Optimal Control, Guidance and Estimation
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Lecture No. # 04
An Overview of Static Optimization - I

Hello everybody welcome to this lecture four of this course optimal control guidance and estimation. So far, we have seen some overview of this course, followed by some I mean some overview of topics related to system theory as well as matrix algebra and in numerical methods as well; that is relevant in this course actually. So, in the another thing that is very much relevant in this course is the concept of static optimization; that means, we are not talking about time varying variables and all, but these are constraint numbers associated with that.

But in general they have some objective, and all that, how do you optimize that actually. So, that kind of things are call static optimization or sometimes it is popularly known as a parameter optimization as well, because parameters are something that typically remains constraint actually. So, anyways those are the topics of discussion of this particular lecture, and also remember as I told you in beginning lecture the very fast lecture that one way of solving this, this optimal control problem is to go heavily to this static optimization ideas and convert the problem entirely to a large dimensional static optimization problem, and solve it; that is called transcription method, and all we will see that as we the course develops.

Another motivation of that is even if you formulate this optimal control problem from calculus variation point of view and all that; the essentially land up with some equation something call Hamiltonian, and all that; and that happens to be like every point it happens to be kind of a constant quantity in its variables state and control, and that you need to be minimize at every point of time in the frame work of static optimization actually is very relevant concept for optimal control in general. So, it obviously, make sense to have overview of static optimization before we go to the dynamic optimization; that means, optimal control in this is now actually.

So, this I will take it in two parts and the first lecture will talk about some overview or some standard results of static optimization along with some examples and all that way, then further topics on it will see in the next lecture, and before proceeding to the calculus of variation ideas and all in that way.

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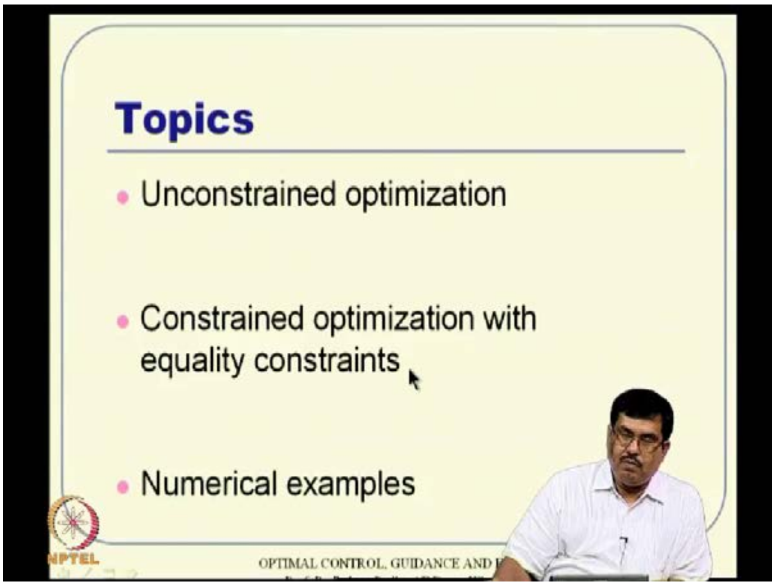

So, the topics of this lecture something like this, we will fast discuss about unconstrained optimization, then we will go to this constrained optimization with equality constraints primarily any quality constraints, and all will see in the next lecture actually. Then we will some numerical examples as well.

So, first is unconstraint optimization and the problem definition turns out to be very clear, that you want to optimization some vary some function of variable, but the variables are combination of some non time varying quantities. Actually that is they are not constraint by any constraint equation neither actually before proceeding with those with then algebra, and then analysis and think that; let us see very simple straight forward idea first actually.

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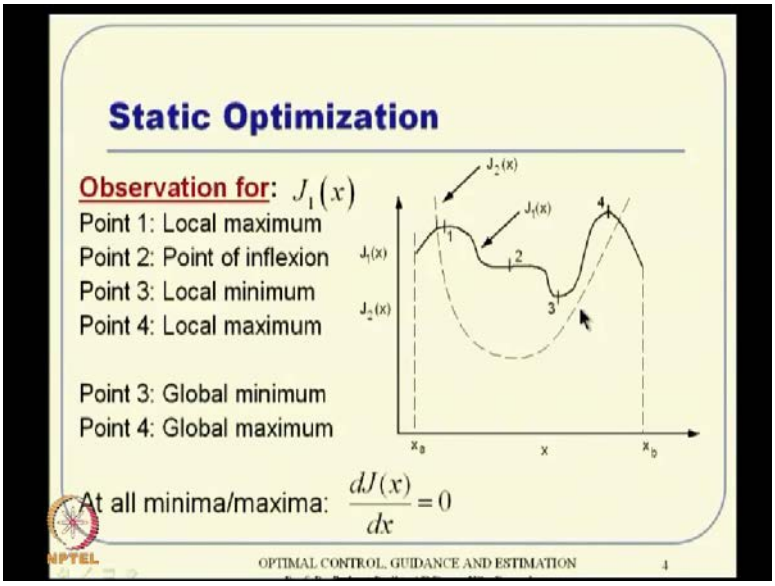

Now, forget it for a second this j 2 dot line, and all this flat was this flat was x verses some functions j 1 it is a, and j 1 turns out be this function this solid function actually. And then very quickly we can see that this 0, 1, 2, 3, 4, 5; they mean something to us in a in just by looking at the flat actually, and very easily it can be argued that this 0.1 happens to be a local maximum, and point 2 is well we cannot say anything about that it is a flat thing, but certainly a candidate for getting worried about, then is called point of inflexion, and all that. And then point 3 turns out to be a local minimum point 4 turns out to be local maximum just by looking at the flat, we can very easily see this is what is happening actually.

And then it also we can see that point three turns out to be global minimum, and point four turns out to be global maximum because nothing else is bigger that in this domain of course, I mean x a 2 x a. And all in that way, but when we talk about global maximum and all typically this x a and x b are also not there; in other words we are talk about minus infinity plus infinity that domain sort of thing actually.

But anyway within this constraint, and then you can think that 4 happens to be the global maximum and 3 happens to be the global minimum actually, and one property that we observe simply observe is that all these points 1 2 3 4, this relationship is true; that means, the gradient of j with respect to x is 0, if I just take a tangent here that is flat this is any way flat if I take it engine flat here is a flat. And then it is if I take it engine at the fourth point then they also parallel to the x axis actually. So, all the points satisfy that.

