Optimal Control Guidance and Estimation
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Module No. # 03
Lecture No. # 06
Review of Calculus of Variations-1

Alright, hello everybody; let us continue our lecture series for this course, optimal control guidance and estimation.

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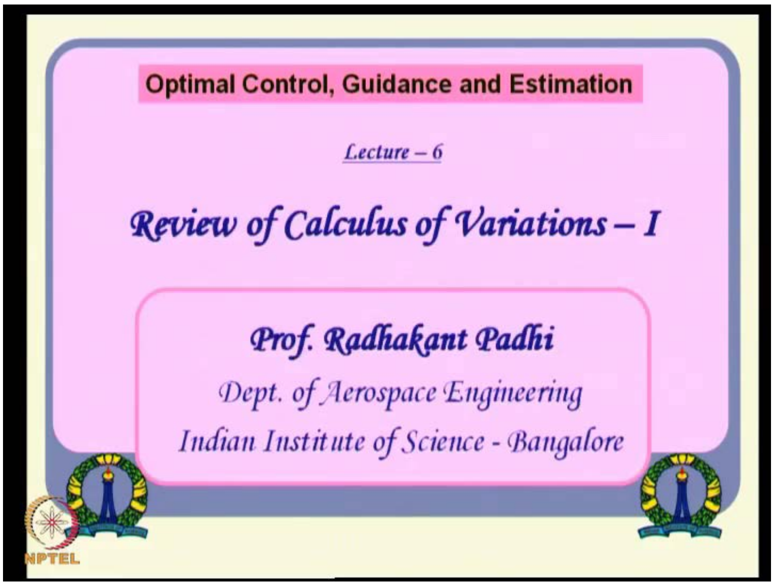

Talking about lecture number six here, where we will review some calculus of variations concept and this a two part lecture series and the first lecture will be taken this time and we will generalize some of the concepts in the next lecture. So, why do we do that? Because, one very strong of backbone of optimal control theory relies on this calculus of variation approach actually.

In fact, the entire problem can be viewed out or viewed as nothing but calculus of variation problem actually. That is what we will do but, typically the calculus of variation is a vast subject also in its own right. So, we will not talk of too much detail about everything that is involved; what will just see are the concepts that are relevant and then we will proceed further for optimal control ideas and of that actually next. So, this is the motivation; why we want to study calculus of variation actually here (Refer Slide Time: 01:11). So, first thing is before going there little bit of motivation we can, we have seen some of these things before; what will have is a re-look at that. What is an optimal control formulation that is what we are, our ultimate aim is actually.

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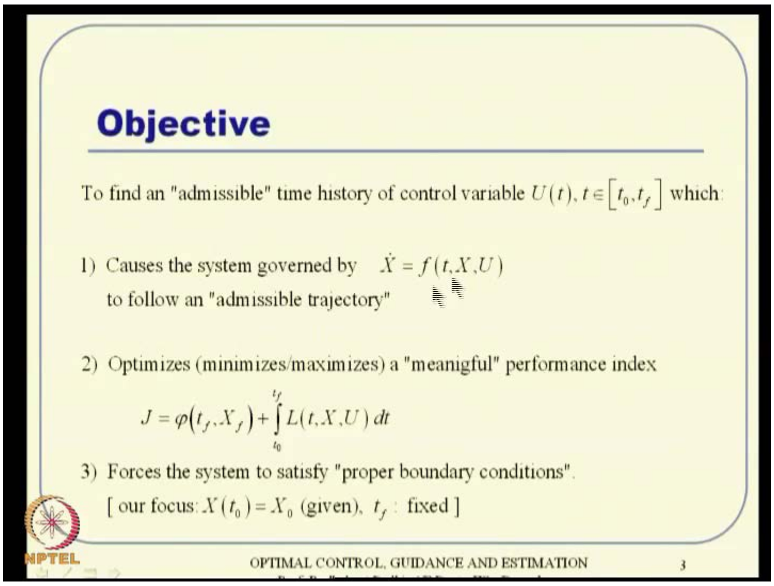

So, in a way it can be summarized something like that which I think we have all discussed about it well before. But our objective an optimal control problem is to find an admissible time history of control variable u of t in the domain t0 to t f; t will also be in that domain. We have to find out all u of t and rather, u of t trajectory within this t0 to t f which causes the system governed by the system dynamics to follow an admissible trajectory ok.

So, it is a control trajectory we are interested to find out; so that the state trajectory becomes an admissible. If you view it as a solution to this I mean (Refer Slide Time: 02:06), this equation or whichever. But, on the way has to optimize; that means minimize or maximize a meaningful performance index which can be given like that in a kind of its very generic form. But, that does not mean it is the only form basically; it can be having here one way of defining it, but it is a very generic way; this can be defined something like that and it also is to set - force the system to satisfy certain boundary conditions ok. So, what is our objective? All that our objective is, to find the some sort of control I mean, control trajectory in this domain.

So that it will satisfy all these nice things actually; it will give a solution of the state which is admissible; it will minimize or maximize a cos function and it also satisfy a certain boundary condition with both initial condition as well as the final condition actually. That is our objective and obviously, it falls in the control variable calculus variation because this is a path dependent on optimization. Optimization is what we are talking about here and then there is a dynamic system involved in the process actually. So, at a point it does not make too much sense but, it is an evaluation in time, it makes a lot of sense actually.

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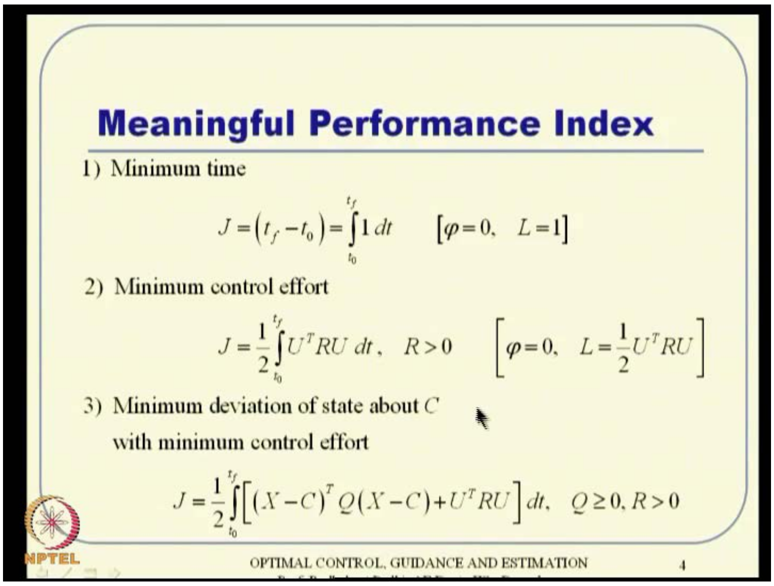

So, looking at this particular cross function you can talk about, again discuss this a little bit before as well. So, if you talk about minimum time then I can talk about minimizing this fellow (Refer Slide Time: 03:22) and hence I can select phi 0 and l one this thing 0 it is one then it is nothing, but t f minus t 0. So, that is want I want to minimize here actually ok.

Now, if you want about minimum control effort and this is this is what I will do this is I will put it as 0 and this one I will take it as something like you transpose r u then if I if; that means, I minimizing the control effort in a way actually. So, this means phi is 0 and l is the nothing more like that. Then, if you talk about minimum deviation of state about c with minimum control effort is this helical probably in certain ideas that I discussed before as well.

So, then you can select something like this; in other words, your l is coming like that and phi is 0 actually. So, like that you can variety of things you can talk about and this particular thing talks about something like a terminal time dependent of a component of the j and this talks about some path dependent t 0 to t f what is happening on the way actually.

So, this is what happens at t f and this is happens at what happens from t0 to t f these 2 components are built around that kind of ideas actually.

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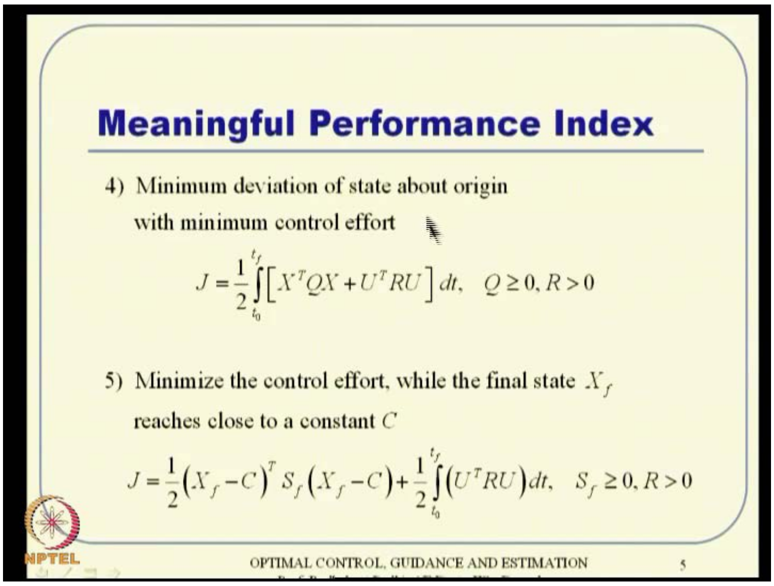

So, this all can do that and again if you talk about minimum deviation of state from origin and think like that then it is it gives a standard kind of quadratic regulators sort of ideas actually. Then, you can take values as something like this (Refer Slide Time: 04:45) and if you want to minimize the control while the final state reaches close to a constant value when this is, this must in a error quantity want to minimize that then you can talk about a phi component now at equal to t f this is to minimum. On the way, you want to control effort to be minimum; so, l has to be like that actually.

