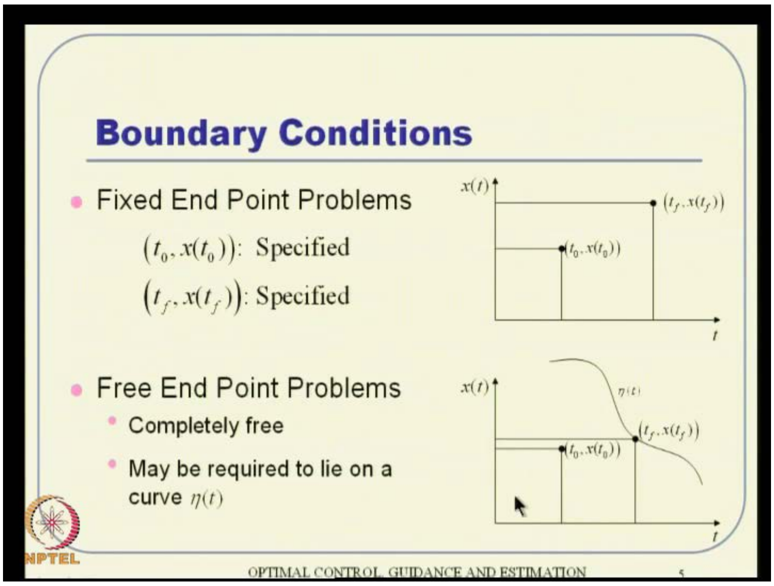

We also gave examples, how to compute? And think like that in the last class actually. So, then coming to the boundary conditions we discussed two class of boundary conditions, and the fixed end point problems, and free end point problems. Fixed end point problem can be kind of fixed both in time and state our time in the free variable; sorry this time is the free variables and x is the dependent variable. It can be fixed at both sense, t zero at t zero x can be fixed at t of x can, x can also be fixed or it can also be something like free end point problem, where the end point can be completely free or it can it can be required to lie on a specific curve and I gave you an example of satellite launch and all that on there.

When you launch a satellite does not matter where you join there we from there onwards it will keep on staying in the orbit anyway. So, all that it matters is us to leave it somewhere in the orbit or orbitral condition actually. So, that is that is how? This problems are we find at way (()).

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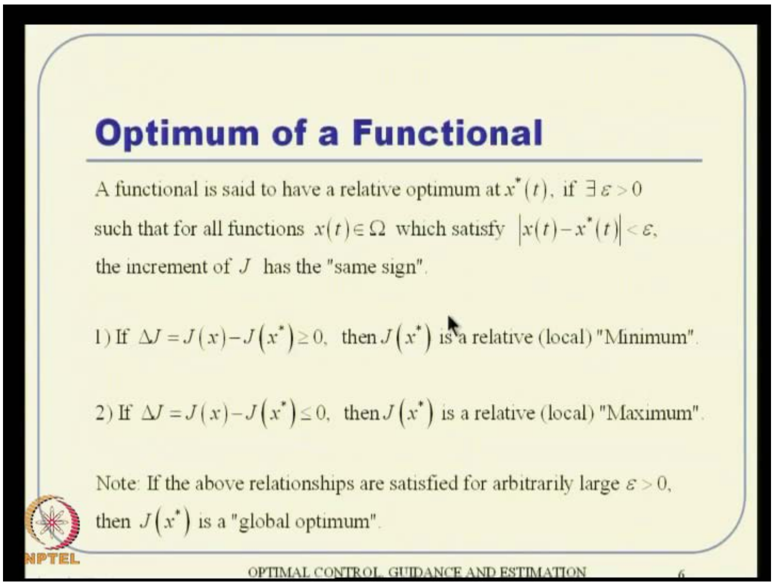

So, then optimum of a functional we also discussed and we had this formal definition that functional is said to have a relative optimum at x star of t remember is not a particular point, but x star itself is a function of time. So, it is functional is said to relative optimum at x star of t. If the; if there exists some epsilon is greater than zero such that the function I mean this x of t will satisfy this kind of a condition. Whenever it satisfies this then you have this increment of J should have the same sign.

That means if we are talking about the minimum then this delta J, J of x minus J of x star should always be positive, no matter what kind of variations we talk about? If whatever variation you have either up or down, whatever it is actually, irrespect to whatever variation you have delta J will keep on the thing positive, that mean J of x star is suppose to have a local minimum. Once similarly, if delta J is less than zero then J of x star is suppose, to have a local maximum actually. And obviously, if the relationships are satisfied arbitrary large value for I mean is satisfied for arbitrary large value of epsilon, then J of x star is said to have a global optimum actually.

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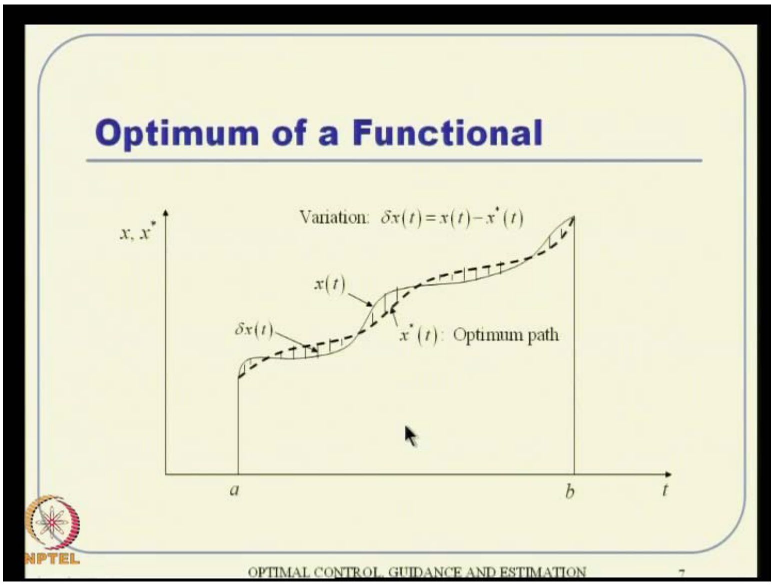

And we also discussed about this variation concept on only x star represents to be an optimum path, then all that we are talking about is neighborhood path of that actually. So, whatever neighbors we are getting that kind of that kind of a thing we are talking about actually.

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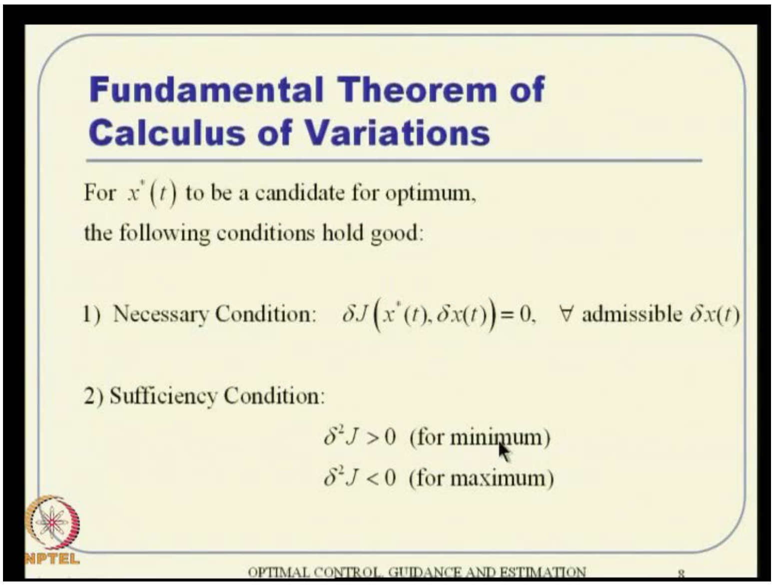

So, for then we discussed about some fundamental theorem of calculus of variation and it turns out that, for x star of t to be a candidate optimum the following condition has to hold good. That means, the first variation has to be zero for all admissible delta x. This delta x turns out to be the difference between these two actually. So, for all admissible delta x, your delta J has to be equal to zero, that happens to be a necessary condition. And a sufficiency condition happens to be second variation should be greater than zero for a minimum or second variation should be less than zero for a maximum. And when it I mean this concept of kind of you can think about peri parallel to the static optimization, what had the ideas of calculus of variation of other.

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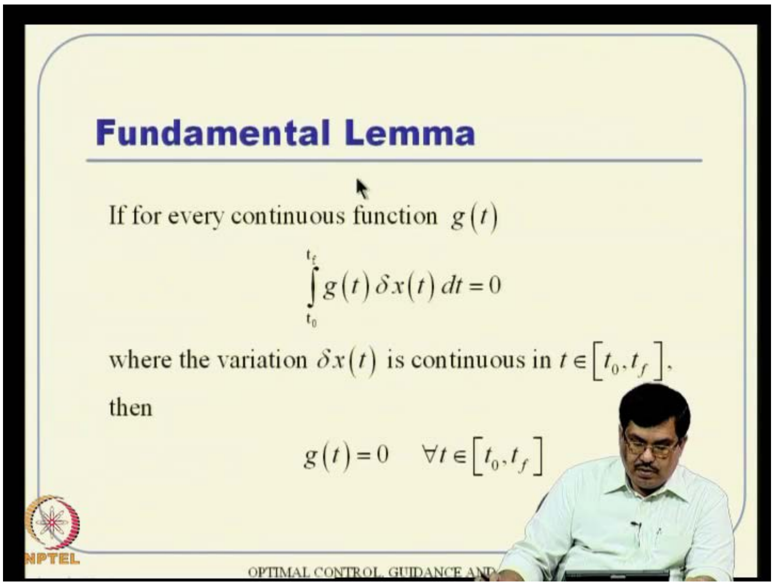

We also discuss this nice beautiful little fundamental lemma, then it will stole that if for every continuous function g of t this equation satisfied, this integral equation is satisfied. Where the variation delta x is continuous in t, then the I mean the only condition that is required is delta x has to be continued that all. Then it turns out that g of t has to be 0 and the entire in to (()). So, that is that is nice property and then many many times we will use it also. And this was a very simple proof also we took at a interval and then in that interval, we took the function everywhere 0, but in that it is not 0.