Now, that it gives is the clue that when we talk about something about optimization then; obviously, the first derivative turns should happen to be 0 that is how to do actually, and does not really happen. So, let us the talk about that that way and also remember that instead of j 1, suppose you have j 2 then really do not have to worry about all these we just have to worry about one minimum; that is what all that will have is one minimum, and that one minimum happens to be the global minimum as well actually.

So, most of the time will take advantage of this observation, and value constructs a function we will want to construct it that way. So, that we really do not want to bother about global things, and all is not relevant also most of the time it satisfy our need also basically is quadratic function and all that we will talk later this is how it turns out to be now coming back to that whether this relationship satisfied for all the points, and all and analyze it little bit mathematical vigorous sense basically.

So, now again we go back to this a this Taylor series, and all which will repeatedly require anyway now remember what is a local maximum local maximum tells you that if point one is local maximum, then very next two it whether go left side or right side either way that will give a function value which is lesser than the function value at point 1.

Similarly, if it is point 3 if it is minimum local minimum, then whether you go left side or right side either direction you are going to get higher values, then what you get it point three actually. So, that kind of constraint is there.

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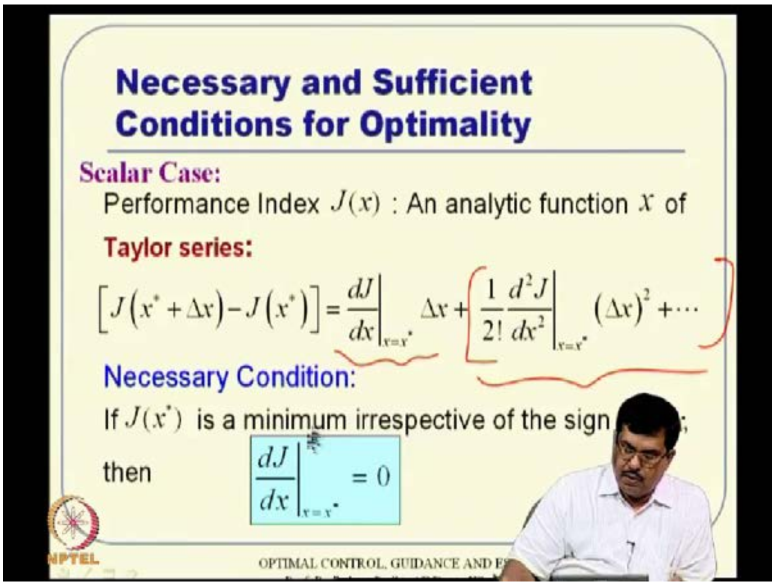

So, let us analytic a point x star and then talk about a value of that function around x star, and remember delta x can be either positive or negative actually here, and also remember we have simply talking about a scalar quantity; that means, x is just single I mean single component j cont j is a function of x, but x does not contain x 1 x 2 components and all that x is a scalar variable sort of thing.

Then what happens x star plus delta x minus j of x star that is the term that will come from Taylor series like this, actually now for any hope to x. What we are looking for a looking for this quantity this j of x star plus delta x should be greater than j of x star if j of x star happens to be minimum point irrespective of the direction of delta x at the sign of delta x really actually. So, that has to be happen; that means, this left side quantity has to be strictly positive for a for this quantity to be on a local minimum or precisely be strictly negative for the other case actually local maximum case.

So, what we are looking at here is a sign independent property really and also remember this Taylor series has a this uniform convergence property, and think like that. So that means, if you right this expression to be something like that if you take a this entire quantity what we what turns out be like this, the infinite terms what we are talking here is a less dominate compare to the first term actually.

So, that is how it will turn out be… So, in other words the sign of it is largely dictated by the first term itself actually not by these terms actually that way. So, if you have any real hope that the this become sign in sensitive, then it turns out that the first term must be going to 0, then only we have some hope actually that is how the necessary condition turns out to be and remember delta x is not 0, but this quantity has to be 0. That means, the coefficient has to be 0 basically that is how it turns out that del j by del x; that x equal to x star is 0 is 0, then x star happens to be a candidate point I mean you can you still cannot it whether it is minimum maximum actually.

So far that we will remain we will go to the next one, and then tell if that that is already there; that means, at this point this is 0 anyway, and then this quantity turns out be that one.

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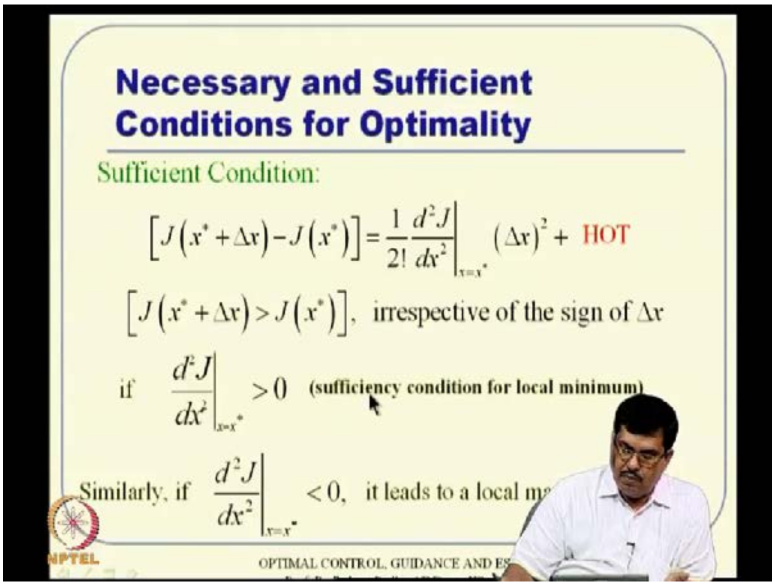

And that one means, I can consider the second order term, and then there is a still higher order terms a from third order fourth order can neglect actually. If this still has be positive; that means, this has to be greater than this irrespective of the sign of delta x remember this square term now. So, this has to be strictly greater than 0 provided this is greater than 0 this is greater than 0 anyway.

So, this quantity is greater than 0 provided this quantity happens to be greater than 0; that means, second order term has to be greater than 0 basically, and similarly the second order term happens to be less than 0 it least to the conclusion that this term is less than 0, and other words this term happens to be maximum actually x star happens to be maximum.