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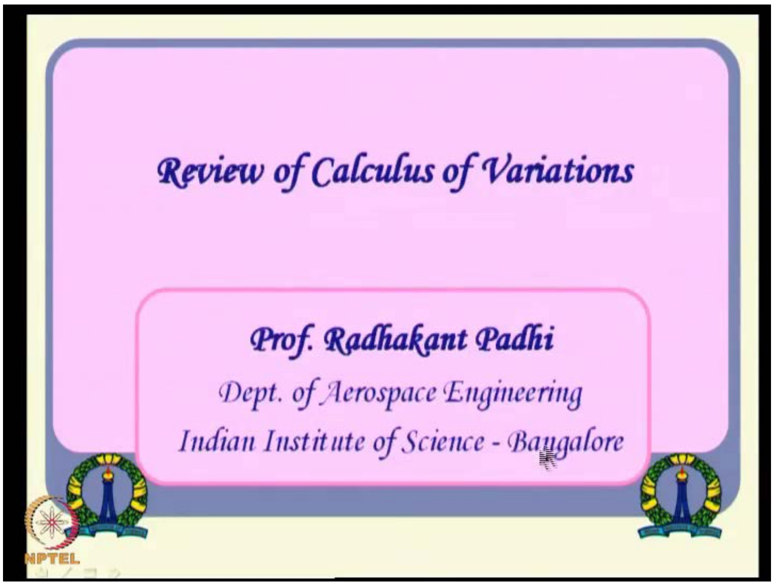

So, various things can be defined they are just taken I have kind of taken examples out of actually. You can think of your own problem and construct a cos function your own way actually. So, if you just see that this particular problem is directly falling on the calculus a various problem because we are interested in finding a state trajectory finding a control trajectory which will ultimately also give you some sort of admissible state trajectory.

So, it is a trajectory finding problem sort of thing and with respect to those kind of trajectories everything is optimal to each other. That means, the cos function is either

maximum or minimum that we want to do as well as it satisfies the boundary condition actually. So, directly it falls on the calculus of variation; that is why we were interested to see

some concepts of this calculus of variations actually – ideas, alright.

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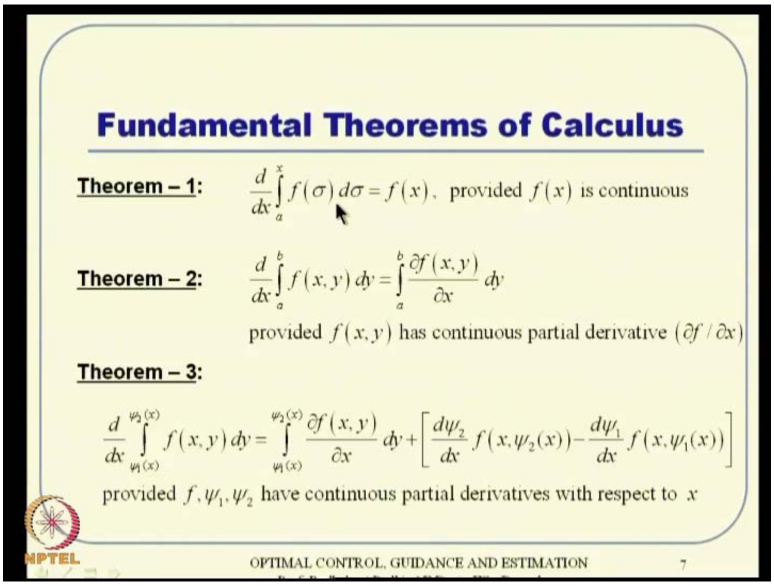

So, before that now let us know a little bit fundamental theorem are calculus as well as usual every talk about something like derivatives or integrals and all that it just good to know that actually; that is what is happening here.

Now that the limits are constant number actually and the integrals do not contain any function of state and thing I mean this function of these x variable then it comes out to be 0, but what happens if the limit itself is a function of state? Actually, the function of the pre variable then what happens, d by dx of this quantity happens to be f of x provided f of x is continuous actually.

These are something directly from calculus not really calculus of variation actually advance calculation actually. Now, what happens of these 2 limits are constant, but inside the function inside the integral nearby function which is a function of the dependent variable then what happens actually. It is integrated over y, but the result is a function of x anymore this integral is integrated over y. So, y is gone actually because that will be evaluated eventually, but it ultimately results in a function of x.

So, that is you can talk about derivative of x actually. So, this result is given something like that provided of this f of x has continuous partial derivative of del f by del x actually then you have something like little bit extension of that and you tell now what if in addition to that conditions my limits are also function of. So, pre variable as psi 1 and psi 2 instead of constants a and b we have this functions not coming there actually. Then, it satisfies something very close to that what you know here plus additional components actually. This will be something; this is what you want to be here plus these 2 components will come here actually.

Some of these things will be used in our calculus of variation analysis concepts also gives a some transposal eddy conditions ideas and all that actually; we see that there anyway. So, this is a quick greens of these theorems 1, 2, 3 which we do not want to prove here anything, but with this back ground in mind about the calculus of variation concepts actually.

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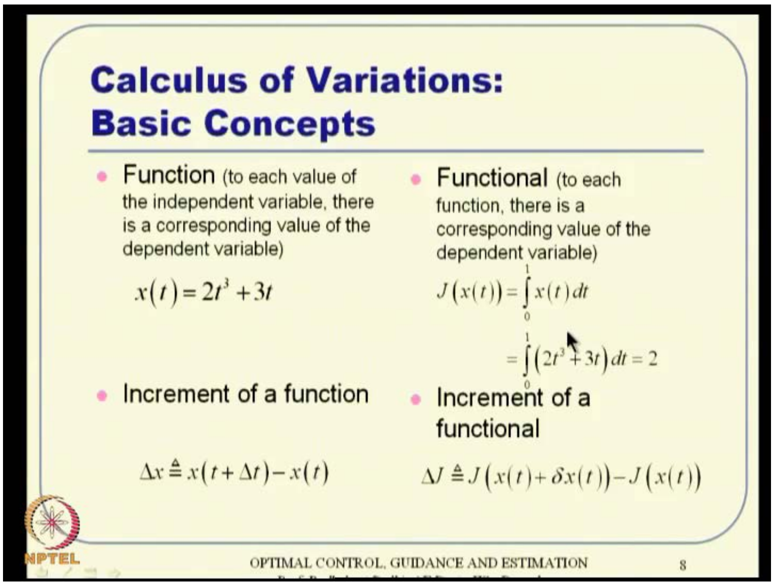

So, what is the difference between calculus and calculus of variations? On one end we talk about functions which is part of the calculus and the other end we talk about functional actually ok. What is functional? You can think of something like a function of function sort of things. So, j is a function of x, but x itself is a function t. So, that kind of thing we talk about a functional basically and typically, the functional are also kind of I mean scalar values and thing like that. Ultimately, it will be integrated; once we integrate it will ultimately result in some sort of scalar value actually. Whereas anyway, that is coming back to that function is something defined like this to its value of independent variable. There is a corresponding value of the dependent variable and we can know these functions from very basic ideas already. Actually, I do not describe too much on that. But, as a functional to which function there is a corresponding value of the dependent variable actually.

That is I mean, in a way you can it feel similar basically; but in I mean, if you just look at a kind of an activity idea sense then this is a function one is a independent variable and we got a function and it is a function of a function basically. So, that kind of ideas you can actually... So, here the differences is to each value of the independent variable there is a corresponding value of the dependent variable and here the reference to each function for this is just the difference actually. To each function, there is a corresponding value of the dependent variable. Here it is to each value of the independent variable there is a corresponding value of a were here is a to each function there is a corresponding value actually . So, that is the difference actually ok.

Now, we have talk about what is increment of a function; obviously, it is defined something like that x of t plus delta t minus x t and what is this here increment of a functional we need to talk about j of x t plus delta x t and this delta is denoted something like this and this itself is a function of t you know that ok. So, and then this will give you some sort of increment of a functional in the sense this minus that will give you some delta j basically ok.

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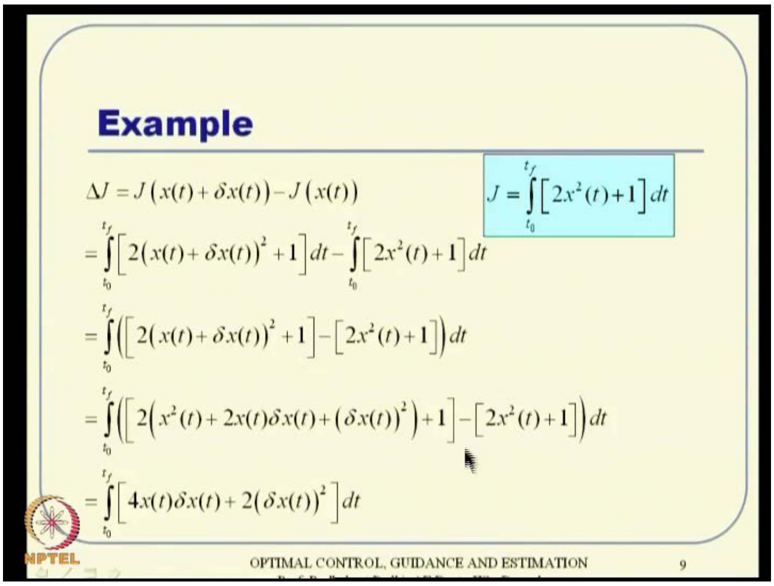

Pictorially also that is possible to denote, we will see that in a second lecture. Now, delta j if we talk about that what about some sort of a small kind of a example talk about j is something like this. The delta j by definition cannot be like that just from this the definition actually. So, we have j of x plus delta x both are one sense of time again minus j of x of t and j of x of t is defined something like that. So, I can put it somewhere like that, but j of x plus delta x I can evaluate wherever x is there I will put x plus delta x and then make it square and thing like that then I will integrate I mean even only a kind of combined the 2 integrals and cancelled out some prompts and think later that way I will this 2 x square and 2 x square and think like that will go well. So, what we will what will be left out with this kind of a quantity actually ok.

As long as you know this delta x of t as a function it is on wait then I can talk about evaluating these integral along with this x of t actually I too I too know both x of t as well as the corresponding variations throughout the time domain then I can evaluate this corresponding delta j basically.