And we can always construct a delta x, which is again non 0 in that particular interval and this integral happens to be non 0. If you do that then this count I mean may thus the kind of I mean counter arguments sort of things. So, were; So, taking help of that we could. So, that it is it cannot happen. So, method proof by contradicts and all that. So, then we landed up with the conclusion that, g of t has to be 0 it is looks as a very simple theorem, but there is lot of great integrations later.

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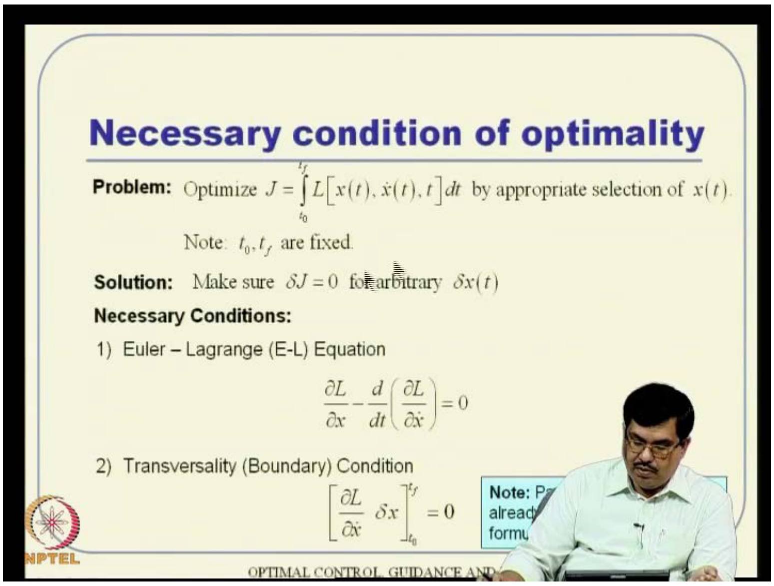

Whenever conditions of optimality then we defined the problem that we have to optimize this kind of a function, where t zero and t f both are fixed and all that you to do is to make sure that the first variation is zero. And after some algebra the, that I mean using this partial fraction of, I mean partial fraction of this integral and all that. We could able to show that, it get ultimately leads to this two conditions and the first condition is called E L equations or Euler Lagrange Equation and the second one is Transversality Condition, which leads to this boundary condition sort of things.

And we also noted that part of this equation might be already satisfied by problem formulation, you have the word if here x zero is fixed it is not a free, then d d x not is already zero. So, this condition you already satisfied actually, whatever is not satisfied you have to, you can extract from there. Whatever you satisfied you already satisfied any way. So, even though these equation is valid for the entire path from t not to t f, but this conditions are valid for t zero and t f only basically. There is a differential equation from this will give a corresponding differential equation as we give a corresponding boundary condition sort of things. So, using this two we suppose to solve the optimal I mean dynamic optimization problems.

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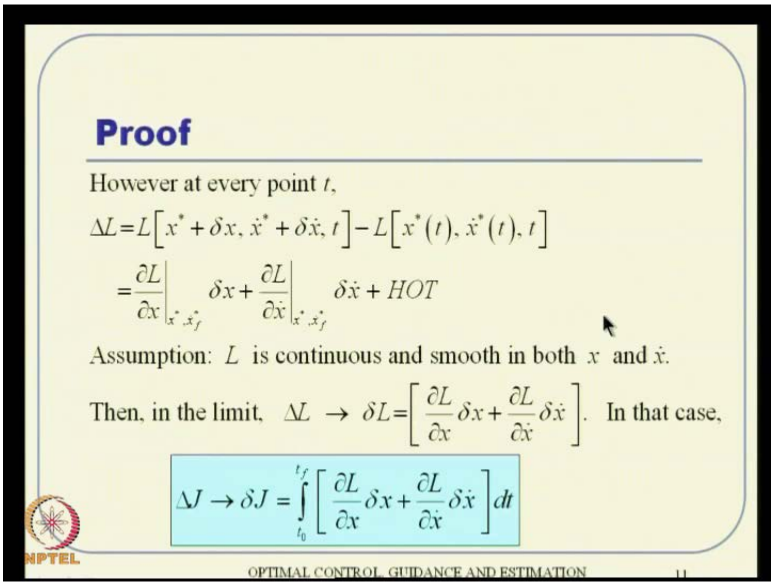

So, now this quickly this prove that that followed you something like this it has the delta L we took at and delta L is by definition like this. And we expanded using taylor's series the this first term, first term in derivative and then either terms. And then if you take in the limit delta L will be become delta L, this delta L and then that turns out to be like that. So, this delta J can be approached matter something like this and then this particular path I mean this second part of the integral.

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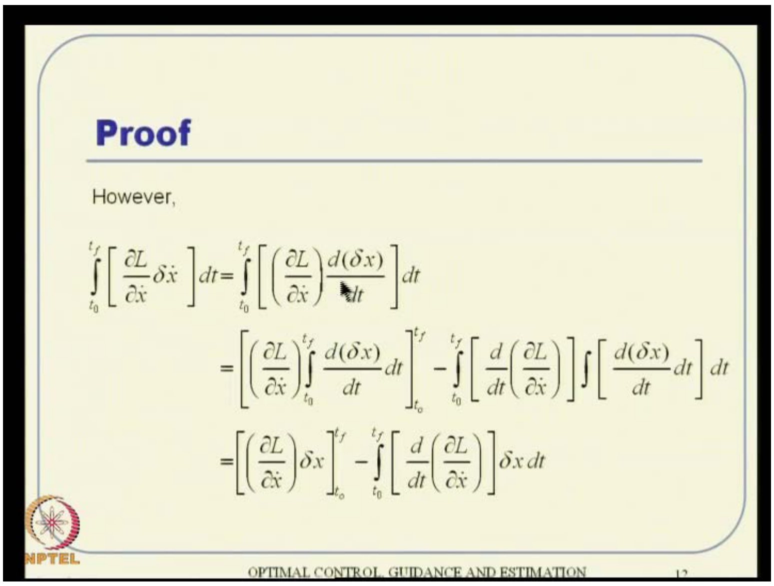

You can kind of integrate by parts. So, then do this I mean algebra and then kind of with this kind of conclusion. This part being like that and then you can put it by here and then ultimately it will list to that. So, if this has to be zero, then this has to be zero and that has to be zero and that is how? We got the; this two condition of optimality that I was talk actually. So, the same trick will all good for vector algebra also, that is why I thought of kind of reversing if again.

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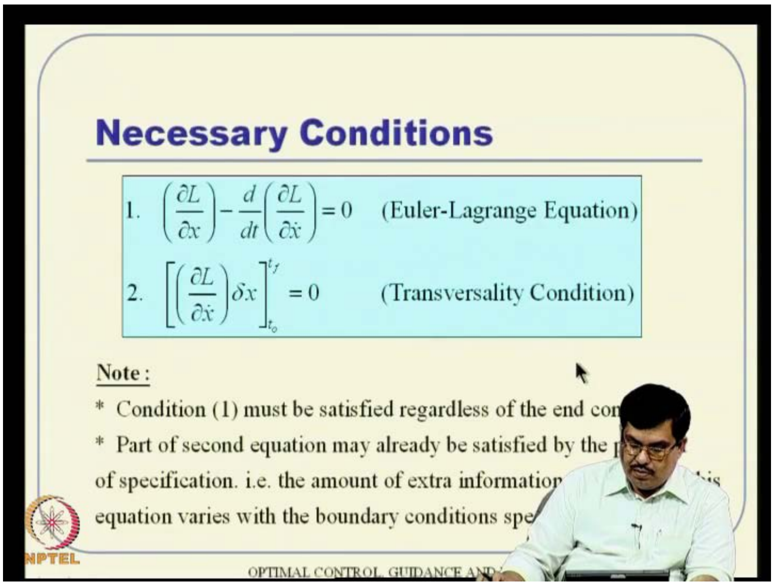

So, this ultimately leads to this two condition and condition one must be satisfied regardless of the boundary condition or end condition. But second part of the equation; that means, this equation we already, we satisfied for the problem specification. So, the amount of extra information contained by this equation, that is with the boundary condition specified actually so; that means, depending on the specified boundary conditions here to extract additional information from this condition.

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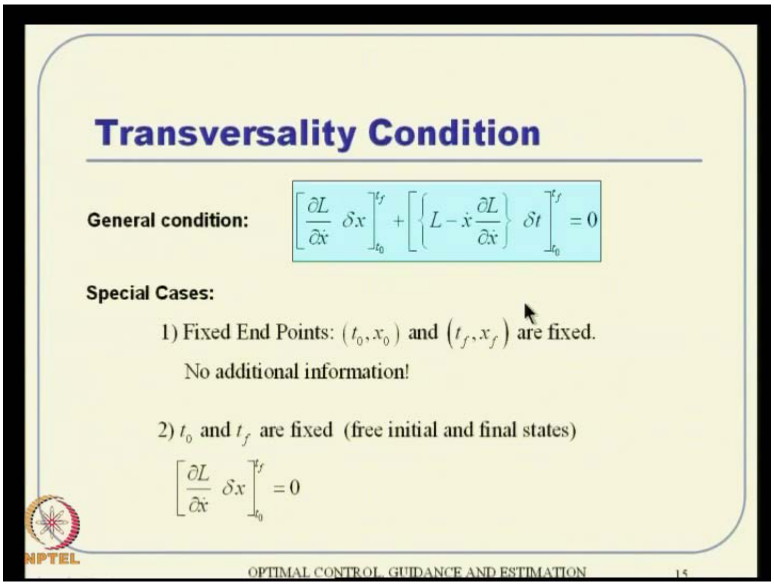

In general we, this was in restrictive sense we try to expand that and then in general we lended up with the, this kind of thing I mean this general transversality condition sort of things. When t this variation of t was also allowed and other variation of initial time and final time that was allowed actually, where you start? Where you stop? Is up to that designers also any way, sometimes the problem when you start? And when you stop? That can be 3 also in a way.

If you bring that into a account then this leads to something like this and then we discussed about various special cases, case one, case two I think up to case five and are somewhere different different conditions. That means, if both the end point are fixed; that means, both time and x naught at t zero and t f are all fixed and they does not; obviously, give any additional information, because every delta x zero delta x of delta t naught delta t f everything was to zero.