So, this is how the how the analysis two to summarize the things what we have doing here is given any function j of x, we will we will put this del j by del x is equal to 0; and then solve this and find out candidate points x start did not be 1 point it can be several point actually and but each of the points

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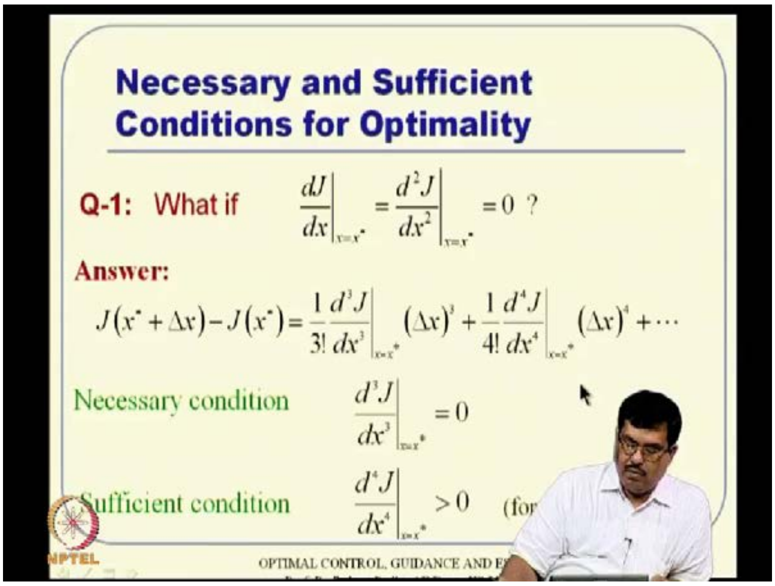

We will try to evaluate this second order derivative and the second order derivative happens to be 0 then it is a minimum point the second order derivative happens to be less than 0 it is local minimum that is the sufficient condition actually. Now, we are carrying out further things, and all let us see this some examples sort of thing, and before even before the example. Now the question is what if this both derivatives of the 0, but third of derivative is not 0.

So that means, this is true whatever we are talked about only if the second order derivative is non zero, and the non second order derivative also happens to be 0, then we have no choice what we go to the third derivative; and the third derivative happens to be non zero. Then we really cannot tell whether it is strictly positive or strictly negative, and hence it turns out to be point of influence actually, and if it is I mean if you still have a hope of getting it something like a positive or negative; it we must have the third derivative also 0. And then go to the four derivative and think like that actually right, if it is second derivative is 0 then this lands up with the third derivative case, and then there is necessary condition of optimality is if we still any hope then third derivative has to 0, then you land of with four derivative. And if the fourth derivative happens to be non zero; that means, positive, and then it is least to be minimum and maximum and otherwise it is maximum actually.

So, the analysis continues it actually toggles between this hard power, and even power sort of thing actually. So, as a necessary condition all hard power has to be 0 and sufficiency condition will come even powers actually.

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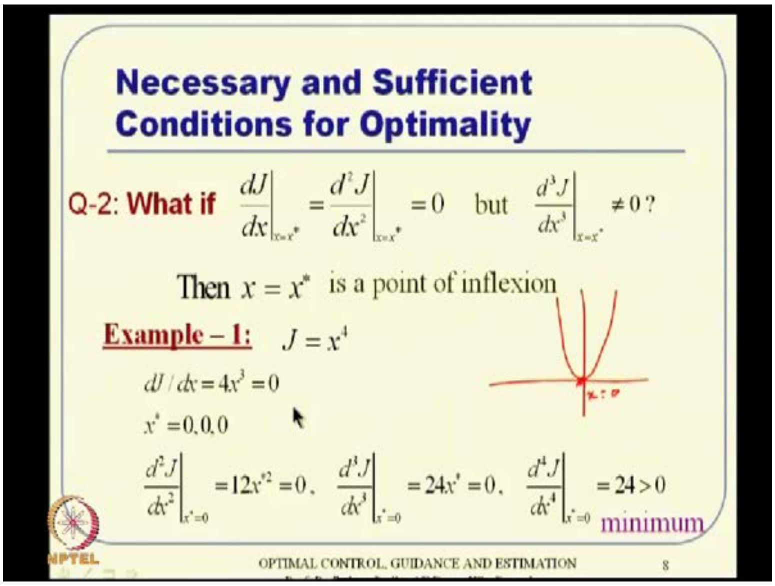

So, this is how the things proceed, and suppose we want to get example here then turns out to be let us take a very simple example j equal to x fourth, and you can see that well the x fourth plot is very, very close to these this kind of very similar to x sort of thing.

So; obviously, we know that this is point of minimum; that means, x is equal to 0 is the answer we know that actually whether we get it or not let see that using this result. So, if you have j something like this, then del j by del x is four x is equal to 0 then solution of that is three quantity is, but all happens to be 0 that x star equal to 0 0 0. And then let us go to the second derivative, and second derivative turns out to be like the taking this term it is twelve x square and x star equal to 0 means twelve x square is also 0.

So, we really have to go third derivative now, and the third derivative personality happens to be 24 x from here twelve into twenty four into x star next star is 0; anyway the third derivative is also 0. So, that means that is still our hope is alive actually, then you go to fourth derivative, and four derivative it happens to be twenty four which is greater than 0. So, if the fourth derivative is greater than 0, then obviously, it tells you that my some of my all of my hard powers also 0, and the first even power that I uncounted is actually greater than 0.

So; obviously, x star is a minimum point actually; that is how we include that the result is compatible to what we know actually come from graphical.

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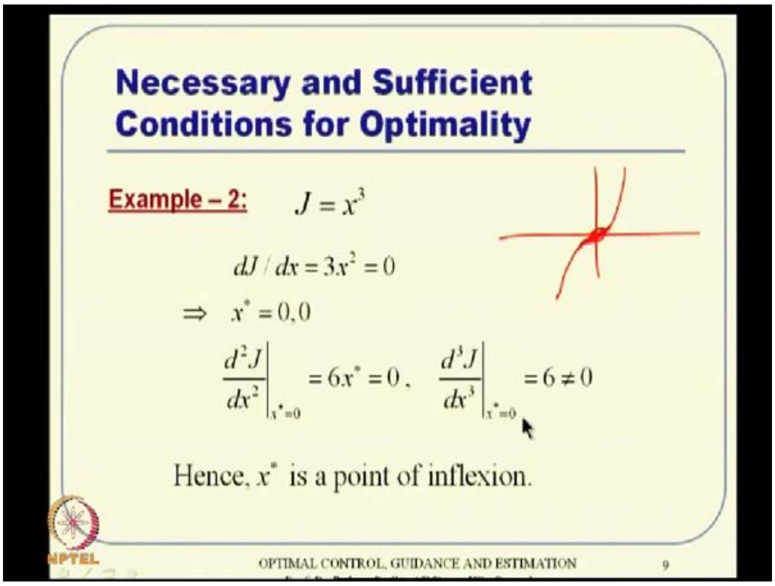

Since, now example to what if x is j is just x q and remember x q, what is the type of I mean thing plot x q is something like this, if it is positive is positive, if it is negative is negative; something like this. So, here you have a point of inflexion really; that means, at 0 what we are talking here is one side, the second derivative will turn out to be I mean one side the function turns out to be increasing other side the function turns out to be decrease actually. So, this will this we know that it is a point of inflexion, now whether our analysis tells of the tells do not let see.