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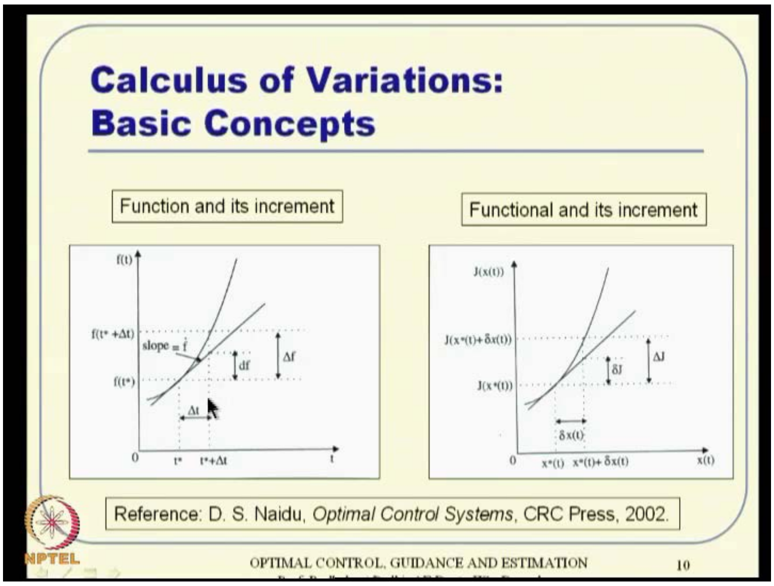

Now, this is j pictorially representing something like this and this is a function and it is an increment you have a function and independent variable in the x axis. You have a functional and the independent function in the yx I mean, x axis. So, you that is what the instead of a value we talk about some set of a value of the function itself basically in x of t. Alright, so, we will in other things will remain same I mean, if you whatever happens here as a as a independent variable sense will happen everything will happen here. In this functions project in the function sense basically, but remember this itself is the function x of t is in be a function actually.

We are just evaluating that function in a particular value of time and then interpreting what is happening there actually. So, we talk about let us say increment delta t we talk about variation delta x and then talk about some there this is the d f and this is delta j inclined that actually. And, typically will even though we know that this is a kind of our I mean suppose you should change the function from here to here this independent variable then the function goes from here to there; but if you put a first order approximation these are just like a kind of elegant approximation of sort of thing and then go up to that point then you will end up there actually will not to the end of there and also that is the defect thing and all that is the error quantity basically.

Similarly, think it will happen also here nearly both there, but this is the error quantity actually ok. So, most of our analysis will begin limited to the first order set of things actually. So, that is why this pictorial representation has on each other and more that you can say see complete from this book actually.

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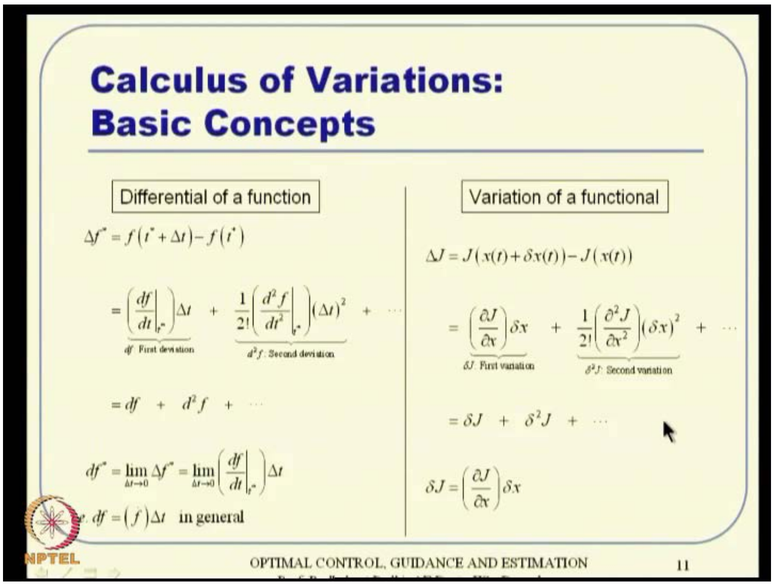

How do you calculate? Whatever you define as a differential of a function it terms to the variation of a functional layer ok. So, what in the differential of a function, delta f star is nothing, but f of t star plus delta t minus f of t star and then you can evaluate that way. So, it turns out to be d f and it's kind of defined d square of anything like that.

So, this is how t is and then if we talk about delta t goes to0 then we talk about the divide these both sides by delta t and then take delta t going to0 then it is represents that the d f by d I is all that is the first rid of (( )). So, you will end up with some approximations with we talk about actually. d f is nothing f dot into delta t in general that is what I told here; d f is nothing but f dot into delta t that is what related that actually.

F dot is the slope and delta t and then d f be back actually. So, this is how it is what we are coming to the other side of the thing variation of a functional you can again talk about very similar and then first variation, second variation like that you can draw and then you will end up with something like this actually. So, most of the time we will be worried about first variation in that will be define something like that way and second variations are important only for sufficiency check and all that later, ok. So, I mean, in one class later we will talk about that (()).

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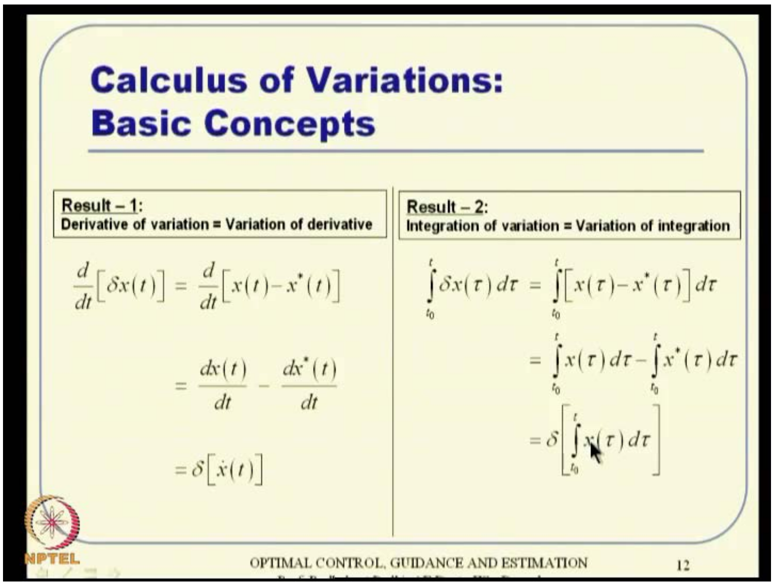

So, what we are talking here is a result 1; what is derivative variation is variation of derivative and result 2 which talks about integration of variation is variation of integration strong trivial and in the proof applies to very trivial also. But, it is a very powerful implication actually. So, what we are talking here? Derivative of a variation: this is by definition like that; this is the variation quantity and this is d y. Now, what we are talking here? All these, let me see whether I have a picture somewhere.

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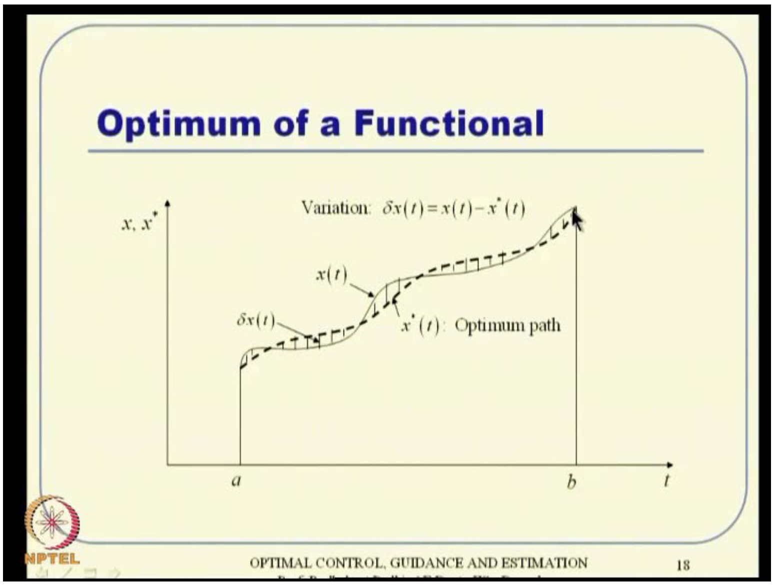

This is what it here talking about. So, it is actually a x of t what we are dealing here and some x star of t and the difference between that is also a function right I mean if you talk about various values of this x axis you will get various values of the difference actually.

The difference is nothing, but delta x actually that is what we are talking. So, if you have a path we are talking about if variation around the path that will, I will take some of the other alternative path around actually that is the concept of variation actually. So, this is what I am talking here actually. So, you can see that this that variation and then derivative of that and then simply use the definition of this variation x minus x star and then it will lies as to evaluate the derivative and these 2 difference is nothing, but the variation of x dot by definition again is that the sum derivative minus sum derivative of star value actually. So, by definition it is nothing but delta of x dot actually.

Again, very same algebra; similar algebra you take the integration and then put it by the definition and then separate it out. It can be done because both is derivative and linear operators are actually linear operator. That is why it happens; actually talk about this; separate it out and by definition from here is nothing but variation of this function; this functional actually. We consider this integration itself is a functional; then it is nothing but variation of that function actually.

So, what we are talking about it is integration of variation by definition; like that is variation of integration which is that by definition like that. So, very simple to line proof at very powerful result as we will see little later actually.

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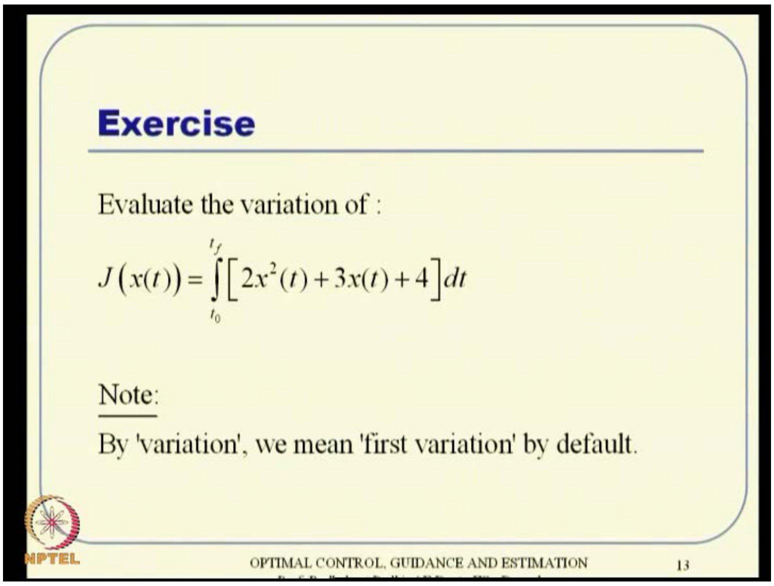

First thing, some exercise problem (()) creation of this particular functional; how do you do that? Also, some small kind of comment that tells you that just evaluate the variation; that means, first variation not many certain power variation and all that actually. Default variation means first variation actually. We talk about this problem; we want if we evaluate the first variation or just the variation of the j then method one you can just go out to definition (()) that then put the j. Then try to expand it; put the definitions and then end up with some expressions like that and remember second order, third order of the variations are neglected.