But if t naught and t f are fixed, but x naught and x f are naught fixed then obliviously this part is also zero, delta t f and delta t naught both are zero will end up with only this part of the equation. Similarly, if t naught and x naught are fixed; that means, t f and x are both are free, then you can consider only the t f part of the equations and then you lend up to something like this. And similarly, if you depending on, what is fixed? And What is free? We kind of take the other part into account because whatever is given, if it satisfies some part of the equation automatically, we considered that is I mean just taken sort of thing.

So, this is how we extract this information contain from transversality condition. Ultimately there the key idea is the number of differential equations and number of un condition would be same, and then it depends on how many free variables are there with us? And how many extra information is can we derive from this equation actually? So, depending on in example and all will talk later, it will be more and more clear how to be; how to use it actually?

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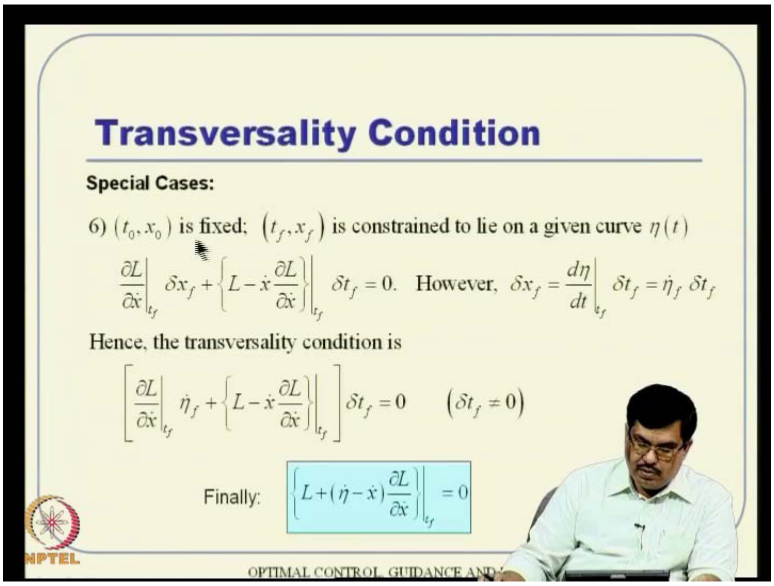

So, the very special case at the end, when t naught and x naught is fixed and t f and x f are constraint to lie on a given curve. Then you to go back to that, they because it is fixed anyway so, this t naught x naught does not give any extra information. But t f x f we cannot take that is zero, this part of it delta x f and delta t f cannot be zero, but they are suppose to satisfies certain constraint equation, So, this constraints equation can be derived something like this, because except x is a function of eta t. So, from that delta x f is nothing but that so, you can substititute that and then land up with this and where delta t f is not really zero, because it is this is constraints to lie on a given curve; that means, your final time is kind of free actually.

That is not zero the coefficient as to be zero that of end up with this addition equation, which will be which is needed for this kind of problem definition actually.

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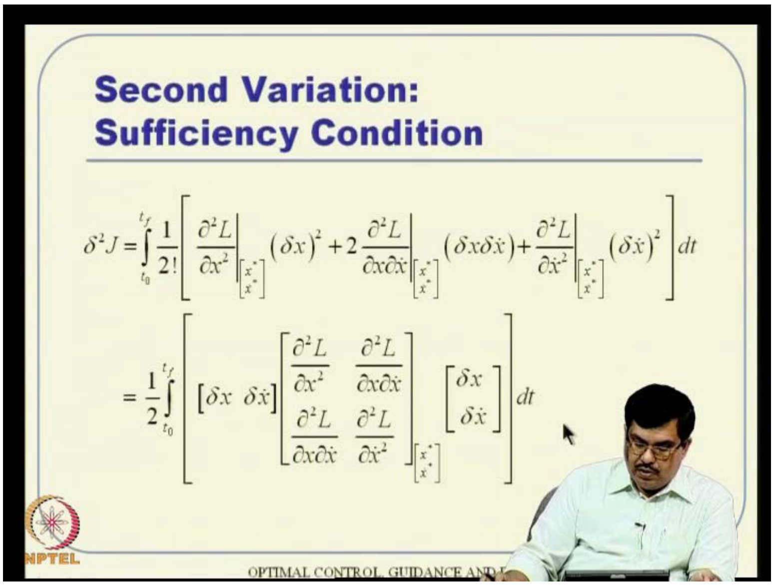

Then we also discuss little bit on second variation and the second variation we expanded the second term of the taylor series and then let of I mean kind of observed that, this lands up this gives us this hexane matrix sort of ideas.

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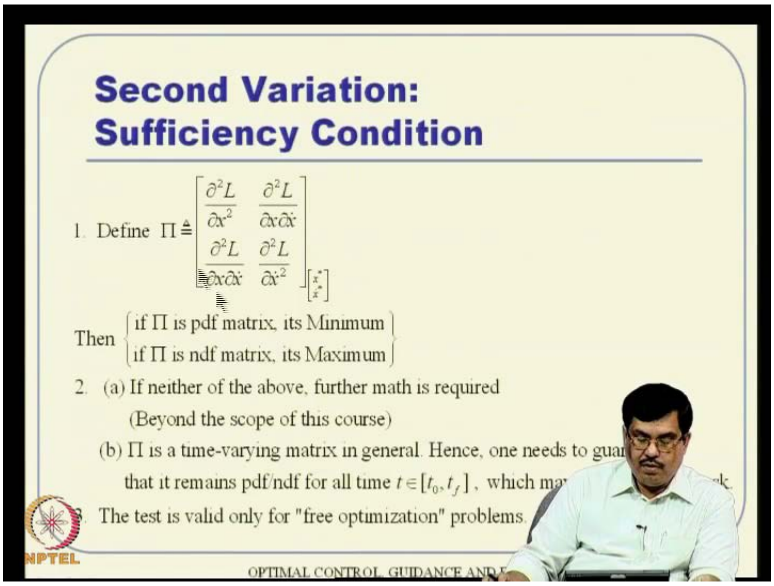

And then (()) to define this pi, (()) pi let that matrix evaluated at the optimum point and if happens to be a pdf matrix it is the minimum and if happens to be a negative different matrix, it happens to be maximum. So, first what you have to do in again problem is to evaluate this matrix phi first and then evaluate whether that the phi is a positive definite matrix or not actually or negative definite matrix. If neither of the above then further math is obviously, required and like static optimization, if you remember? We have this third term fourth term anything like that and; obviously, those things are beyond this beyond this scope of this course.

And also remember unlike static optimization in this particular thing x star and x star dot both are time dependent, they are not really a specific point value. So, they are x star is a optimal solution which is a function of time basically, thus the path trajectory that we are looking for. So, this what you will evaluate phi is not really a constant matrix, but it is a function of time. So, when you talk about whether this matrix is positive or negative is to remains positive or negative for all time actually.

So, for in the entire interval this matrix should turns out to be a positive different matrix. That means, one easy way of doing that is, just evaluate the eugene values kind of symbolically as the function of time and then plot it in this interval actually if they all remain in positive then you have done sort of thing. Also remember as a last comment that is just that we are source of that is we are just talking about is valid only for free optimization problem. The moment you bring in some sort of a state equation constraint or any other algebraic constraint I think like tha,t then if the constraint is active then; obviously, this is the condition does not (( )) actually.

So, this is what we discussed in the last class. How about extending some of these concepts for the vector problem? Because that is where we are have a main interest to like whenever state variables are a typically of one dimensional control is n dimensional thing like that actually.

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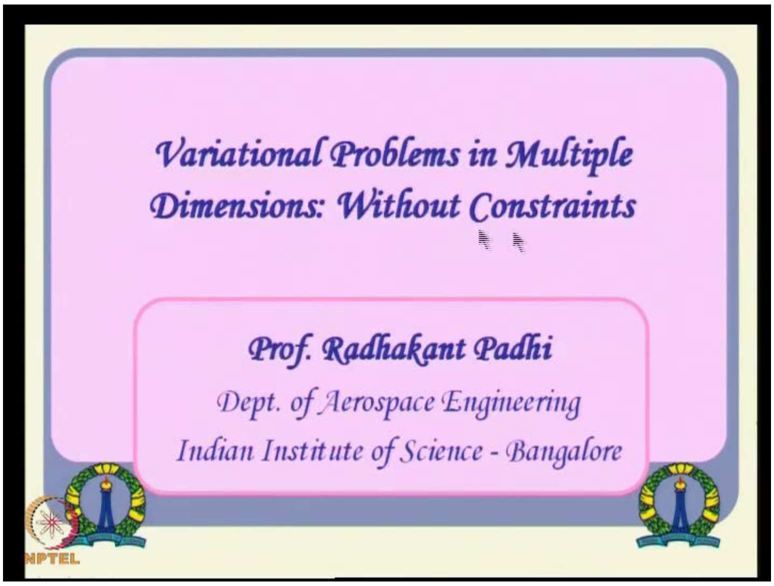

So, what I mean the consider the same problem, but in multiple dimension and to begin with we consider the problem without any constraint.

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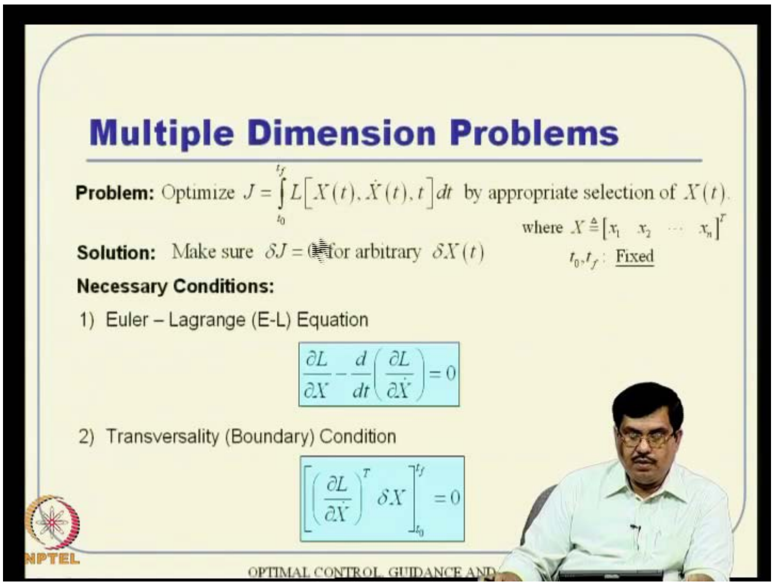

So, just an I mean what we see here is very similar concepts the only difference will be this small x will be replaced by capital N know, where this capital X contains n dimension I mean n components, so, x 1 x 2 up to x n actually. And obviously, again the similar sort of ideas will take t zero and t f both are fixed and our objective is to make sure that delta J goes to zero for arbitrary delta X t. So, again the similar algebra it will do that in a second, but using a similar algebra it will lend up a very very similar local equation.