So, if you have j equal x q then del j by del x is nothing but three x square is equal to 0, x star turns out to be 0 0, now we have but two candidate solution, but both of happens to be anyway. Now, the second derivative happens to be six x square is also 0. So, we cannot conclude anything this point of time this is three x square. So, it will be six x. So, six x evaluated x star that is six x star x star is 0. So, it is still 0 actually.

Now, we go to third derivative and third derivative del q by del q j by del x q from coming from here is nothing but six, but the important point note is not whether six is positive or negative, but six is not equal to 0 remember this third hard derivative actually hard power derivative, but this hard power derivative has to be non equal to I mean not equal to 0 that is more important, actually if it is not equal to 0 your hope is not alive it conclusion as a already in there and the conclusion is the x star is point of inflexion.

Because if I take this series this series this terms will start dominating and delta x is positive defending on whatever sign is there it will lead one sign, and defending on whatever sign is here if delta x is negative to the exactly opposite say conclusion here.

So, is just one side is a either increasing or decreasing, but other side is just the positive actually. So, will turn out be point of inflexion. So, that is how it is there. So, if j equal to x q then x star equal to 0 turns out to be point of inflexion point actually a point of inflexion really. So, this all comfortable to what we know basically now, but this is not kind of story this is just very beginning it is talks about as scalar variable functions sort of thing actually, but we are not interested in scalar variable function.

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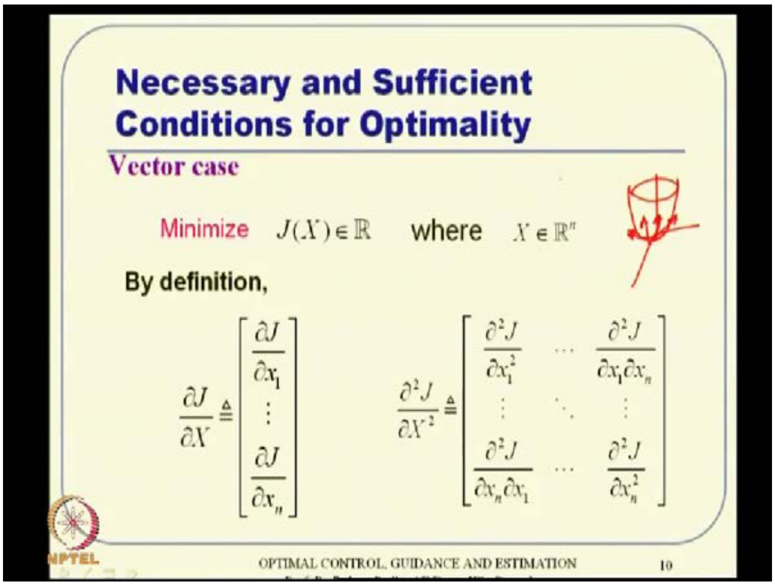

In general because we delay system theory, where state can be n dimensional control can be m dimensional like that.

So, we takes it is a mixer of this components of the state variable really. So, that is what we want analyze then leads to this vector case, and the problem is turns out to be like that minimize j of x where x is n-dimensional vector, and remember the objective function is always is a scalar quantity always otherwise is talk about multi objective of optimization and all that and we are not talking about that at all in this course actually. So, when somebody talks about standard of optimization problem then it turns out to be that the optimization function is a scalar quantity, where is the variables can be n-dimensional there can be several variables playing is playing out different contributes, and all we want minimize this scalar quantity. Finally, and also remember this is a this is a non constraint problem; that means, this component of x are not constraint by any constraint actually you can they can take any values what so ever actually that way.

So, you want analyze that problem and see what is going on actually and also remember phi definition what is saw in this matrix theory is reviews, and all that also that by definition this is del j by del x equal to like this by this is nothing but the gradient vector and del square j by del x square hessian matrix some sort to be like that by definition actually, and this things we will need. Now again going by the similar concept this optimization ideas, and all now we are asking a question that we want a value x star where from x star, you do go whatever direction you want no matter whatever direction left right top bottom whatever is actually which ever direction we go you will get some value, which is strictly greater than value of x star or you will get a value which is all of which strictly less than value of j x star. Basically then only you can talk about whether is a minimum or maximum quantities sort of thing actually, and picture really again if really wants picture then they two d sense, you can think about this kind of a picture this is not this kind of picture actually.

So, if you have minimum points some where no matter which direction, you will look it you can think of it something like a increasing function in all direction sort of thing. So, no matter which direction you go in you will land of with higher values compare to the minimum point actually. So, that kind of questions we are talking about actually.

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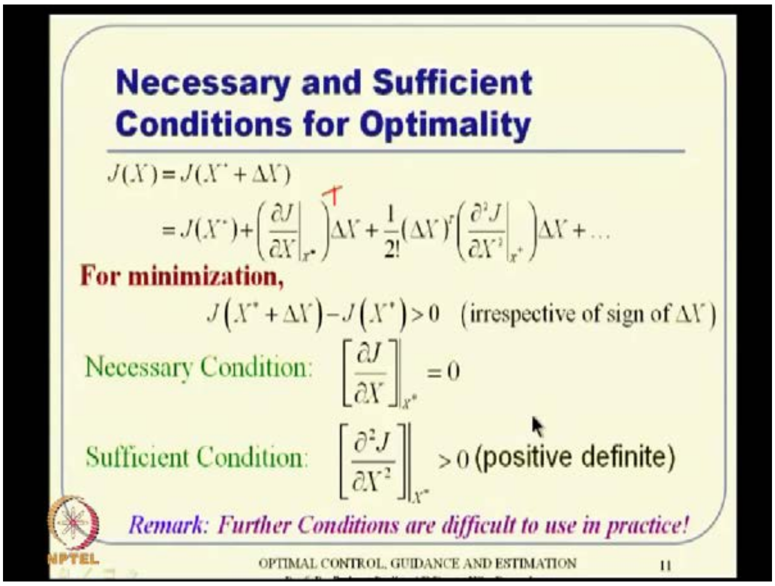

So, again going back to this ideas Taylor series, we know there is this vector variable also we can use this Taylor series expansion. So, we talk about j of x is nothing but j of x star which is x star happens to be candidate for analyze for maximum or minimum, and then j of x star plus delta x can be expanded that way remember this is the first term j of x star plus the linear term evaluate del j del x evaluated x star into delta x plus this second order term which is 1 by 2 factor delta x transpose into this session matrix times delta x and then the order term. Actually and this case will not very adventurous go to this, third order fourth derivatives and all that still possible to write.