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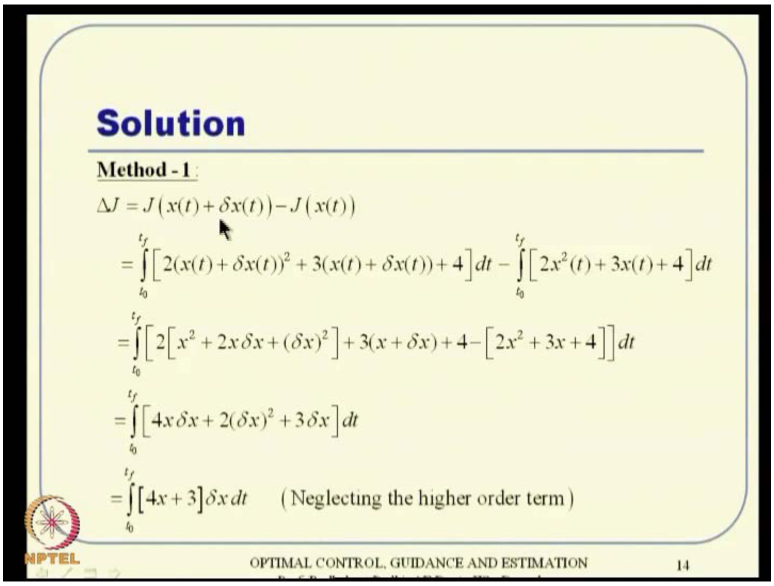

So, you can neglect all these terms and you will end up with something like this actually. What about method 2? This is you can directly go to the result; this is the result that we discussed; this one then you can simply put it back here then, evaluate the del j by del x directly. Then you can the except this area of variation of integral is integral of variation here; that means, this is what we are talking about. It will be like this; this one can be pushed inside, now this integral actually.

Then you can end up with something like this. Directly we are taking help of del j by del x evaluation actually. That is the difference and that simplifies value of algebra and you will end up with something like this actually. This derivative can be pushed inside; this limits are constant; now can be pushed inside and then you can have a simplification term 4 x plus 3 here. So, that is how the reference there actually

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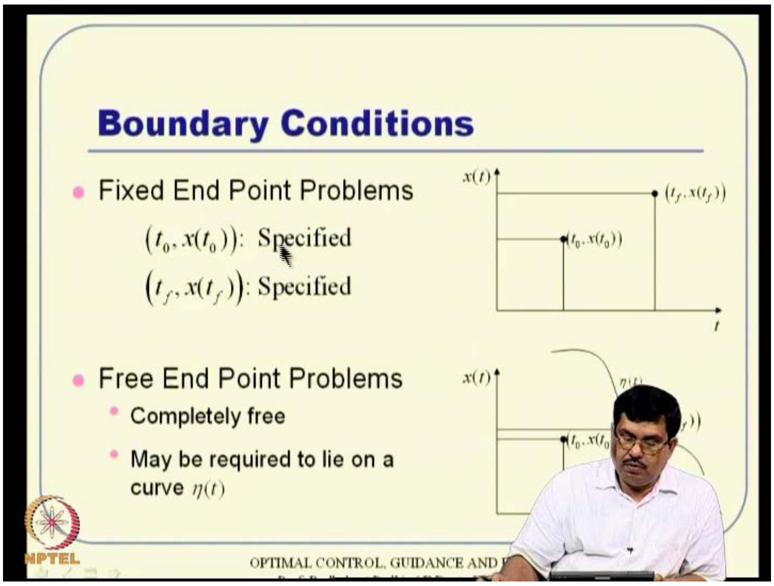

Now, what are this some boundary condition ideas and all that? I remember this optimal control problem also talks about boundary conditions and here you can talk about a variety of boundary conditions. It can be a fixed end point, free end point, partially fixed, partially free, something like that actually. So, when you talk about fixed end points what we are telling is t0 my x of t0 is a fixed value and t f my x of t f is also a fixed value. So, this is the like you too have a solution as to satisfy these kind of end points. Actually, t0 x of t0 is a number I mean, specified number and t f and x f is also a specified number actually.

So, that is a kind of fixed end point problems of and free end points problem is can be both these; that means, it is either completely free. We do not really know about where it lies and things like that or it may be required to lie on a curve actually. Many of these things practical examples in probably demand that for example, as if you really want to launch a satellite. When you go and launch a satellite anywhere in the orbit is fine actually. That is, once you have, once you satisfy the orbital conditions, it will keep on evolving in the orbit anywhere. So, that in your job actually does not matter with may which point (()) with the orbital condition said the point; obviously, if you consider that they we nothing but a function.

So, orbital equations will happen to be an elliptic and then that elipse as a (()) equation, governing equation sort of things. So, if your variables happen to merge with that governing equation then you are fine if the location does not matter actually. So, this kind of problems are something like this it may require to lie on a curve eta of t actually; or it can be simply completely free. You do not really worry about where it goes something like that as long as it it satisfies the other. Objective is basically with infact to the end points happens to the completely free will have some term in the cross function which will give, which will kind of while the problem in a loose constraints sense. In other words, soft constraints sense other thing. If you have a soft constraint term in the cross function then you have a final boundary condition can be really free, but in a way it is indirectly kind of directed towards that actually required.

Now, what are the specified points can be like I mean these t0 x0 and t f x f? Specified means it can be either hard constrained or soft constrained. If it hard constrained we are demanding that if this value x of t0 has to be equal to certain value and x of t f is be equal to certain value and suppose it is not a hard constraint, but a soft constraint. Now, telling x of t f can be somewhere around that value ok. So, the soft constrained part typically goes to the cross function and hard constrained typically becomes kind of a end point problems sort of things so; that means, boundary condition sense it will come actually.

So, all that I means these are the concepts that we are talking about. So, the boundary conditions can be fixed end point conditions or free end point and if it is fixed this way if is free that way actually ok.

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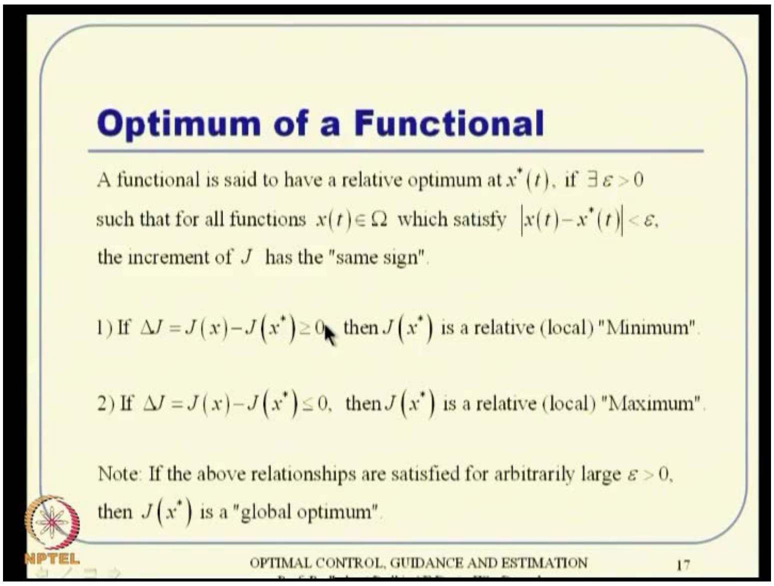

Now, what you how do you defined optimum of a functional actually. So, that is where will a final objective will lie there. Now, this mathematically it can be defined something like this a functional is said to have a relative optimum at x star if there existing epsilon which is greater than0 such that all functions x of t belongs to omega that is the domain of interest what you are talking which satisfy this conditions and; that means, you are not along this variations to be arbitrarily large.

If satisfy this particular condition if if epsilon happens to be a small quantity in general then where along those variations in a around x star actually ok. So, for small variations any small variation you take then what happens here the increment of j has the same sign actually ok. The same similar concepts that we derived from the static optimization and all if it is a minimum point any direction (()) the function value is supposed be more than that minimum point value ok.

Similar thing it is the any variation you take around a around optimum path the optimum I mean the ultimate crossed function that is of interest to you has to be either (()) either more than that value if it is a minimum problem or less than away that value that a maximum problem ok.

Ultimately the cross functional and most of the time we will talk it as a cross function also both with the assumption that we are talking about functional anyway. The most of the time what will happen in it is if you take any arbitrary variation around optimum path the function is oppose to either increase or either decrease or decrease basically that sense we talk that is optimum value is basically ok.

So, this is what you written here. So, if you take j of x minus j of x star if it happens to be greater than equal to 0 for all variations delta x we satisfy this kind of (()) of this is not the minimum delta x actually delta x found absolute value of delta x at the ending point of time is less than equal to less epsilon actually for all small. So, also small variations this is true then its called a local maximum for also variation if it is this one is true then it is a local minimum actually and if it is satisfy these conditions are satisfy for arbitrarily large values of epsilon then; obviously, j of x star happens to be a global optimum point also visited ok.

If you relax yourself than the various need not be small variations can be large is large as possible then it leads to the concept of global optimum actually.

This is the concept that I have just. So, new also of the suppose you have actually found an optimum path this dotted line dark doted line actually that is the that is the optimum path then if we talk about any variations around that can be repellended like that.

So, if I evaluate my functional around this solid line I will get I will a value which is more than the integral value are at the cross functional value if I evaluate on that path actually. So, there are 2 paths here and remember the cross function happens to an integral valve of this functions basically.

So, if I evaluate the integral taking this function what about solid line thing I will get a value which is higher than the integral value if I use the other one instead actually that is the already other.