But remember these two equations now, are not exactly sending what we have sent before. Especially this particular equation contains n equations really, del L by del X one minus d by d t del L by del X one dot equal to zero, then you can substitute by x 2 x t 3 like that. So, it really contains about n equations actually. About an here, transpose is also must because this itself is a vector and that is also a vector. So, what we are talking here, is very closely what we have done in scalar problem, but not exactly one to one sort of thing. And proof also will this particular condition derivation proof will also be various very close, but we just be careful about here algebra that is what.

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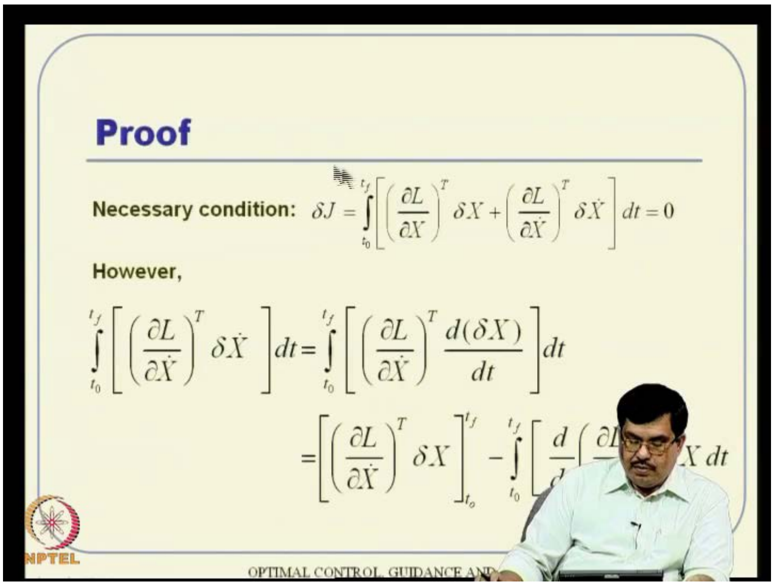

Because vector matrix algebra is not very similar to scalar algebra, that is the only thing actually. So, what we are looking for is delta J? The has to go to zero and delta J is something like this and obviously, the second term again is a problem. So, we expand it try to kind of simplifying the second problem and second problem this delta X dot is nothing, but d by d t of delta X by definition. And now we integrate by parts and then you just tell this is the first term this is second term. So, you will keep that is first term into integration of the second term, which is delta x minus in differential of the first term in case like this is one.

So, this is how it is think must perhaps, this is like this is one term I missing there probably may be this is, this is the transpose here. This is just note the transpose I mean this a because this is again, these vector metric algebra this has to be noted actually, because we just cannot do cannot refer to lose this transpose and all that. So, this is the vector and that is the vector ultimately multiplication is to be a scalar actually. So, this is the how it is and then we clear with this, with this same thing I mean we this, this term lended up with something like this, this leads to that. So, you take this term and substitute it back and this expression and then see that this, this these two terms and then substitute that and then lend up this with something like this actually.

Exactly same just that if this transpose cannot be omitted and then left cannot go to right think like that actually the multiplication out of (()) transposes to be emplaced think like that way.

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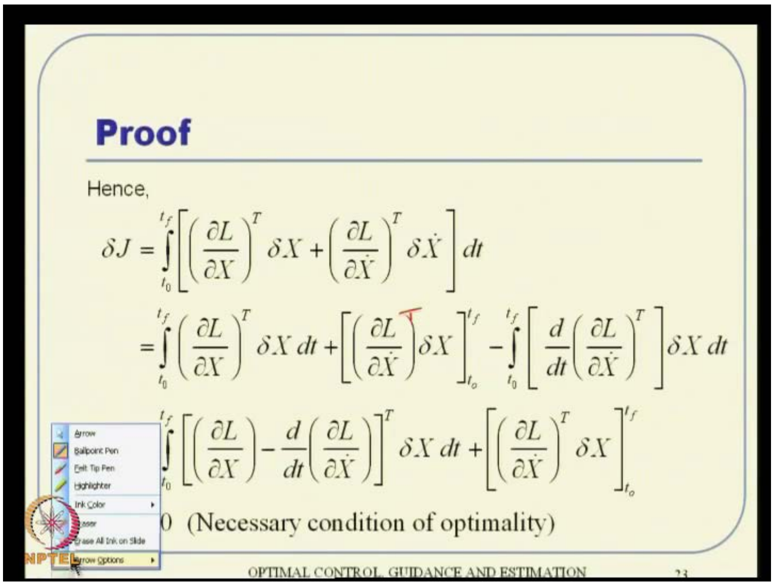

So, (()) here one more transpose is missing looks like. So, let me correct that, this is where I think probably here anyway. So, this how algebra proceeds and then same thing it goes to zero. So, obliviously this coefficient has to go to zero. So, this first term will give us this E-L equation.

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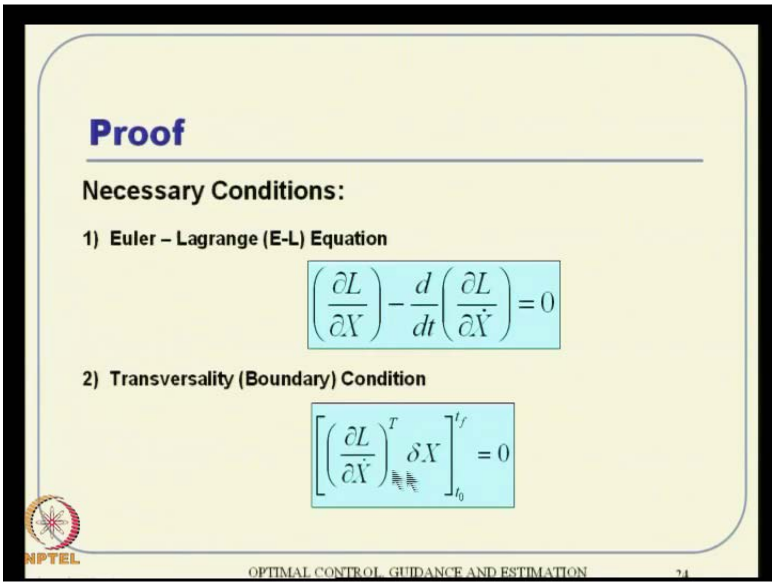

And the second term will give us the transversality or boundary condition basically. So, in any given problem we have it is avoided d t to satisfy both the conditions and carry out the necessity algebra, which will make sure that the these condition are satisfied.

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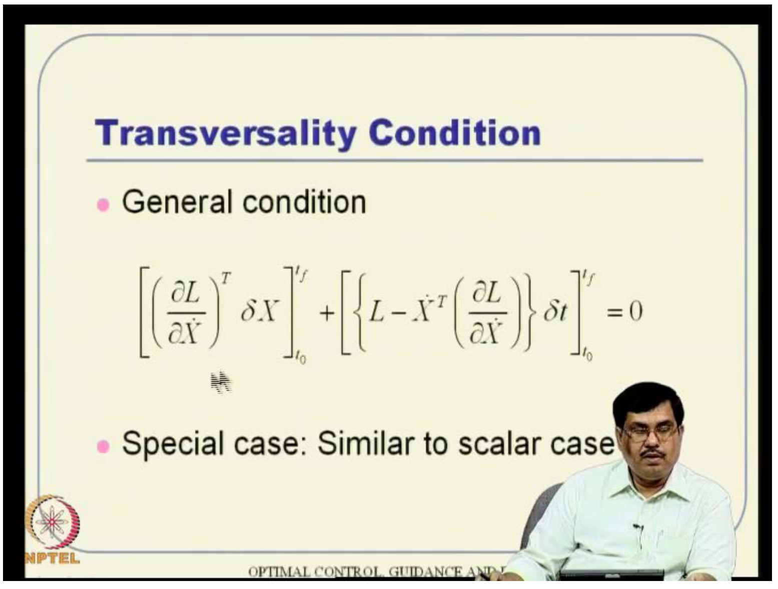

And again extending the transversality condition we can go for this similar ideas del L by del X transpose delta X something like that. And special case is very similar to the scalar cases (()) these are the cases that I am talking actually. These are the cases that I am talking; that means, either fixed end point or partly fixed of the t naught and t f both are fixed or the states are free like that each other. So, that kind of conditions we are talking here and everything will be very very parallel to what we have discussed there.

Such that it you have to be slightly careful about here algebra actually alright. So, this is the condition that is there on now, that is all about free optimization which is very parallel to the scalar cases. Now, about constraint optimization so, that is more important because many of our problems will invariably have an equality constraint in the form of state equation later. So, we are is more interested with this constraint of variational problems and that to with multiple dimension and how does it go? Even very close to this static optimization sort of concepts here.

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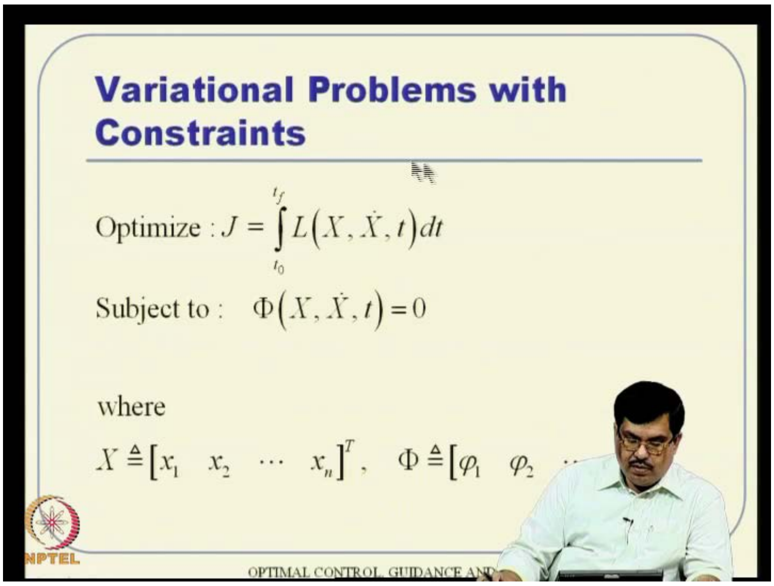

So, what you are looking for the problem is to optimize this cost function, where are the cost functional J, which is; which can be a non-linear function (()) cotaining both X X dot as well as time. And it is subject to this equality constraint and if you notice a little bit carefully and suppose, the X is really the state of the problem state vector of the problem. Then this constraint is nothing but state equation, sort of thing because normally we have X dot equal to normally we have something.