But mathematical they are quite complex actually to try to in longer algebra sort of thing actually. So, let us see the idea first in this context most of the time this is efficient for us also basically. So far minimization problem what we are looking at is this quantity this a j of x star plus delta x minus j of x star that quantity has to be strictly positive; that means, no matter which direction you go the delta x can be any direction component by component, they can take any positive negative any quantity what is our actually? Then it turns out that if my value turns out to be have in that then it is a minimum point actually.

So, obviously if you want to satisfy, this and then going by this idea for direction sensitive information, and all that we can always tell that my hope is analyze; that means, I really want control having a conclusion something like that very free precisely then it turns out my this term what is fasted out terms has to be 0; otherwise if I have a something non zero, here the coefficients then depending on this quantity delta x which can be positive negative whatever component. Since I will land of with a idea with the this quantity either positive or negative. So, it becomes direct sensitive actually and which we do not need.

So, it turns out that a if you really want to have a minimum point then this fasted are Jacobian matrix Jacobian vector other sorry it is gradient vector. So, gradient vector has to be 0. So, that this multiplication has to be I mean turns out to be 0, basically this is a small probably print mistakes sort of thing this is transpose actually because we talk about this del j by del x being a column vector and all that actually. So, the multiplication we define only we needs vector actually anyways this is what it is. So, del j by del x evaluated x star transpose and think like that actually. So, for minimization again this again some raises for minimization we want this to be strictly positive. So, the necessary condition turns out to be something like this, and the sufficiency condition again going whether earlier ideas with wants this is 0. We land of with this term and this term has to be positive strictly positive for all delta x non zero, and if you remember this del square by del x square once it is evaluated x star it is nothing but a matrix of numbers basically.

So, this matrix of numbers multiplied with some vector transpose, and then write multiplied with same vector with non trivial vector and all that it has to be strictly positive quantity; that means, by definition of what we saw in matrix theory the equivalent idea is this matrix what you seeing here by definition has to be positive definite by because the positive definite definition tells us that anything that we pre multiply some vector transpose, and post multiply same vector which is non trivial vector, and that has to be strictly positive for all search effect actually by definition this in that case this matrix turns out to be positive definite.

So, and also remember positive definite slightly into the Eigen values, and all that and del square j by del x square is guaranty to symmetric matrix; that means, does not values not guarantee to very well. So, if it is real and it is all Eigen values guarantee to be greater than equal to 0, because it is positive some definite I mean it sorry I take it back this is symmetric matrix. So, Eigen values are really actually.

Now, it turns out of that if it is a positive definite matrix by this analysis as I told then Eigen values are going to all positive actually. So, just by looking at the Eigen values this matrix, because it is constraint matrix after getting a evaluated x star; if all Eigen values are guaranty to be positive and remember it is guaranty to very really any case then it turns out to be positive definite, and if it is positive definite then this term turns out to be strictly greater than 0 for all possible delta x no matter whatever direction I will takes is, and then hence this relationship will be valid actually as a summer we turns out that the this is necessary condition which is can give like this. That means, you are gradient vector evaluated x star has to be 0, and then whatever candidate solution you get from there you go to the sufficient condition, and tell if I evaluate this is Jocobian matrix set values x star all possible x star, and then I see where the jocobian matrix turns out to be positive definite are not actually.

And if it is positive definite the x star is a minimum solution, if it is negative definite on the other end it is a maximum solution actually, and also is a remark further conditions like third order fourth order and all are difficult to use in practice, and typically it is not required as well in most of the applications actually.

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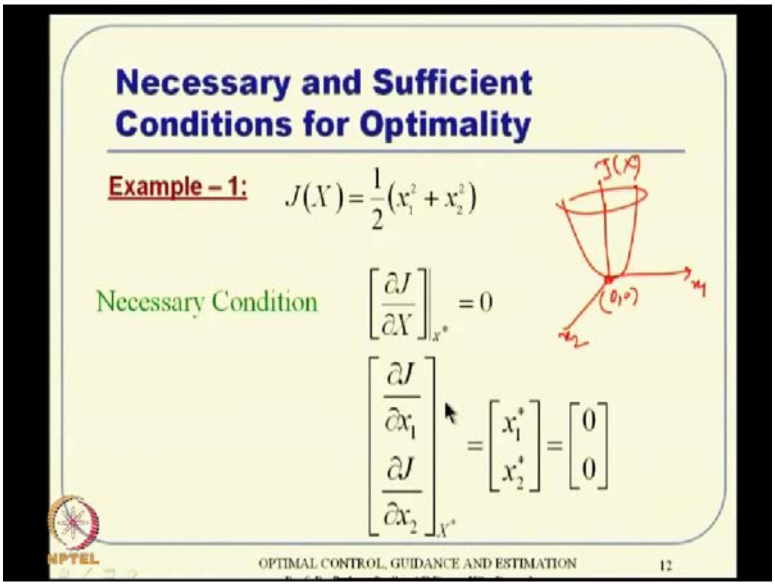

Anyways this and other most of engineering application I will say that then many time we land of with such cases it turns out that; first order second order turns out to be sufficient actually, but that does not mean that the mathematically we cannot develop further and all that actually that can be always there, anyway can back to the example suppose we taken example of two-dimensional case. Now half of x 1 square plus x 2 square you want see where the minimum lies, and all that way then has a necessary condition I will first go through that this del j by del x, we evaluated x equal I will put it equal to 0 remember this vector equation; that means, it has to equations actually in (( )) that.

So, essentially tells you that del j by del x 1 and del j by del x 2 equal to 0 0, but del j by del x 1 from this expression is nothing but x 1 and del j by del x 2 is nothing but x 2. So, evaluated x star; that means, these star things all. So, this has to be 0 0; that means, as I told again this picture really represents what I shown already this is picture really we know this result actually x 1, x 2 and think like that; this is j and we know the solution already here actually 0 0 turns out to be same actually from (( )).