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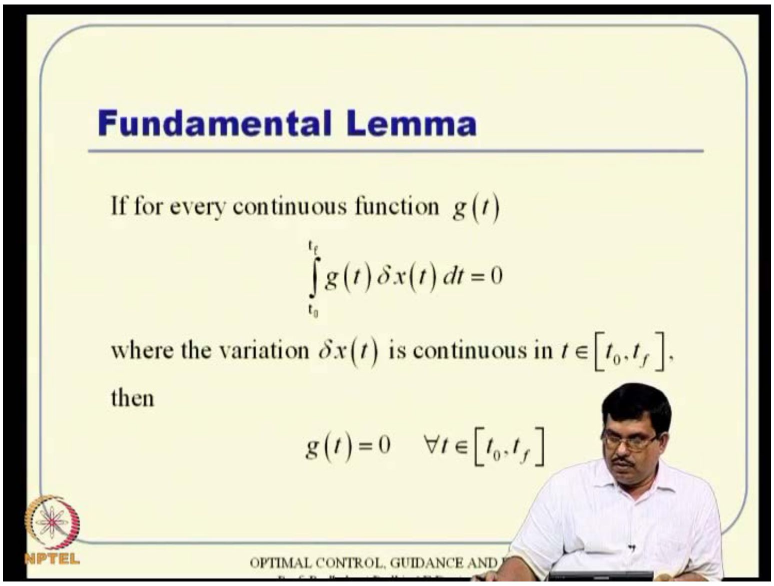

Now, what is this fundamental theorem of calculus of variations this is a very nice fundamental theorem here it tells us that for any x star to be kind of a candidate optimum solution we should have a necessary conditions which tells us that the first variations is equal to0 sufficiency condition tells us that the second variation should satisfy this (()) either positive definite or negative definite actually ok. Close to what we know in static optimization ideas theories, but extensions of that into dynamic optimization ideas

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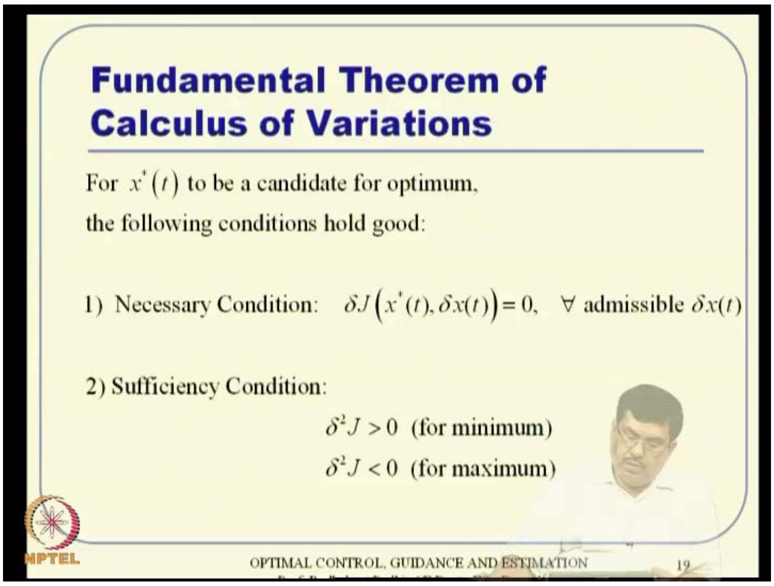

Then, there is a very interesting fundamental lemma as you settle with this calculus variations which tells that this kind of things. So, for say, if for every continuous functions g of t this is true then this has to be 0 for entire into whole actually the create theorem again if for every continuous function g of t if this is true this every position actually this is true ok. Then this has to true for all for entire interval actually ok.

And, only condition is delta of x that is the variation of x has to be continuous and delta domain as long as the variation of the delta x is continuous in the entire domain in addition to that this is true this integral is true then the integral value is to be true for the entire interval is a very powerful theorem actually ok. When integral value being equal to0 for the entire interval is something there is a big implication actually that is what we will exploded heavily in deriving this optimal control necessary conditions and all that (( )) ok.

So, we will see that. So, were again that theorem is like this and little proof associated with that also let us see because it is a very important theorem as well.

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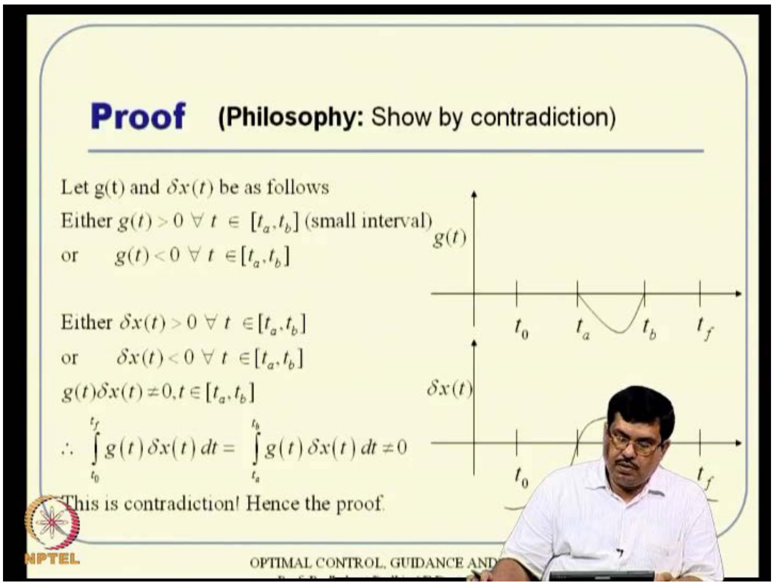

So, it is actually kind of a very easy to solve by using this contradictions ideas and all that actually. So, what it tells you that let g of t and delta of delta x of t be something like this; that means, g of t is (()) hum I mean greater than for (( )) all t which belongs to that small interval if it is it is less than 0. We are not interested in other things it is and either delta x is greater than 0 or delta x is less than 0 and this interval each other ok. But, but it is. So, (()) if it is true than g of t delta x of t is not then I mean is not equal to 0 that is the over idea actually.

What we are talking here? We are talking of some contradiction basically. So, what we are showing while interested to solve this one ok. So, what is contradiction here that g of t is not 0 am when g of t is not 0 then it can he either positive or negative right that is only we are telling actually right. This is the contradiction g of t is not 0; if it is not 0, they can be either negative or positive I mean that is the only 2 conditions that will happen.

So, this is what is written there is actually and we are telling do not worry about the entire interval, but limit our analysis is to a small interval it can be 0 everywhere else we do not care about that one particular example where within this interval t a t b the functioning either completely greater than0 strictly positive throughout or strictly negative throughout is possible right (()) may long as t a t b is small possible we are taking that particular type of thing actually.

So, remember that then remember this is something very interesting actually. So, delta of x this delta x is can be arbitrary right. So, somebody can choose the delta of x something like this delta x is actually variation of h x (()) right if can take any arbitrary thing and it talks about this function is continuous and it can take any variation sort of things.

So, we will choose a particular variation which is non zero in in that interval that is what that is what is matters all these math equations as I said we are considering that particular case where g of t is non zero and a small interval at least it is everywhere else it is 0 and we are we are selecting a particular variation for which for in this interval t a t b the value the function the variation value is non zero. And if is strictly one side it is non zero does not mean it can change sign in all that actually that is the type of variation we are selecting, but because where free to select any variation (()) each other ok.

So, as long as this is completely one sided and this is complete one sided there sort the integral of that whatever integral we are talking about here it can turn out to be something like that. It can turn out to be something like that and it means it is non zero because, if you multiply any positive quantity or negative quantity with any positive or negative quantity here throughout basically.

Here, then we end up with a non-linear value that is what (( )) need be positive or negative value something like that. We just requires that the integrally is non zero, but what tells you where it also that do not allow basically right because this is the requirement of theorem it tells you this to be true and here is the case where this not true basically ok. Why it happens because it fundamentally we assume that g of t as some non zero value in this way interval actually. So, as long as a very within a very small interval this g of t f ends non zero.

We can always this want select some side of variation which is non zero and once that particular toll and play in that this integral in non zero. So, using that concept and I mean, using that idea this you can tell that if we that also not allowed then the only way it cannot appear easy (()) the function is continuously0 through its actually. If it continuously0 there is no interval at t a and t b in between where you can take some value like this in one sided of the value where I can this (( )) some small variation around that; thus, the whole idea there. So, it is a very interesting theorem which tells you that this happens to be included. This happens to be true for every continuous function g of t. Thus, this is happening to be true where this variation delta x is continuous from some of in that then this g of t must be equal to 0 throughout the interval actually; very interesting kind of ideas there actually, ok.

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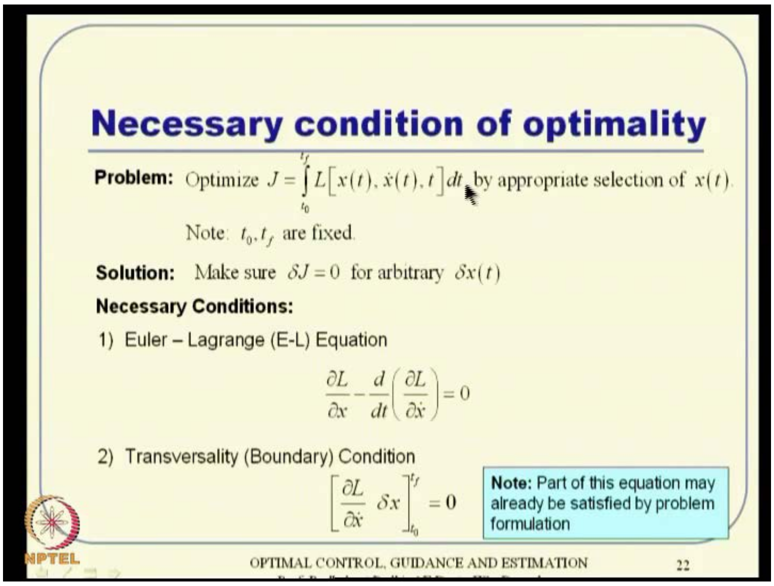

Now, coming known for this we talk about this necessary condition of a optimal eddy. So, what is the problem here problem here is to kind of optimize this gauss functional now it for this functional what we are talking about here function of x and x dot as well as independent variable t x t actually. First idea here, we want to optimize this gauss functional here gauss function by appropriate solution of this x of t and here, you will consider this t0 t f are both are fixed and what the solution the solution tells for using for fundamental theorem in all tells necessary condition tha's the first variation is to be equal to0 ok

So, if you this in the first variation equal to0 in the that part will not be able to prove in this flow probably just take it (()) and it is very interesting concept if you see this pictorial ideas and all it is vary (( )) any orbital variation you take a function is to increase and if I really have a optimal value then that will attain either the variations will not lead to any improvement actually; that means, the first value assign has to0.

So, that kind of value of this (( )) if you if you talk about that and use this region first deviation is equal to0 then we will end up with 2 interesting equation is called very famously called Euler-Lagrange equation and other equation is called something called transversality condition essentially that includes (( )) something like a boundary condition actually ok and we remember that this boundary condition can like part of the equation might have already be satisfied by problem formulation itself that we suppose you talk about initial condition being fixed you know the result condition (()) you know variation along that then you talk about delta x is 0 any way.