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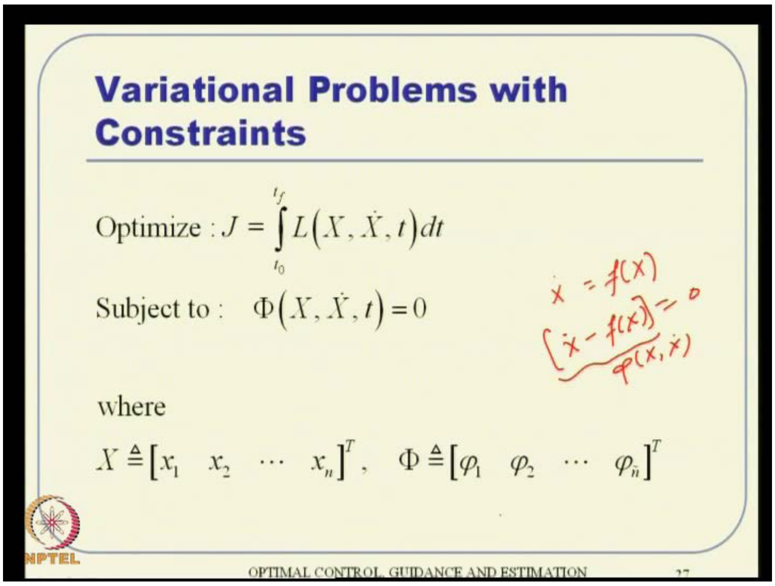

Like X dot equal to f of X, X is sort of thing what about you can take that out for a second and then tell that is have it is may let us X is I mean u is we are not talking right now. So, let us not talk about that we talk like this then I can consider that X dot minus f of X equal to zero. So, this is nothing but my phi. What we are talking this is phi of X and X dot. So, this is alright. So, what we are talking here is some optimization of a cost functional subject to an equality constraint in the form of algebraic equation or in the form of state equation, nearly it can be a function of both X and X dot actually.

And remember this dimension of this constraint may not be same as the dimension of the state, it can be very different and then that can be very different actually. But typically when we talk about the state equation then obviously, both the equation are same. So, the equation until and then will be a same actually in most of the cases anyway. So, convert to this we are interested in optimizing this cost functional subject to this equality constraint now. How do you do that?

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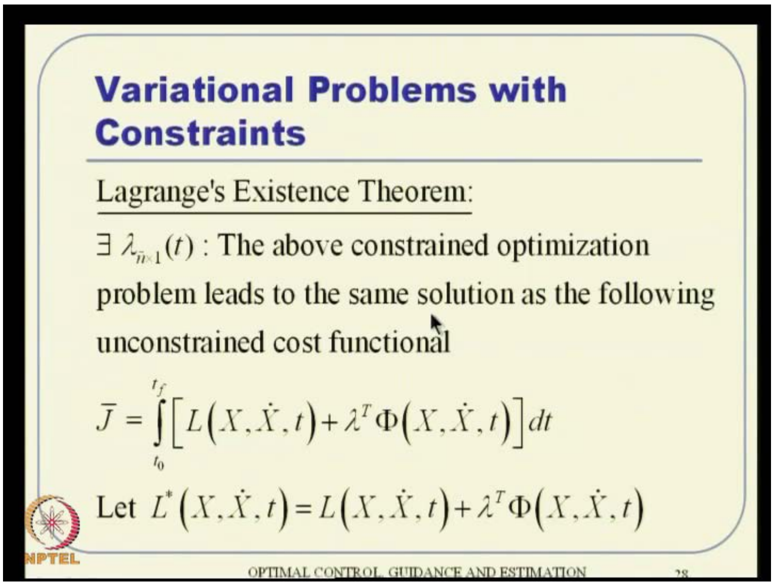

Then I go back to this lagrange's existences theorem and this existence theorem tells as that, the this constraint optimization problem. What you are looking at? What it tells is there exists I mean this same dimensional costate vector now, lamda is the function of time variant very very close to X actually whatever we need that. So, n dimensional is X rather n dimensional is this constraint until there other. So, what it tells? It tells there exist the variable called lamda f t. So, of same dimension is the number of constraint. So, that if you construct this J bar this way and try to optimize it as, if it is a pre optimization problem then it is equivalent to solving this problem that is all it tells.

There exists a lamda of t, such that the above constraint optimization problem leads to the same solution as the following constrained cost functional unconstrained cost functional. Thus now, all that you able to do is construct this J bar and then tree it as if it is a pre optimization problem, but the variables are increase now; that means, you do not have only freedom in X you also have to sort of a freedom in lamda X. So, that is a kind of what is mathematically called as (()) of the problem. So, you are actually taking it to some sort of a higher dimensional problem, because lamda itself is not a constraint vector, it is a time variant thing. So, you really need a differential equation for that at all, will talk about little more on that we go long actually, especially in the next lecture anyway.

So, the lagrange's thoerm the lagrange's existence theorem tells us that, all that you have to do is to construct a J bar like that. And then treat it as a pre optimization problem in the form of in the pre variable be X and lamda both an L start, which is L plus lamda transpose phi.

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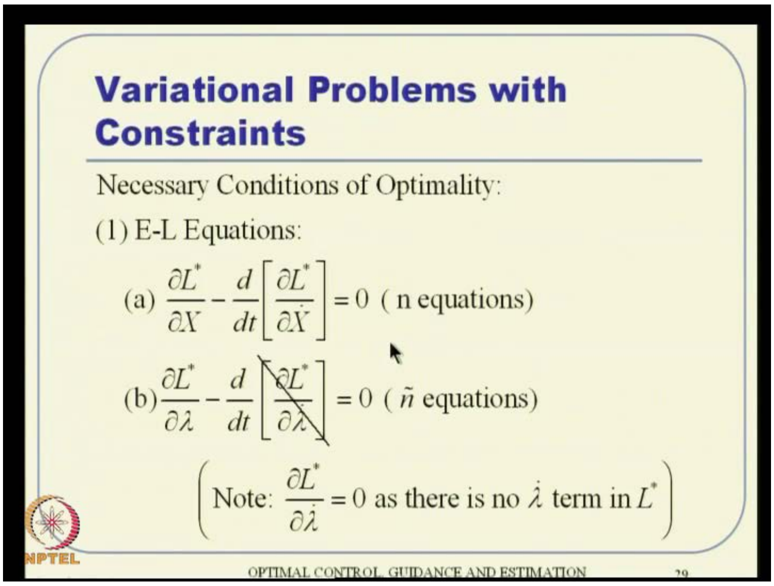

And then we simply here we can tell, we will apply the E-L equations directly. We do not have to really keep on deriving the E-L equation again and because we know the function of L star is a function of X X dot t and probably lamda also. So, let me that is mistake here, this is a function of lamda as well (()) L star containing a lamda actually and lamda is here. So, what you were doing here is a considering that and then applying the E-L equation to a sort of thing two equations actually. First is with respect to X variable and second is with respect to lamda variable with, because we always defined something like a if you; if where not comfortable with this idea.

Where you can always defined some of the vector let say some capital X which is actually X and lamda with first is X variable and then next is lamda variable. And apply the E-L equation with respect to the capital X, then it is equivalent of applying it twice with some lamda separately, because the write because by the definition this will be like components of big X. Components of big X will first contain X and then second contain lamda actually. So, (()) applying writing it separately basically an and also any more this equation L star does not contain any lamda dot expression, that because of that this partial derivative with respect to lamda dot happens to be zero.

So, time derivative of that is a also zero (()). So, that is how, it is leads to this del star by del lamda equal to zero actually.

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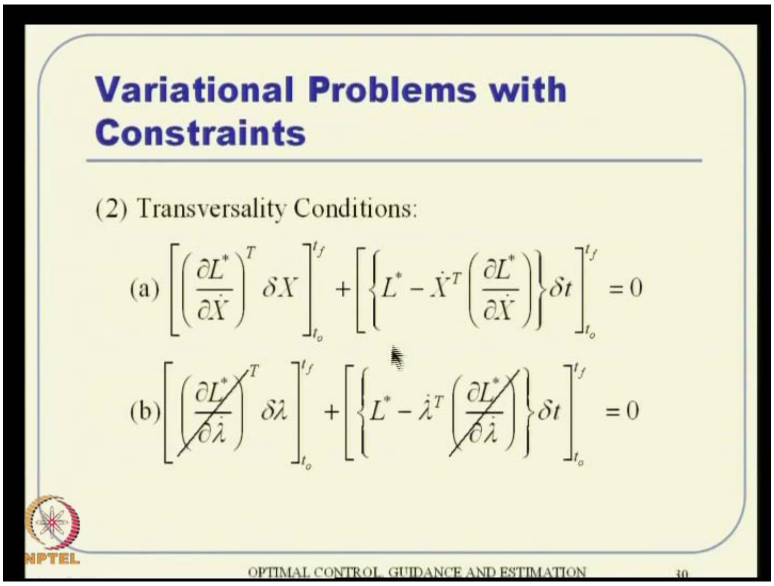

Above transversality conditions again same thing, there are there two things here, first is with respect to X and second is with respect to lamda and again because lamda dot is not there. So, any partial derivative with respect to lamda dot is zero.