Now, the question is can we verify sufficient condition and answer is very much yes we can go to see once we have the j once we have the gradient vector we can talk about the hessian matrix actually. So, del square j by del x square evaluated x star turns out to be to be something like this and remember this is the gradient vector.

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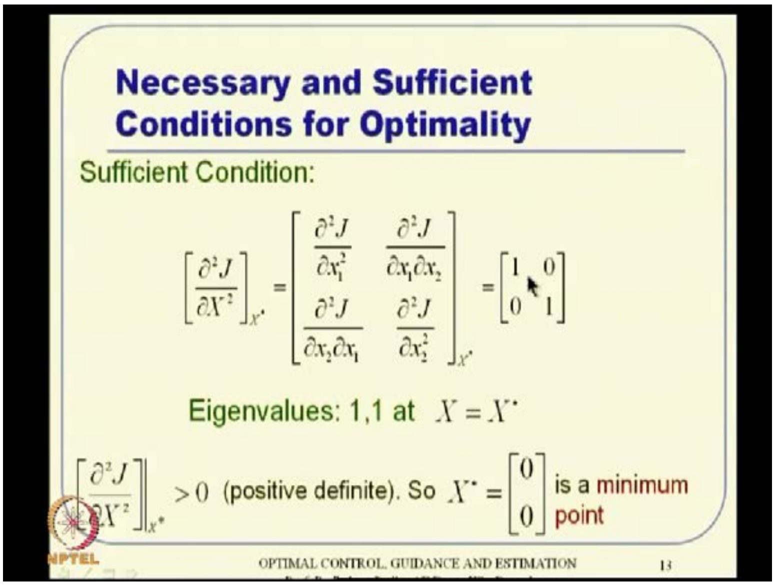

So, take second order derivative this variable this is only x 1 this is only x 2.

So, if I take with respect to x 1 one more time I will get one if I take vary derivative with respect to x 2 there is nothing there, because of function of x 1 only that is 0; similarly, if we talk this one this turns out to be again this all this function of only x 2. So, if I take partial derivative with respect to x 1 again then turns out to be 0 here, and if I take a x 2 this is one. So, this great this hessian matrix evaluated of x star turns out to be nothing but identity matrix actually here. So, identity matrix is certainly diagonal matrix and we all know that is diagonal matrix all diagonal entries actually nothing but one all that actually that way

So, Eigen values this particular hessian matrix is nothing but 1 and 1; both are greater than 0, and hence this matrix evaluated x star which is 0 0 evaluate this matrix evaluated x star happens to be positive definite matrix. So, what we turn out to be like we got solution 0 0 a candidate solution, and the second order analysis; that means, the hessian matrix evaluated that point happens to be strictly positive definite matrix because all the eigen values are strictly greater than 1 greater than 0 basically.

So, because of that this point is guaranty to a minimum point a satisfy the both necessary condition coming from here, and sufficiency condition coming from here which tells us that the it is a diagonal matrix, and it is a Eigen values are diagonal entries positive numbers, and hence it is a positive definite matrix basically already.

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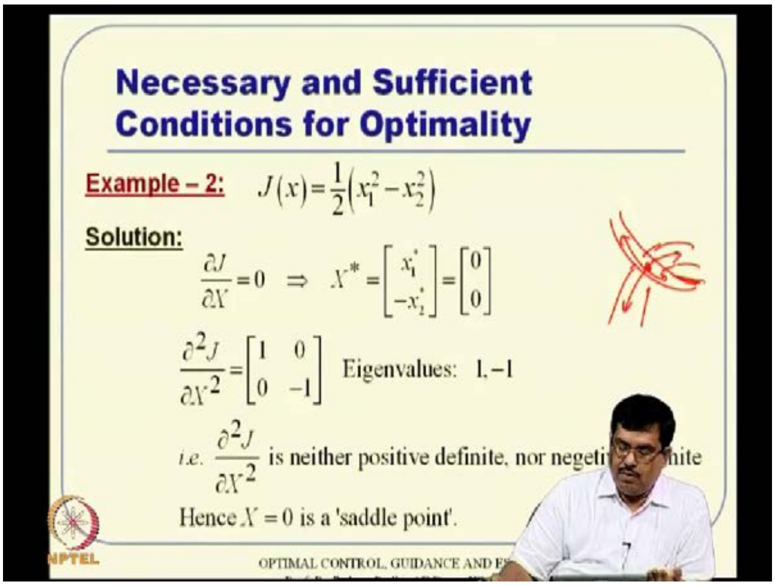

So, let us move further what if a this kind of example (( )) x 1 square minus x 2 square then the algorithm is completely different, and it turns out to be similar algebra you get a candidate solution a 0 0; however, we go to the second derivative we turns out to be 1 minus 1. So, Eigen values are 1 minus 1 So, it is really neither positive definite nor negative definite, but strictly speaking you can think if we go long x 1 direction something happens, if I go long x 2 direction the exactly the oppositely thing happens, and typical example is this I do not know this horse bike, and all that we talk about right I mean this seat on the horse bike, and all that that picture really I do not know I can write that way this kind of thing function will decrease this way and this will increase other way round actually.

So, if you imagine something like this direction the function will be increase, but this direction the function will be increase sort of thing actually. So, this is the case we are talking about not a point of inflexion and that is a idea of scalar variable. So, here it is something called saddle point actually. So, this is the point where in one direction in x 1 direction things will increase, but in x 2 direction the function will be decrease actually.

So, in this case mathematically turns out that del square j by del x square evaluated the x star is neither positive definite nor negative definite, and hence but the first gradient vector is 0 0 anyway. So, in such cases this is a nothing but saddle point actually.

Now, moving on what about constrained optimization, and then when we talk about one talk about free optimization like this. What we saw there again this are not very meaning full thing, because most of time will be constraint by some relationship between the variable and again and again I keep I keep on emphasizing the in optimization, and optimal control everywhere constraint has priority actually; that means, your solution will be absolutely no meaning, if it does not satisfy the constraint.

So, you have their constraint; first you satisfy the constraint, and within that constraint only you try to optimize actually, and in general it happens in really real life. Actually we have to talk about constraint first, and then talk about optimization around that basically that is our dealing with when we talk about constraints the constraints can be two type; one can be equality inequality and other can be equality, and then we will in this particular lecture we see about equality constraints, and then any quality we will talk about the in that next lecture actually.