So, delta is 0 this equation is 0 at t0 basically that is part of the equation as already satisfied and now part of the equation we can we can get it from this transvaersality condition actually. Now, little bit interesting history part of thing this e l equation is first actually derived by Euler and who represents to be quite kind of senior to Lagrange actually about then regressive comes with some sort of I mean, the Euler equation. All about discretization and all if we descritize and then (( )) is it. So, the c l equation in simpler of simple of algebra and all that but, something it terms this like discretization are not comfortable with discretization process itself actually.

So, the like this ideas of calculus actually that is rigorous something like that. So, using this calculus ideas and all Lagrange which are who was that kind of follower of Euler they can have a if it an alternatively derivation which will see in now (()) and that derivation Euler kind like it very much. So, he prescribed to the world that approach to be followed.

Now, the discretization approach pursued basically. So, the people all that kind of history, but ultimately it will kind of result to the same equation anyway. So, this is the kind of famously known as el equation or Euler-Lagrange equation actually using the negatives.

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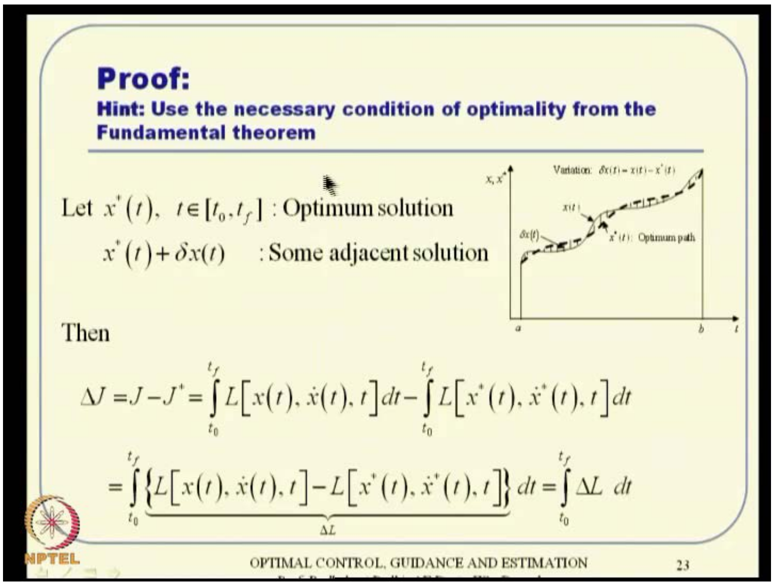

So, so that is why we can try to say what you done in this a. So, the proof part of it is what I am talking about. So, this is the problem we have got an x of t and we have got an star of t which we client that it is an optimal path it is an optimal path. For the corresponding variation function what we are talk about here expect t minus x star of t the difference part all over actually.

So, this is what I am including x star is an optimal solution and x star plus delta x is some adjacent solution actually. So, what is the delta j j j minus j star it can be end up it will like that j represents to be this one by definition and j star refers to be that by definition. So, if I if combine this 2 I can talk about l minus l of all these minus l of all these star values and that it will nothing, but delta l actually ok. So, sincerely I can represent delta j as an integral of delta l times g t.

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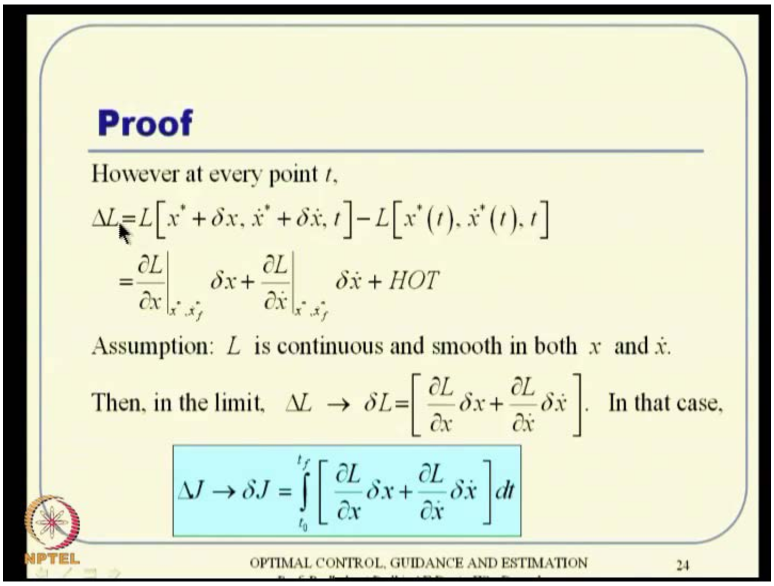

Now, what a delta l way to analyze that and delta l is nothing, but l all that may not this is x x dot n t x is nothing, but x dot plus delta x as similarly like this actually. So, delta l is like this and I mean l of all this star values and l of sorry all the star values plus delta x and plus delta x dot like that and minus l of this value actually.

So, if you see this expression this is again gives us a scope to analyze it using taylor series actually and I keep telling this taylor series appreciation by over by more number of engineering ideas is it. So, we will repeatedly keep on using this Taylor’s series ideas.

So, now coming to it this delta l is everything of del of that minus l of this ok. So, using this taylor series ideas will end up to something like that with all the terms and neglecting again this high hot term and all we can now tell that this delta l turns to this variation delta l first variation sense if I come if I came only the first variation quantities and always nothing about that because I am neglecting the (( )) here ok.

So, here now what happen to this delta j is tends to that approximately equal to this delta j other is the variation of first variation of j and that can be represented like that we get this one delta j anything about that and delta l it turns to be something like this. So, using this 2 you can write the delta j times to delta j variation j is nothing, but that is what we get actually.

Now, at the time to analyze this one in the variable to include that, the problem here is this term as we know what is variation of x we really do not know variation of x dot l and this 2 not really extremely advertising quantity. You cannot select one extremely independent of the other because one is the function of time in the other one is derivative and the variation that actually for that kind of relative to each other actually anyway. So, we take on for that actually.

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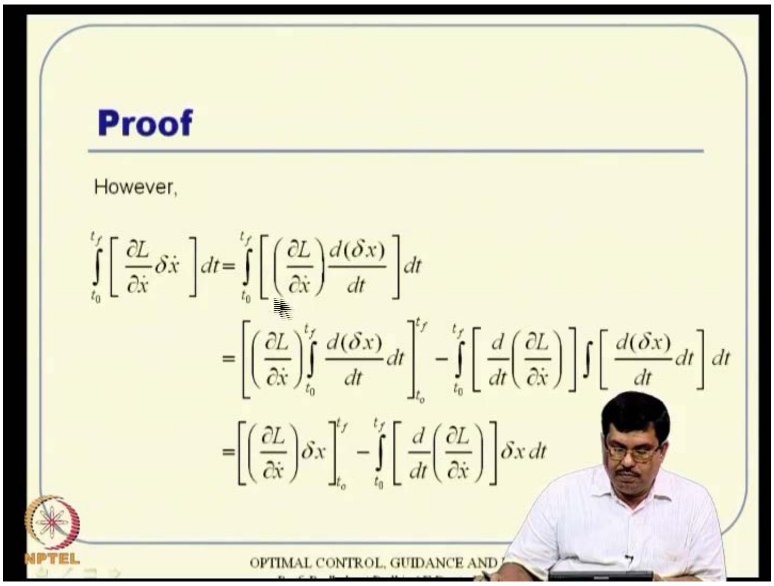

What happened to see is this quantity is what is our concern and this quantity by definition what we are talking here this is that term by definition this variation of x dot is nothing, but d by d t of variation of g just definition. When we talk about of delta of x dot is nothing, but d y d t of d x, I mean delta x is equal. So, now here is an opportunity to use this integration by part now take about this is one function in this another one.

So, let me do that integration by parts. So, I talk about using that integration by part. So, first function times the integration of the second function minus integral the derivative of the first one into integral of the second one and all that actually integration of everything just nothing, but integration by parts actually. So, if you do that what happens now is going to some d t d t is kind of varying here to various derivatives and integration no need to do that.

So, it resulting only delta x; so, del l by del x dot into delta x ok. We were nearly this integral now and similar thing we will also appear here. So, what we learn of it d by d t of this term multiply by del x into d t actually. Now, if we look at this term by this term this term contains variation of x dot and all that this term does not contain variation of x dot.

It contains partial derivative with respect to that is ok that it will be very well and the reason to all types of variation of x n s 0. Anyway, nicely going up this inter dependency of this various n and its derivation that is if now, go with this is the 2 term one term and second term. So, second term we got some ideas. So, we will put in together and then del j nothing, but first term plus second term and first term we will keep as it is second term will put what we got right now.

So, once you put that this is how it is actually. So, this is the first term this is the first term in that this quantities. So, that comes here and that is the second term of that quantity which comes here ok. So, what will getting here after that we can we can talk about combining the first and last term first term and last term you can combine and then put it in the integral from here. So, and this one is not a function of n this is not an integration process. So, it will keep it aside actually. So, what you are talking here that you talking about just an simplification of delta j in terms of that we talks about only variation of x and integration of with respect of dt basically.

So, we have a term which is an integral term and you have a term which is outside the integral actually, but what you tells us the fundamental theorem what does it tell us it tells us that the first variation is to be equal to 0 ok. So, first variation equal to0; that means, this quantity has to be equal to0 ok.

Now, you will go away and find out this quantity can be equal to0 provided this is independent of this integral process. So, this has to be equal to 0 these 2 are kind of very independent quantities actually ok. So, in in a way this has to be equal to be 0 that integral terms is to be equal to0 basically thus that is what we are demanding basically anyway. Now, what happens to this quantity this is clear actually this quantity has to be equal to 0 that is clear.

However, what about this quantity now we will tell that if the integral is 0 throughout and the variation is continues then we will exert this theorem that we just discussed with proof and all the actually ok. We will tell the integral value has to be 0 throughout the throughout the interval and that is what to lead us to this integral value whatever integral is these that is to be 0 throughout the throughout the interval t0 to t f actually. So, that is how we will get this thermoseal equation we will get this real equation is nothing, but this integral quantity actually ok.