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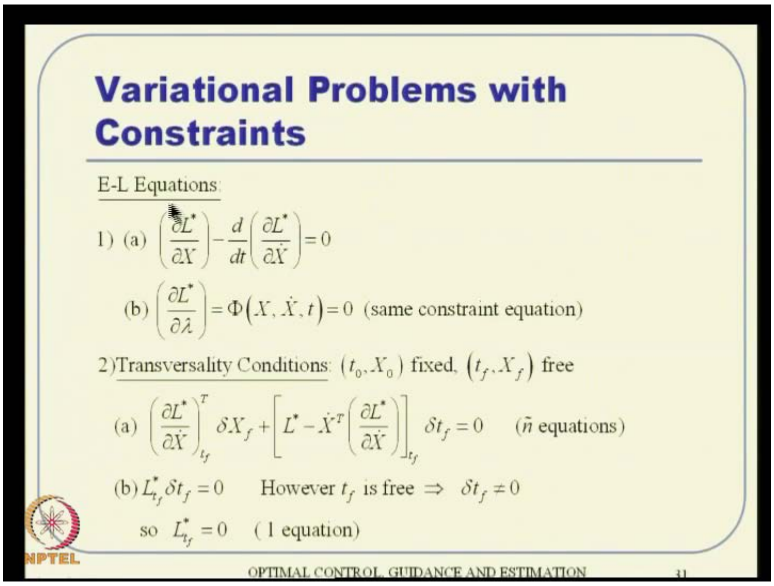

So, essentially lend up with this kind of equation fist equation is a del L star by del X minus d by dt of del L star by del X dot equals to zero and the corresponding to that, what we called as costate equation in general. So, del L star by del lamda is equal to zero and even though when (()) dell star by del lamda this L star is like this. So, del L star by del lamda is nothing but phi. So, what you are telling is phi equal to zero and as nothing but equation. So, the constraint equation appears again (()) a state of necessary condition, a part of necessary condition actually.

So, constraint equation gets embedded into the (()) of the solution basically (()). So, what you doing here is a this kind of thing first is a with respect to X variable then with respect to lmada variable. When you do this with respect to lmada variable constraint equation reappears actually same constraint equation comes. So, this is how it is. So, what we have looking at, we are looking at prolong with some n dimension from x and n tilted dimension for from a constraint equation, and one freedom if you are t f is free then, one more freedom t f actually.

So, essentially we have this boundary conditions for this n plus n tilted plus 1 sort of thing that way. So, n conditions are already there because t x naught is fixed, t naught x naught that kind of fixed and everything can be view out from here and the transversality conditions tells something like this. So, when you apply this leads to this, this n tilta equations sort of thing and then we have one more condition that L star of t f into delta t f equal zero, since t f is free it leads to this L star t f equal to zero. So, you have this one a freedom and also equivalent boundary conditions sort of thing. So, it should be able to solve it actually.

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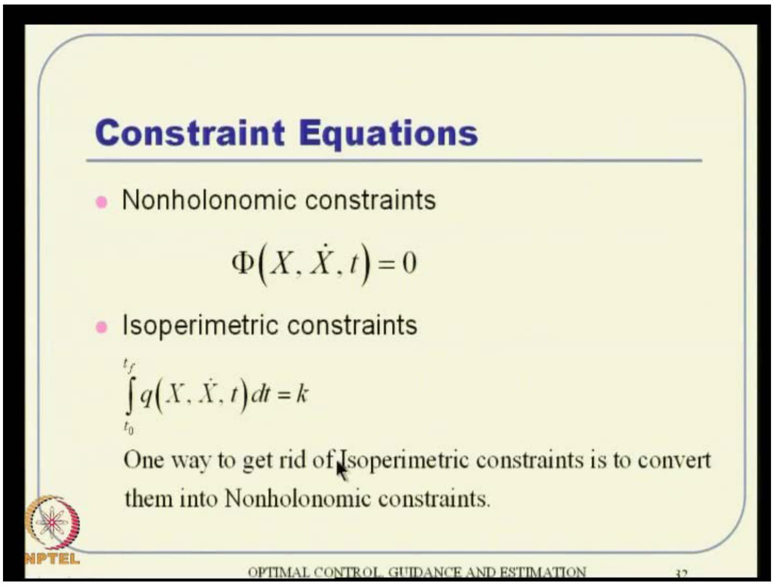

So, then there is another cause of constraint that (()) application problems, this nonholonomic constraints and all that each other. That this similarities, this nonholonomic constrains appears in the form of state equation, there is also another class of constraint is (()) is this is called isoperimetric constraint. That means we are not interested in particular valve for say, of this particular function what we are interested in a integral valve of that function has to be some value X actually, how was on example? An example probably like what I can think of it is.

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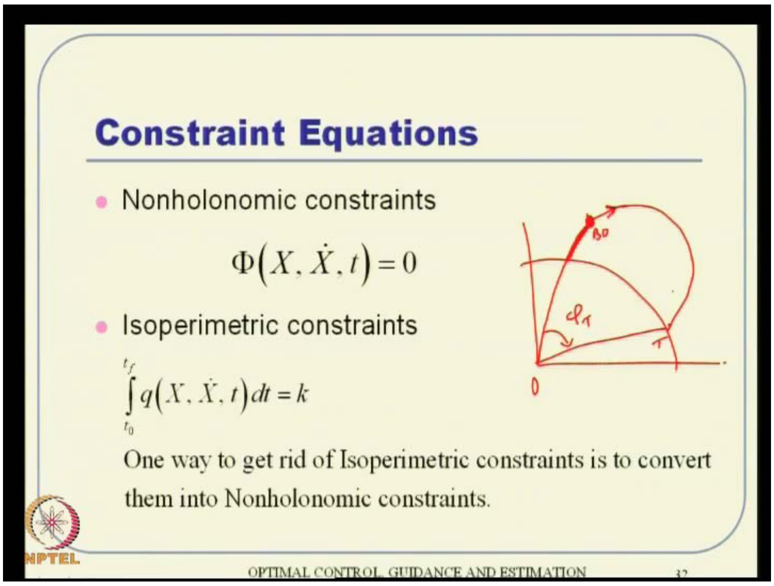

Suppose, we are talking about, let say some sort of a calculate entry (()). So, you have it here and the vehicle was there and then it they something some where the target there. That means, this is a centre of (()) say then what we are looking for is, this angle is called something called range angle actually. So, what you are looking for is, you to guide this vehicle in this segment, this is you suppose to guidance and set at the point one out you guide it in such a way, that it will follow the same trajectory that you where looking for this.

That it follow the same trajectory that you are looking for and ultimately it will end of that target. And equivalently tells you that, this angle that is getting covered that this range angle and all at the end not not in the guided segment, not in the this. This t naught to t f for a power guidance purpose can we ending here, but you extend that. And then tell after so much of time, it will finally, going to falls some were on. If in that particular thing I mean, by that time whatever angle I covered is to be equal to some value is it not? Then only I am I mean vehicle will be reaching there, otherwise it will be somewhere else actually.

So, that is not acceptable. So, that you can that kind of problem is called something like isoperimetric constraint and you can think of many different applications as well actually. This is just a small I mean example sort of thing (()). So, there are cases where it will? Which? Where it will appear this kind of constraints? Now, how do you handle that? As we are not talk about anything about that so far basically now, if you are a little bit I mean, clever then, you can think that integral it is kind of a an counterpart of differentiation actually.

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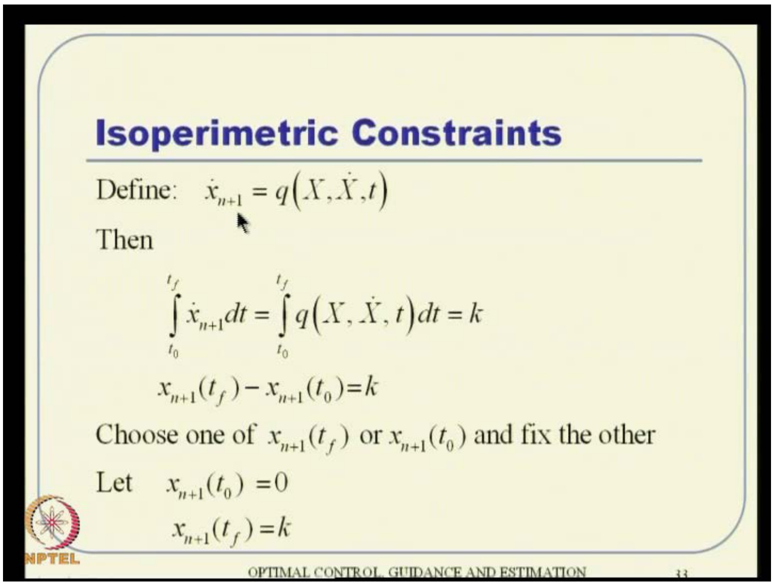

Then what I will do? I will define x another kind of state equation, sate variable sort of thing another free variable x n plus 1, which is not part of n (()) X contents x 1 to x n. So, I will define another variable x n plus 1 such that, x n plus 1 dot is nothing but that. So, what you do? If I integrated both sides from t naught to t f, what i am? (()) We can that, this is by definition x n plus 1 dot is nothing but that and that is nothing but this constraint equation basically. So, this integral of x n plus one dot d t is nothing but equal to k.

But this one can be expanded now, as x n plus 1 t f minus x n plus 1 t zero equal to k. So, this equation that we are looking for this particular equation actually two free variables and one equation, we do not know as long as the different k then we have got k. This appears as a I mean, this non homonymic constraints sort of thing. So, this constraint is accounted for only the boundary conditions sense we need something and a we got it actually, what it happens to be like two free variables and one equation actually. So, what we do now?

So, essentially the idea here is we can choose one of these two and fix the other one because any way, it is a free variable sort of thing as long as this differential equation is I mean satisfied, then you have done any way. The boundary condition does not matter that much, as long as a difference happens to be k. So, you can select one and fix other and whatever you are selecting can be interpreted as something like a tuning variable. That means, if you select a different value, the results may be different (()). So, it happens thus the one that you select happens to be at any parameter, the rest of the will go as apart of the elevation actually.

So, if you not is you not sure what value to select and all that; obviously, an idea is select the x n plus 1 t zero equal to zero and then x n plus 1 t f will automatically become k. That is a guide line sort of thing actually. So, (()) because many practical problems you may lend of some constrains like this for the objective to be met actually. So, in those situation one idea is to define an additional pre variable x n plus 1 and do this. Bringing this additional constrains, equation in the form of a non holonomic constrain actually that associated boundary conditioning will be derived from the constant equation (()).