The analysis goes to will little more complexes something like that actually, but also remembers as a combination of equality inequality constraints is what is what will happen in real life problem actually. So, we should know how to handle equality constraints and we should also know how to handle any equality constraints as well in general basically anyway.

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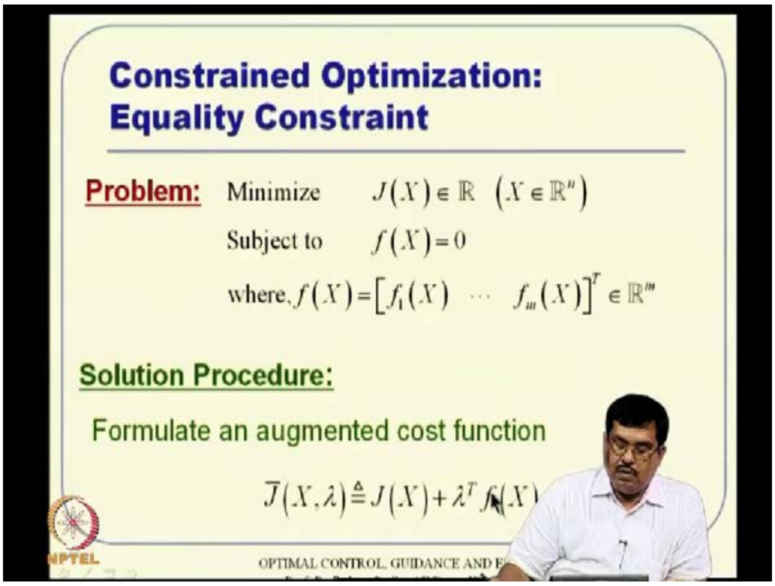

So, let us talk about equality constraints here, the problem formulation is like this we want minimize some j of x again always r of scalar, but x is of vector subject to this the sort of equality constraint is not just one constraint actually remember that. So, this equation actually is m equations because f is components to f into f m.

Now, this is the problem we want minimizes the scalar quantity of f function, where the function itself n-dimensional and that subjected to m constraints actually m algebra n constraints now the solution processor here relation this. So, called existence theorem all that which tells us that, if I formulated another cost function like this or another this objective function all that j bar which is nothing but j of x plus lambda transpose f of x.

So, the theorem tells us that if I introduce function like this and interfluent for second you forget the that f of is equal to 0; that means, this is 0 anyway basically no matter any about, but for second forget, and then tell the what if lambda is free variable and I do not worry about this constraints really I just want put of this f of here, and want minimize this function other j bar of x consider lambda is a free variable actually. And also just small command before that the most of the time talk about minimization, and most of the text books and most of the lectures also talk about that different courses the reason is if something we want maximize the negative of that something like that, we want minimize basically.

So, just introducing a negative sign the minimum can be interfluent is maximum viceversa actually. So, we does not that much of a great importance to aggregate this talk aggregations all time actually. So, we will confine our self to discussion around minimization all the time without loss of generalative; of course, anyways coming back this is theorem which tells us that if I formulate an augmented cost function like this; that means, j bar equal to j plus lambda transpose f then if I consider j bar is a function of x and lambda both are equally free basically, and I consider this free optimization with respect to x and lambda both. Then it is equivalent solving this optimization problem subject to this constraints it is a great theorem, I mean it is a little bit math heavy also, but those of are interested can see some I mean some formula some space optimization books, and all that actually that way.

We will not go to that that much of thing again this course is primarily meant for engineers and engineering clever and all that way. So, let us coming to that the theorem essentially tells us that if I constituted a j bar like this j plus lambda transpose f of x lambda is a sort of variables, and again and again emphasizing, because similar concept we will use in optimal control also basically right. So, if I constituted this function and consider this is a free vary free optimization problem in terms of x and lambda both then it is a equivalent of solving this problem this is a j of x minimize subject to this actually.

So, using that gives us to the solution procedure, and the solution procedure tells us that first the formulate this j bar like this, and then carry out it as if it optimization problem and remember the number of variables are now lambda.

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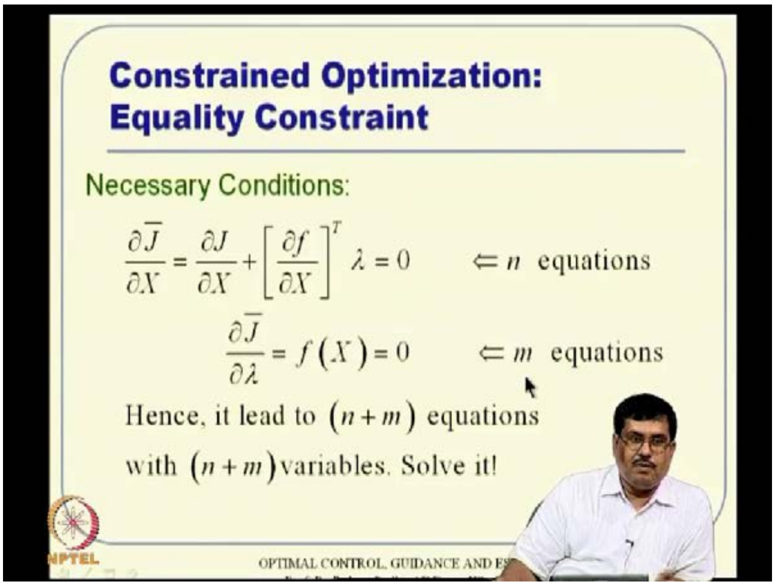

That means, I have to satisfy simultaneously this del j by del x equal to 0, and del j by del lambda also equal to 0, now when we talk about del j by del x del j bar by del x it turns out to be from this definition this is nothing but del j by del x coming from this term, and this second term and remember; if you do this algebra del j by del x of this using this one of the standard results of this vector matrix calculus that we discussion in review class before this lecture. Then it turns out to be del j by del x of these happens to be this result actually; that means, this derivative del f by del x transpose happens to be the left hand side the shift form right and to left and actually.

So, this is how will get it del j bar by del x is this 1 equal to 0 and remember lambda is also free variable; that means, del j bar by del lambda is equal to 0 what is del j bar by del lambda by definition this is nothing but f of x.

So, f of x is equal to 0 which is which came from here is part of the necessary condition anyway. So, we are accounting for the constraint as that is how is equivalent actually in a way. So, what we are doing here is this one del j bar by del x which is equal to 0 and del j bar by del lambda this is also equal to 0 this will give us n equation, and this will us m equation lambda is m-dimensional remember that this lambda is same dimensional f f is m-dimensional actually.