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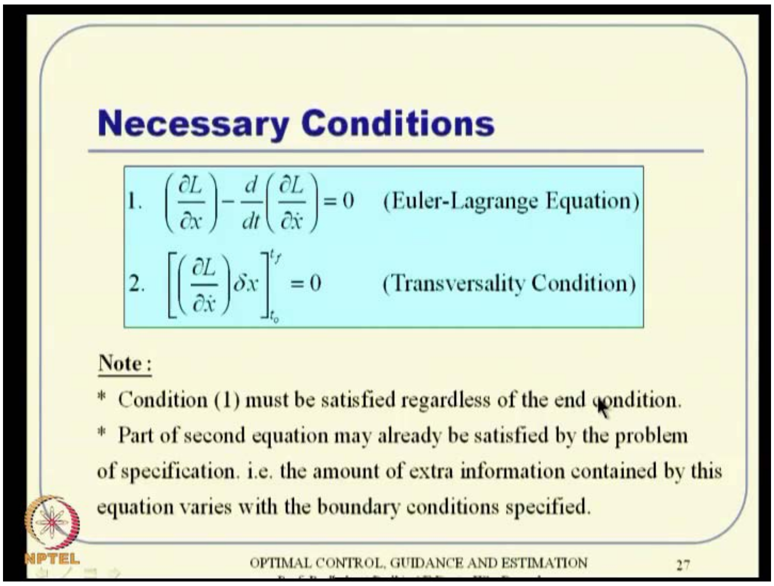

So, del l by del x minus d by d t of this quantity is equal to0 is famously known as e l equation and then the second quantity is nothing, but that refers to be a boundary condition equation or transversality equation actually.

Now, couple of comments says that the first equation that we are talking about must be satisfied regardless of the end condition you remember this condition comes from the integral quantity does not talked about the end condition actually. So, this condition has to be true throughout the interval and that condition has to be true at the boundary points actually and as I told you before part of the second equation may already be satisfied by the problem of by the problem of by the problem of definition actually.

If we talk about the fixed end point condition like you start with as certain initial condition then delta x0 is already basically ok. So, that is not required. So, how this is utilize by the way this small evaluated t f minus this one of the same one evaluated at t0 that is equal to0 that is how this equation is interpreted that way the this value all evaluated at t f minus this the same value all evaluated at t0 is equal to0 ok. So, that is the best way of interpreting actually; obviously, time for a small example problem to get our ideas in more clear actually.

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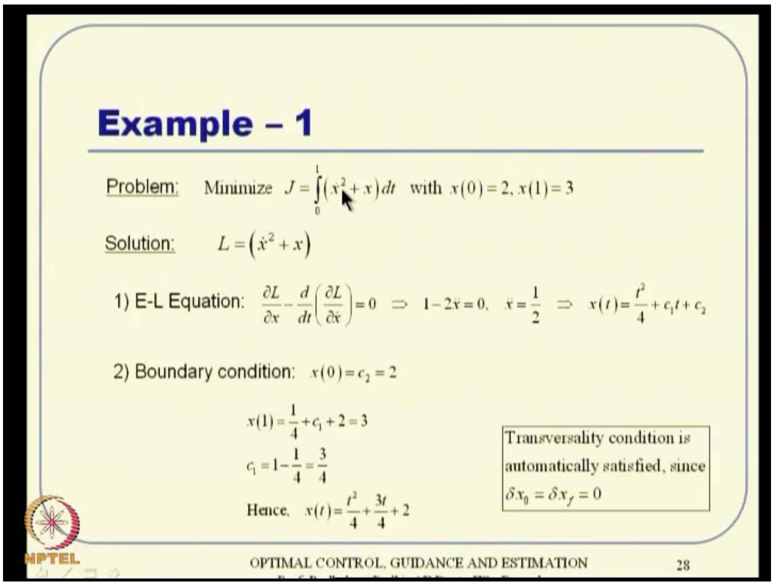

So, the whole idea here I mean the problem definition here is something like this we want to minimize this quantity x dot square plus x and into dt this is a cross function to minimize this with these boundary condition we know this boundary condition of fixed actually. X of0 is 2 and x of one is 3 both are tightly fixed there is no variation around those particular values basically.

We want to do that this minimization we want to do. So, what is the solution here we will follow this e l equation approach then first to do define this l l is whatever is inside the integral. So, that is l is nothing, but x dot square plus x in this particular example then e l equations tells that this condition is to be true del l by del x minus d by dt is to be 0 ok. So; that means, if you plug it back this l definition here then del l by del x nothing, but one it come from here ok. So, this is one then d by dt of this one. So, what is del l by del x dot this is nothing, but 2 x dot and d by dt of 2 x dot is nothing, but 2 x double dot.

So, one minus 2 x double dot equal to 0; that means, x double dot is equal to half and then you can integrated very easily when it is double dot is half then x dot is nothing, but something like half t plus this is easy right t mean it is that you want something like this happens then x of x dot of t is nothing, but t by 2 and x double dot of something like I mean x sorry x of x of t is an integral of that (()).

So, this is something like c 1 and x dot is nothing, but t by 2 plus c 1. So, x of t one more time integration. So, that is t square by kind of 4 t square by 4 plus c 1 t plus c 2 that is what you done there. This one again there is already there anyway. So, from this equation x double dot is half you get x dot is t by 2 plus c 1 and if you integrated one more time you will get t square by 4 plus c 1 and c 2 that is how we will get actually.

So, it is 2 constants c 1 and c 2 which make sure evaluated at the boundary using the boundary conditions. So, what is the boundary condition available x of0 is 2 and x of one is 3. So, you put x of 0 is 2 that will give you if you evaluate that at x of0; that means, t equal to0 this is 0 this is 0. So, this is nothing, but 2. So, 2. So, 2 equal to c 2 that is what you directly get it here and if you evaluate the other one that x of one equal to 3 then x of one is t a t a is nothing, but 1. So, it is one by 4 plus c 1 plus c 2 one by 4 plus c 1 plus 2 now because c 2 is 2 has to be equal to 3 in the best condition itself hmm.

So, you put that back and evaluate c 1 and c 1 reference to be something like this 3 minus 2 is 1 already minus 1 by 4. So, that it reference to be 3 by 4 the final solution reference to be this one. Actually, that is how you solve it this very simple scalar problem with 2 fixed boundary problem boundary conditions actually.

But you do not worry; this may not have this kind of lecture all the times actually and also a small comment the transvesality condition is automatically satisfied because there is no variation around that this are this are tightly fixed there you know variation there 2 are 0.

So, this quantity what y p u see here delta x t f is 0 delta x t0 is also 0. So, this condition is automatically satisfied you do not worry about that you do not get anything any extra information from that is already invited while we talk about fixed boundary conditions like that. What happens to the next equation and let will see an example that variation problem very similar looking problem j is said and x of 0 is 2 which is also same, but now x of one is where it would be free it is not fixed actually then we will see to get the solution nature changes actually,

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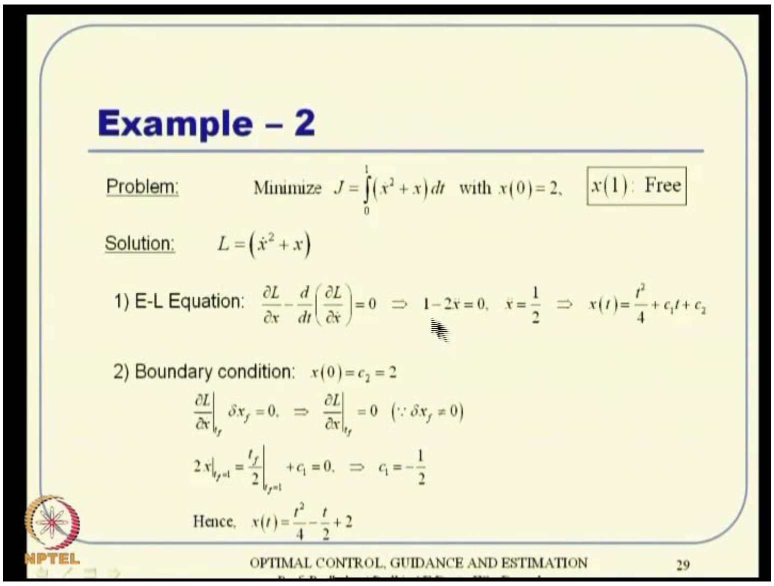

How do you do that? How do you see that this part of equation remains is same and the a l equation is same this equation is same and hence the generic form is same, but when you start evaluating the boundary condition again the first boundary condition is same. So, again c 2 is 2 that is fine, but the second boundary condition you cannot evaluate anymore.

So, that it obey to that sociality condition and tell del x by del x dot into d x f minus del l by del x dot into d x0 I mean whatever the this equation is 0 nowd x0 is 0 that pact is alright, but first pact we will have to establish because that is that is not 0 in general because of x of is free. So, it can take any value. So, delta x surface can take any value as it good hmm. So, we will. So, that part is that happens then this coefficient corresponding coefficient has to be 0 that is at what you will give us certain the value actually.

So, this appear as an to be true, but this is not 0 is general. So, the coefficient has to be 0 the coefficient is 0 then del l by del x dot del l by del x dot is nothing, but 2 x dot right ok.

So, 2 x dot evaluated at t f equal to one do you know that is o more important actually this happens only at the boundary. So, t f is one here o 2 x evaluated at t f equal to one that has to0 and what is 2 x dot you remember x dot is that ok. We just derived that I mean its x dot because x dot is that because 2 x dot is something like this mh mm alright. So, will get some value there actually. So, it is just evaluated that I mean evaluate that del l by del x dot equal to del l by del x dot reference to be something like things. So, 2 of that and then put it equal to 0 to 0 sort 2 things actually hmm if 2 x dot happens to be mm; some 1 second 2 x dot happens to be 0 mm then x dot is also 0 ok.

So, that is what I a happen here and get this small algorithm mistake here. So, let me try to correct it; then, the 2 x dot is 0 then 2 is not necessary x dot is also 0 ok. So, when the x dot is 0 then x dot is nothing, but that is what we write here. So, that is what is getting used here actually. So, c 1 is nothing, but minus of one because t f is one you remember that, so, c 1 reference to be minus half.

So, what is the solution? The solution is this. So, is one case we will end up with a solution something like this and other case we will end up with a solution something like this very different (()) just because one case the bond the final boundary condition was fixed at a value the other condition tells above its a free value actually and remember free means it can be forward from this value we do not care actually and that is why it gives a different solution actually required. So, this is what is the fun of I mean in kind of very interesting to see the in calculus the various an ideas actually ok.