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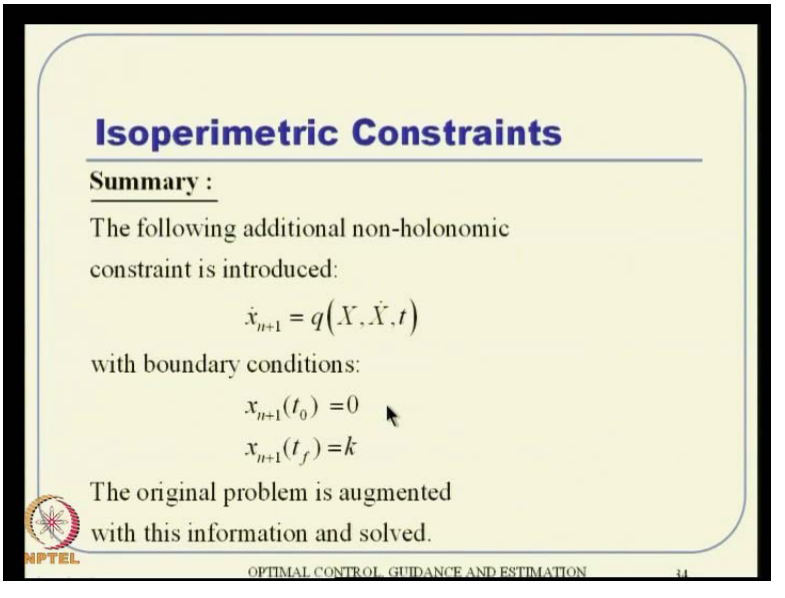

So, what is the summary again? The summary is the following additional non holonomic constrains is introduced, which the boundary condition like this. This zero (()) some other thing I mean, if you select some other number, then this capable as that number will appear here actually. And that happens to be a design flexibility sort of thing actually. So, what is essentially done is the original problem is augmented with this information, this constrain equation with this boundary condition and then it is attempted to any way attempted to solve this actually (()).

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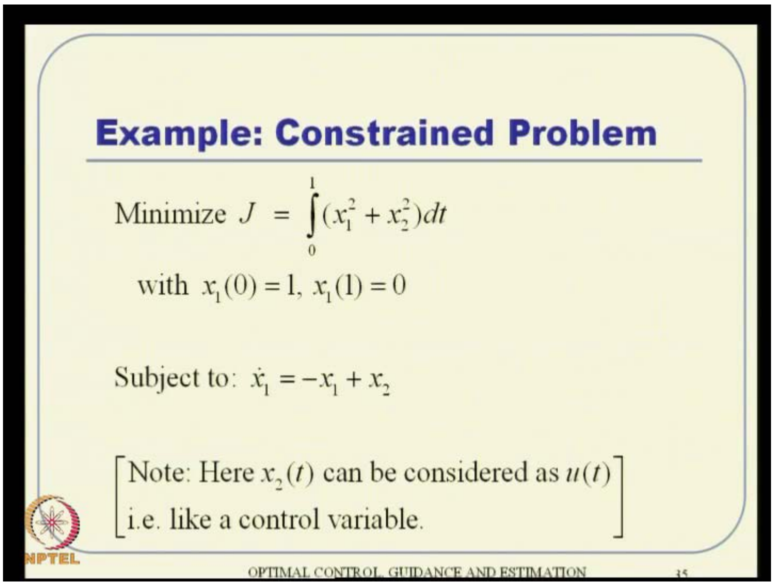

Let us go for a small example problem now which will make over ideas slightly more here probably (()). So, this is an standard very small example problem I will encourage everybody to you solve it usually using kind of basically here so, because this problems will get our ideas clear actually. What you are looking for? We are looking for minimizing a cost functional, like this it is standard cost functional quadratic cost function with x 1 of zero equal to 1 and x 1 of 1 equal to zero.

We know that and there is absolutely no constant term, x x 2 for say where the this equation is constant, this x 1 is further constant where this differential equation x 1 dot is minus x 1 plus x 2 and x 1 boundary conditions are known to us actually. But we are now interested in minimizing this cost function. So, for a second if somebody is clear, you can think well x 2 of t it can be interpreted something like control variable u of t. So, if x 2 t happens to be control variable, then it is this is the way the control problem do reappear actually.

This will be something like x square plus u square well let me do that probably this is J this suppose, I define x 1 as nothing but x and x 2 as something like u. So, then J this J will be nothing but zero to 1 sorry (()) zero to 1 zero to 1 x square plus u square into d t. Subject to x x dot equal to minus x plus u and then boundary condition is x of zero equal to 1 and x of 1 equal to zero. Let me see any time; think, if we talk about x, we are talking about x of zero equal to 1. So, you have this one if a zero equal to 1 you have this one and x of t equal to 1 and somewhere here it has to be zero. I do not know how it? How it will develop the entire (()) on be a straight line. So, we can develop whatever we can want actually (()).

You can develop whatever way if you like to you have, only condition is it has to optimize this corresponds on the way it is it to minimize this cost function actually. This is if you interrupt this problem what you are looking for here? It is actually the same problem is getting invaded in the form of this x 1 x 2 variable these actually. So, in other words we are trying to actually, we are trying to solve an optimal control problem here in a indirect way itself. Thus the message there and also remember this path that, I put we do not know, How it look like right now? We will see whatever the equation pops up accordingly it will evolve actually.

But it is a satisfy your boundary conditions that is they is they know conformation that actually. So; that means, x of zero has to be 1 x of 1 has to be zero. They are the two conditions actually, and also remember we are mainly interested in the interval zero to 1. We are not interested anything beyond that. So, we have this fixed final time problem, sort of things actually where initial time is fixed at zero, your final time is fixed at one. You know variation of those value actually and this initial value of the x is 1 and final of x 1 is also 1 actually. So, that they know freedom of that. So, you can think of it is something like fixed n point condition problem.

So, we proceed further. So, what do you actually? Will not worry about x and u anything like that we I will continue our discussion with x 1 and x 2, not even in introduce an optimal control problem, which we will do it in next class anyway. I mean looking at that, will just go back to our E-L equation and try to get an answer for this actually. So, what you are doing? These are two ways of doing that, one is follow method one first here, I will be slightly clever I mean, then one is follow method one first here I will be slightly clever. I mean then it then and look at this equation and I know everything all the algebra that is necessary in the form of x 1 x 1 dot and all that actually.

So, however simply eliminating x 2(()), but x 1 dot plus x 1 and I gave at and substitute that there, one part is x 1 so, x 1 square. So, I will keep this x 1 square has x 1 square and x 2 square is x 2 is nothing but x 1 dot plus x 1. So, I will substitute that x 1 dot plus x1 square. Now, it is a problem of clearly in the form of x 1 2 is gone actually, and x 1 boundary conditions are also available. So, we will able to solve it actually we say how to do that. So, write if the E-L equation E-L equation tells us that, this is true and also I mean this one is take here is both x 1 and x 1 here.

You put this del L by del x 1 minus d by d t of del L by del x dot equal to 1. I mean equal to zero than, what you are doing here is nothing but this del L by del x 1 the L is like this. So, what is del L by del x 1 2 x 1 from here, and this is nothing but this (()) I mean this chain rule of derivation you replied, then it turns out to be two into that. And then partial derivation of del x 1 by del x 1 that is one actually. So, 2 x 1 plus 2 of that minus d by d t of partial derivative with respect to x 1 dot now, you need to x 1 dot then this one there I will get coefficient is 1. So, we will end up with that actually.

So, when you do this partial derivative with respect to x 1 dot, do not consider x 1 is dependent on that, you know that they are two different quantities actually. So, we end up with something like this equation. So, now it is time to simplify a little bit now, you simplify it comes out to be like this and essentially you can see, that this is nothing but x 1 double dot. And this line you will see out this x 1 dot and x 1 dot cancelled of that plus and minus. And I will left out with this differential equation x 1 double dot minus 2 x 1 equal to zero.

So, this is rather easy to solve, even this is linear equation sort of thing second order linear equation. How was in this? So, characteristic equation turns out to be lamda square minus 2 equal to zero and then lamda equal to plus or minus root 2. So, x 1 now; because x 2 is already eliminated, this is what? It is the moment we know x 1 we can construct x 2, form this algebraic x 1 dot plus x 1, the x 1 is really a function of time now. So, I can construct x 2 very easily write there. Now, what is the boundary condition? Boundary condition tells us that, these two boundary conditions x 1 of zero is 1, x of 1 is zero.

So, (()) when you substitute t equal to zero, then e to the power to zero is one anyway. So, what it gives us? This is 1 equal to c 1 times 1 plus c 2 times 1. So, this is nothing but 1 equal to c 1 plus c 2 then putting that here 1 equal to c 1 plus c 2 this is an derivative, the second 1 is at t equal to 1; that means, when this coefficient will be coming to the power root 2 and in the power minus root 2. That is what, it will come here that values will become zero.

So, hardly if you are not comfortable you can just derive this equation again. So, this first equation leaves to 1 equal to c 1 plus c 2, the second equation leads to zero equal to c 1 c to the power root 2 of root 2 times one, that what it will turn out to be. So, this is e to the power root 2 plus c 2 e to the power minus root 2. So, this same these two equations are are written in a vectometrics from here alright. So, this is how it is and then it is easy to complete c 1 c 2 c 1 c 2 is a inverse b sort of things.

So, complete that and we get it something like this, we can further simplify the algebra we can stop here, computability value whatever actually each other, but you got what is the value of c 1 and c 2. So, this is once you get the value of c 1 c 2 we are done because x 1 t happens to be like that, and x 2 will happen to be like that we are done with that actually. So, this is one way of getting actually the smart a very good way of doing that, this method of elimination and all that for all damage some problems will not be possible. So, you need to have a direct flow of the (()) like that.

So, let us go to see this method two, which we just studied that lagrange approach sort of thing you know this is the constrained equation right. This is the; this is a cost function that I want to minimize subject to this constant equation. So, you can introduce this lagrange variable and then introduce additional equation and all that actually, this what again the here. So, L star is nothing but l plus lamda into that phi, whatever phi is actually so, this is this will end up like that. Now, we have E-L equations in three variables because x 1 x t 2 and lamda, you will not eliminated x 2. So, x 2 reference to be a three variable also visible, just that x 2 dot is not there and lamda dot is also not there. So, these two quantizes will go to zero.