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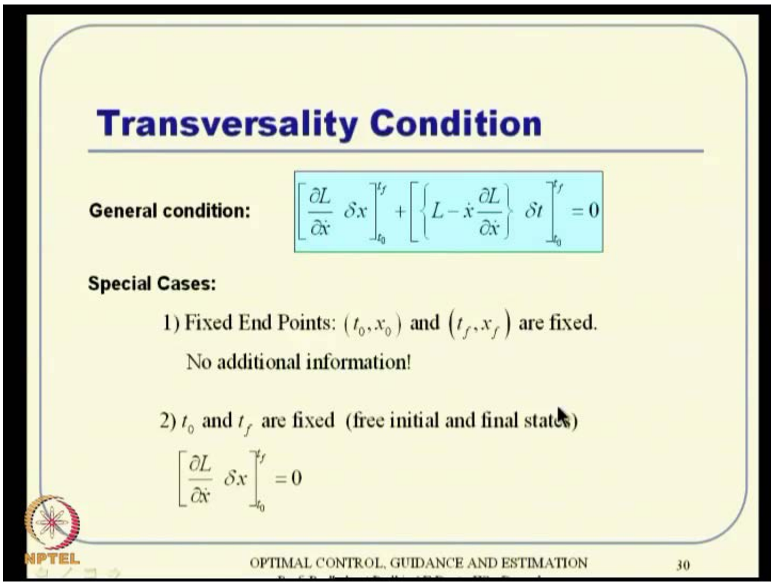

Now, transversality condition in general to an one can talk about something this. So, first we derive this part of the condition, but in general if t is also free also variations can happen the initial time as well as a final time. So, in a all oven derivation here, we assume that this r0 and t f there are fixed actually, but suppose there are also I mean they can also vary; that means, you do not know the initial time it you allow some flexibility for that and you do not know the arrival time you also allow some flexibility of that; that means, t f is also kind of flexible than you can have I mean that for those problems are also allowed in the frame of calculus a variation and in those derivations you go through that when will you learned of something like this ok.

Alright and also remember this minimum time solution flow domain all that this applies to be a very critical information actually ok. When we talk about except I mean t f is 3 and you want to minimum time also the end that kind of formulation will see as you go long later what you will see the this transfers are transversality conditioned is very (()) actually.

So, the fixed end point curl thing these are fixed. So, we does not give any additional information if you give this condition there t0 t f are fixed what we us to drive than these 2 quantity this t0 and t l both are 0 there this is gone. So, will end of with that we derive; however, it t0 and x0 are fixed, but free final time and free final state then you are talk about there now because t0 are the I mean delta t0 and delta x0 those are0 what the delta x f and delta t not these are not 0.

So, end of some equation like this similarly this 3 are fixed at at the free final time then will end up some equation like this and if all are fixes then you learned of something like this actually. So, this (()) general transversality condition kind of in builds a lot of cases of cases that are of interest rows actually ok. Alright, and there is very special case that we discuss in the beginning that initial dimensed ik mean 0 are fixed; but this tf xf is constrained to lie on a given curve eta t then what happen. So, you invoke this and this one mean transversality condition, but remember the delta x t is free, but in a constrained way that mean it has to be it is lie on this curve, where delta xf can ne can we kind of expanded this way delta d eta by dt into del t f follow thing and so, ti gives us some delta x f is nothing, but that basically ok.

So, now you put brake and then we talk about this is the whole thing. Multiply by delta t f is 0, but delta if is non zero because you are allows some of freedom that here. So, this coefficient is to be 0 that will give you this transversality condition when we talk about our final state or constrained to to a lie on a curve on a known curve it (()) basically ok. So, this one 2 3 4 5 six we are the different kind of special cases and all these thing can be derived from the, from this general transversality condition actually.

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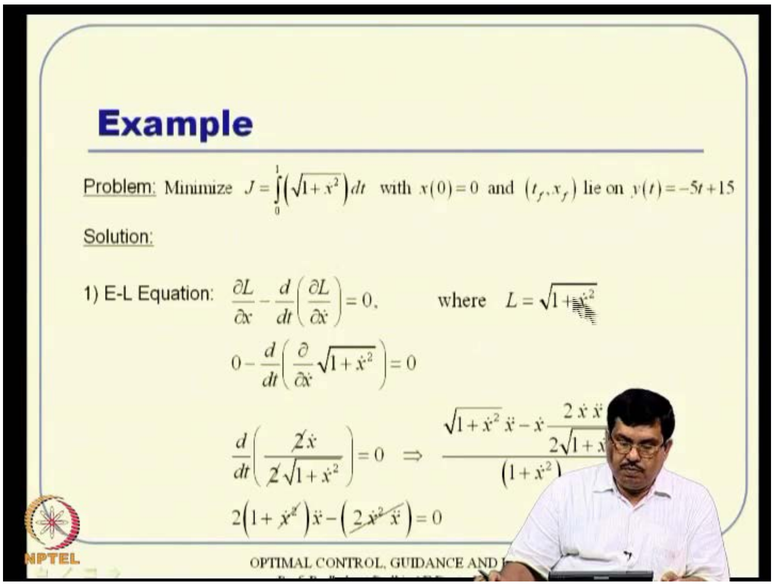

Alright, so one more example it talks about minimization of this functional lets with this condition x of j is 0 and t of x of with should lie on this curve this occurs that I am talking here el equation is very straight forward del l by de x minus d by dt of del l by del x is out be 0 where l is nothing, but the integral quantities that is that is quantities and in integral value what about it is there.

So, it just (()) the values d l by dx is not function of x. So, the partial derivative with respect to x is 0, but partial derivative with respect to x dot is not 0 because. So, you would you would take the partial derivative with respect to x dot and then put it here and also the d by d t of that we need to talk about. Remember, this is just the partial derivative that quantity, but then you talk about derivative that quantity with respect to time again that is to be equal to0 and when you do again this partial assigned do that you can being n and tell this is square of that this is like that then talk about this quantity to the derivative that quantity minus the same quantity into derivative other function like that actually this is some f of 5 x to the x dot divided by g of x dot shadow things each other.

So, use that formula properly when you take that derivative that an ultimately will end up with the cades here the one plus x dot square is always a positive quantities. So, we will ignore the that the multiply that side it is 0 and you learned up simplify this impression you will end up with some equation like this actually and this quantity what you see here and the quantity will cancelled out to this case over there and you will end up with this 2 x dot I mean, 2 x double dot equal to 0; that means, double dot is 0. So, this first term, this to second term and this term will cancelled out. So, will be left out with 2 and 1 is still there. So, 2 x double dot 0 so; that means, x double dot is 0 e l equation gives of that. So, x double dot is 0 means again c x x of t is nothing, but c 1 t plus c 2 basically.

Now, the first boundary condition tells you that x of 0 equal to 0; that means, if I put0 then c 2 is nothing, but 0 will end of with x of t is c 1. Now, the transversality other transversality condition will tell us something like this one what is derived here and if you log in that gives a some condition like that ok and if you if you work out I mean work it around that given us this constrained. Actually, remember x of t is like that. So, x sot of t is nothing, but x dot nothing, but c 1 basically ok.

So minus 5 c 1 t f is also c anyway; so, minus 5 c 1, like that. So, c 1 is something like that. So; that means, x of t is nothing, but just t by 5 actually and remember it is a free final time problem. So, tf is also unknown what if you really want to find t f then you put the constraint equation t f by 5 except t f is equal to that minus 5 t f well plus fifteen that is to lie on that curve. So, if we constraint that equation and you find it value of t f also basically ok, (Refer Slide Time: 57:00). Now, before stopping this lecture, if we talk about what you, what sufficiency condition, now will talk about second variation of that first variation is all about necessary conditions anyway.

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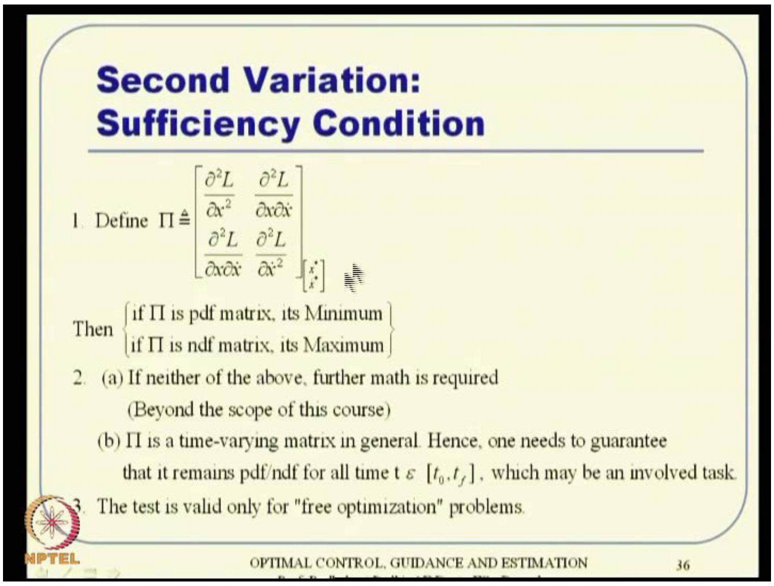

So, use this second order Taylor’s series go through that second order expansion and all you can write it that way. So; obviously, this gives us a condition that you can define this (( )) matrix lot of ideas and if this matrix applies to be positive definite its minimum condition; if it applies to be negative definite it is a maximum condition actually and neither of the above weapons than that means, further math is required on of a will not talk about this course actually. But, remember that this matrix is time varying in general and hence, one needs to guarantee that it remains positive, definite and negative definite throughout the time unlike static optimization. Static optimization there are various now there are function of time; x star itself is a function of time x star dot is a function of time.

This quantity is a function of time. So, when you talk about this matrix is first to define a matrix; that means, that is first to definite throughout the time interval. Actually, that is what we are we are mentioning here actually ok. So, one is to guarantee that is remains first to definite on negative definite for all the time interval actually and also remember, this test what we are talking about - second variation and now is valid only for free optimization problem. Now, because entire thing we talked about free optimization itself constraint optimization is not valid and further required things are equal to each other. So, with that I think we are ready to move for what little more general concepts of vector related things and I will talk about in the next class. But, this particular lecture I think if you got sufficient analysis of some ideas of calculus, various associated mathematics around the lecture.

So, I suggest that you kind of understand the concepts well before we proceed further alright? I will like to stop here, thank you.