So, that we are looking for is this equation and these two equation actually. Let this L star will contain L plus lamda transpose f or transpose phi, but it is a single equations anyway. So, we do not need that transpose is (()) and L is nothing but that L is whatever comes inside the integral. So, this is how, it is constructed L is whatever comes inside the integral plus lamda times and ingenerality lamda transpose time phi what here, it is a one circular equation anyway. It take lamda times phi, phi is nothing but that, this why definition. When you start applying E-L equation that is nothing but individual component value will apply, first we apply with respect to x 1. So, del L star by del x 1 minus d by d t del L star by del x 1 dot equals to zero.

Similarly, replace with x 2 and then found out this equation an and that by lamda and from the equation. And it turns out that in this L star there is no x 2 dot expression anywhere, as know lamda dot expression any where is not there actually. So, that is the reason why, these two equals zero (()). So, you lend up with this equation as well as these two equations. That means, to be solve actually, what does it leads to? del L one del L star by del x 1 minus d by d t of this quantity.

So, this will lead to this equation lets and then del L star by del x 2 and then lead to that equation and then this del L star by del lamda will lead to this some constraint equation. That using before del lamda del L star by del lamda equals to zero; that means, this equals to zero. This x 1 dot plus x 1 minus x 2 equal to zero that is nothing but the constraint equation, that we started here (()). Now, let us. So, this will appear basically this algebra sense well, it is very simple the other relates try to do something actually (()). So, del L star by del x 1 if you look at that that is nothing but 2 x 1 plus lamda actually because 2 x 1 a from the first come and lamda will come similarly, del L star by d x 1 dot if you talk about is nothing but lamda basically.

So, what you have here is 2 x 1 this is 2 x 1. So, when you substituted this equation, this 2 what you learned up with 2 x 1 plus lamda you have from the first 1 plus d by d t of lamda, that is lamda dot equal to zero. That is what I getting here, very first equation that is 2 x 1 plus lamda from this side, then the lamda dot from the second one actually that is equal to zero. Similarly, when in (()) this algebra of this del L star by del x 2, then all that we having is here 2 x 2 minus lamda. So, that is 2 x 2 minus lamda equal to zero and similarly, when will do this of the one is very straight forward, this equation has to be there and all the coefficients of lamda is to be appear that nothing but the same constraint equation, that how? We get it here actually.

Now, it is time to (()) time for us to solve this three equation and this three equations can be solved in variety of it is obliviously, you can solve it whatever where ever you want actually. This follow one approach, where you can (()) I can eliminate lamda from equation (1 b). So, I will eliminate that at a get lamda equal to 2 x 2 and x 2 is nothing but 2 in to x 1 I mean x 2 is nothing but x 1 dot plus x here this equation (()). So, two and as if now you can vary (()) I will use this equation 2 and (1 b) using that kind of equation basically (()) alright lamda lamda here eliminated.

So, what you are getting here? Lamda dot anything, but the 2 x k well write down 2 1 a probably, I think this will be probably one just check it. I mean and do it yourself properly if you use this now, whatever you done here. So, we have we use this equation only in way; any way and then when you x 2 then x 2 we have use this also right, in this putting this, but you not use is this equation actually. So, this equation is substituted by ultimately get lamda dot here nothing but 2 x 1 plus lamda.

So, 2 x 1 plus lamda basically so, you substituted here you get this any way. So, in any algebra is there you can; I think is not that had to wait actually, otherwise equation. And then we are looking for it, let me correct myself also little bit here, you do not do that let is a does not worry about this algebra either actually (()). So, what you can think about is the lamda is nothing but that so, what is lamda dot here? Lamda dot if I just take a differentiation here, nothing but 2 x 1 double dot plus x 1 dot.

So, this is the this is something like I mean, then you can think of using whatever you want to use basically and then you can try to simplify this actually. So, ultimately the point here is, in the this now lamda dot know you can use this one basically. I mean equation 1 lamda dot is nothing but 2 x 1 plus lamda and then again lamda is substituted like this actually. So, you play round with this the equation after taking this derivative and then you will that kind of this something like this. So, what you are getting here actually ultimately? If you looks at this 2 equation, this 2 equation now here, now 2 x 1 and 2 x 1 gets cancelled out, this 2 x 1 dot and 2 1 dot gets cancelled out. So, here left 2 is getting cancelled out any way basically.

So, here and then x 1 dot gone basically so, what you left out is something like x 1 double dot is nothing but 2 x 1 just look at this algebra just have to cancelled out the term that gone and split the remaining terms actually. Incase 2 x 1 double dot will happen to be 4 x 1 and hence x 1 double dot is nothing but 2 x 1 actually. So, this is the what you got before algebra write down that, if you look at this equation before will end up equation any way, x 1 double dot is 2 x 1. Then you carried the simplification and solution of that in all.

So, this is also similar thing will ended up with this equations, this exactly same equation as before as you can proceed the same, I do not have to do the further algebraic actually. Finally, this expression comes out to be like this, remember we have solved this equation that way. So, we want t is also available as number basically. So, finally, I have this x 1 is that and x 2 is nothing but x 1 dot plus x 1. So, you carry out the algebra that is necessary for this x 1 dot and then had the same expression 1 x 1 basically we will end up something like this.

So, that is how we get a solution basically. So, as I told before if you consider x 2 of t is nothing but u of t, that is the control variable rather than when the process, what you have done is actually, we have solved an optimal control problem already basically. So, this is not to simple like that, we have to want to generalize to generalization and then no power full things and things like that. And this will also gives you some sort of connection between calculus of variation and optimal control problem (()) that is the motivation of the even this example actually.

So, this is what we are hearing of towards that in the next class. So, before there before going to close, I have been I mean our main objective in the entire course is to lead towards this optimal control and estimation concepts everything. So, this is again like that I will just put as few words about the optimal control problems (()). We discussed that before actually anyway. So, variety of optimal control problems then, we thought about putting it is something like this, that you our objective is to find an admissible history of control variable now.

Thus primarily more important what you have? Which is cause? These three things first you should call the system governed by this differential equation. Now, it is a state equation, it is the whatever the you take the solution u of t and substitute it here and kind of solve this equation either close form or numerical whatever way. Whatever results in x of t will get, this is called state trajectory and that state trajectory should be admissible actually, it is not violate any constraints on the state trajectory too.

So, the ojective here is to find an admissible time history of the control variable from this segment t naught to t f such that, it causes the system governed by this differential equation to follow an admissible trajectory. On the way it should also optimize that is minimize or maximize a meaningful cost function and then it is also satisfy this systems boundary condition actually, whatever boundary condition u and impulse. Later there it can be a projection constraint or it can term constraint or whatever you want to do the primary thing is to satisfy the (()).

And on the way it is to satisfy this cost function, which contains partly the path different n cost and the final boundary condition. Cost of that, and you have disclose about this also what let me quickly summarize it again. This cost function that we are look at is, kindly kind of early (()). It can involve various class of problem actually, first thing we discussed if there is, how do you talk about minimization of something like operational time? The time taken from reaching it a goal point from starting point, from even point.

So, that is here minimum time we shall wait and think about the problem actually. So, then this problem this phenomenon you can think, this is zero and this is 1, then it will lead to that. So, phi is zero and 1 then it is nothing but integration of 1 d t is nothing but t f t f minus t zero. That is what we want to minimize. So, if you take phi equal to zero and it is equal to 1. You will have (()) of the minimum time problem sort of things. Then minimization of the control variable if we talk about that, then it is U transpose R U that mean R 1 times U 1 square R plus R 2 times U 2 square like that.

If R is a diagonal metrics of R 1 R 2 and all that, then you have this cost function which contain these terms. Half of is R 1 U 1 square plus R 2 U 2 square and all that. And each of the entries we have positive thing that, will lead to something like a positive definite function basically. So, we want to minimize that then you will ultimately learned up the minimum control effect. Similarly, if you talk about in other words, if you talk about this, then again still phi is zero in this thing and happens to be this function actually.

Similarly, if you have a minimum derivation from the state minimum derivation of state from fixed value C with minimum control effort I talked about that thus a kind of helicopter (()) problem and all that then well it is X minus C is the error that we want to minimize. So, X minus X transpose of time X minus C 1 term, but there are error of X with respect to C here is minimize and then you have U transpose R U, where the control guess minimize on the close actually. So, what you are having, you are still having this phi equal to zero and L equal to all this things inside. And along with that half is there well the half is also like I mean their half does not play major role, but actually it helps us to simplify the algebra later.

Many of these you see this real equations and all will talk about derivations when you take derivatives a partial derivatives all that, then this half term; well these are quadratic terms. I mean the half term is half and 2 will go, you do not have to take if this number to all the way and it does not really (()). I mean that will be quality of a solution and the solution where is supposed to be I and half of J will have the solution of the same point that is more important actually. What you are looking for is the solution of the projector itself. So, that means J or half J minimize.

So, 1 is on the same (()) actually then an different examples if you want to minimize the deviation of state from origin with minimum control effort and you can think of this delta S and X and delta U then it is nothing but regularity problem. Actually this variable instead of that C t, you consider that delta x. Delta x means some deviation with respect to some known (()) sort of thing and similarly, the deviation control delta V then these corresponds and happens to be a regularity problem actually.

So, what you are looking for phi zero and it happens to be like that and what about this this kind of a function well minimize the control effort on the way. But you do not worry about minimization of the state on the way, but you worry about reaching a final goal point C. So, that means, X minus C happens to be at end actually, then you will think of you minimizing this kind of a close function well of phi happens to be like this not zero anymore and l happens to be like that.

So, we have the class of problems that, we have talk various class of problems that you can involved within this generally close from sort of thing that does not mean you have to confirm yourselves within this class all the time. But the you can think about your own coefficient as well actually and then along with this cost function, we have two at these boundary condition and as I told before the boundary condition happens to be either fixed end point free end point and a variety class and things like that.

So, this will ended with the an optimal control problem and very quickly I mean next class we will see how do we take advantage of this calculus of variation to solve this optimal control problem? Where we will leave it actually? All that that is all is this particular lecture thank you.