So, this is actually leads to something like a n plus m equations with n plus m variables basically, and these are n equation these are m equation. So, totally n plus m equations, and free variables we have x, since n-dimensional free variables and lambda is m dimensional free variables. So, we have this n plus m variables actually

So, we have a comfortable sort of a equation we need to solve it together, and also after the solution we get the solution we typically not bother about lambda, we are bother about the value of x actually this is something like think of it is something like a catalyst operator in chemical reaction, and think like that actually.

So, cartelist x self kind of facilities the reaction what ultimately the cartelist is a not a product of that interested in all that actually, but without that you cannot carry out that reaction probably many times.

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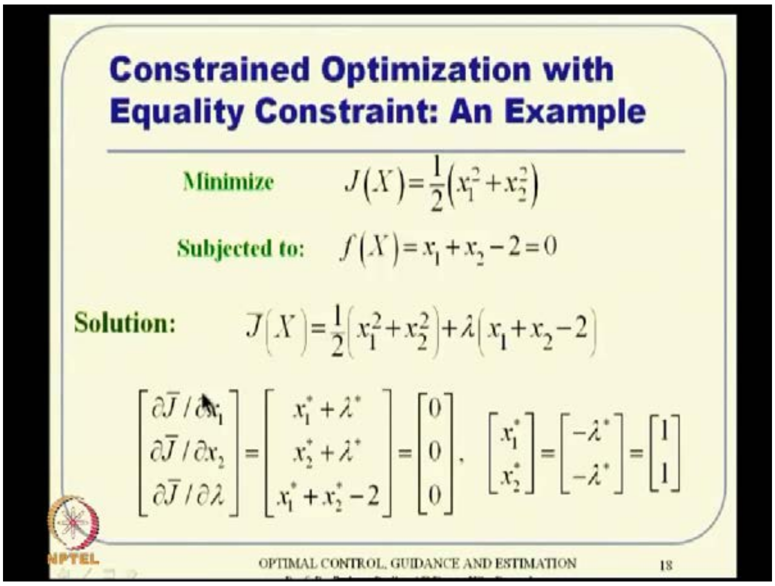

Anyways this is a how it is. So, let us go to the example again. So, we want minimize the same thing whatever example is saw before, but not a free optimization problem that we know that 0 0 happens to the solution, but this times is constraint to a linear equation let us say x 1 plus x 2 equal two that is nothing but something like a surface something like a plane equation sort of thing actually

So, this example I gave here this one is now subjected I do not know the what weight is inclined, but let me just conceptually draw one, this is plane what we are talking about and this plane actually this plane goes, and curse this a I mean this parable and all that way. So, we will land of with some like an inclined parables sort of thing, and inclined that we talk about what is maximum minimum actually I am not interested 0 and 0 any more.

So, this is type of problem that we are interested in and then the result turns out to be I mean the processor turns out to be like that j bar we constitute first. So, this is coming from here followed with remember this one equation, this is one singular lambda again. So, lambda into this f of x this minus j bar, and first I have to do del j bar by del x and these are two equations here because x contents x 1 and x 2. So, it is del j bar by del x 1 equal to 0 and del j bar by del x 2 equal to 0 and the third equation has to be del j bar by del lambda and lambda is scalar here. So, del j bar by del lambda is nothing but that equal to 0.

So, you carry out this del j bar by del x 1 if you see this expression one is coming x 1 is from here and this one more term here. So, this is lambda star lambda here, and of course from here to here we seen some time this star notices are because the star candidate solution these are generic expression and all that. So, do not get confuse that way.

So, we get del j bar by del x 1 this term going from here and this term going from here and that is equal to 0; the similarly, x del j bar by del x 2 this terms comes here and then that term because that term here both are equal to 1, and del j bar by del lambda if I talk about this expression lambda means this equation. So, the x 1 plus x 2 minus two is equal to 0 sort of thing.

So, I have to solve these three equation with three unknowns, now x 1 and x 2 lambda basically. So, if I go and solve it from this first equations I get this kind of relationship, and from the last equation if I put it back when we turns out to be that lambda star is nothing but minus 1, and hence x 1 star and x 2 star both of tends be one actually.

And that also a prince to be kind of if you imagine little bit here, then I do not know it requires little bit imagination, and all that defines and plane inclination and think like that. So, this point what sorry this point happens to be our point this is projects and if you take here something like this one full form actually.

So, this is a depends on the plane inclination what with what inclination the plane cost the parable sort of thing actually alright. So, that is that is the kind of (( )) that we are looking for now generalizing this results little bit further

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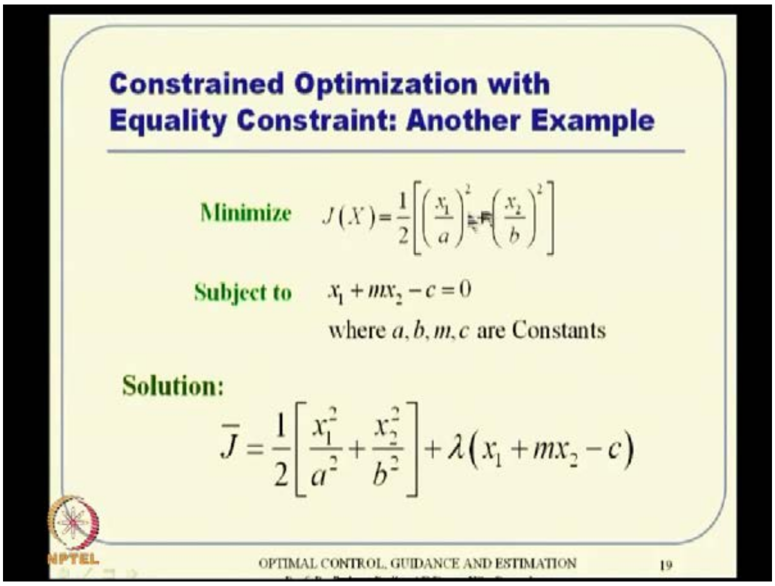

You can also thing of a minimizing some sort of generic expression were in we variables can be thought about it is weighting parameters, and all that and subject to generic state line kind of non state line generic equations sort of thing actually.

So, where this if in this expression a b c m n c; all these are kind of constant quantities, but if you take different numbers you get it different solution ultimately actually. So, this is a kind of generalization about you saw here actually.

So, again you go back solve it j bar is nothing but that in the, you can have a lambda plus this quantity, and then I carry out with algebra.

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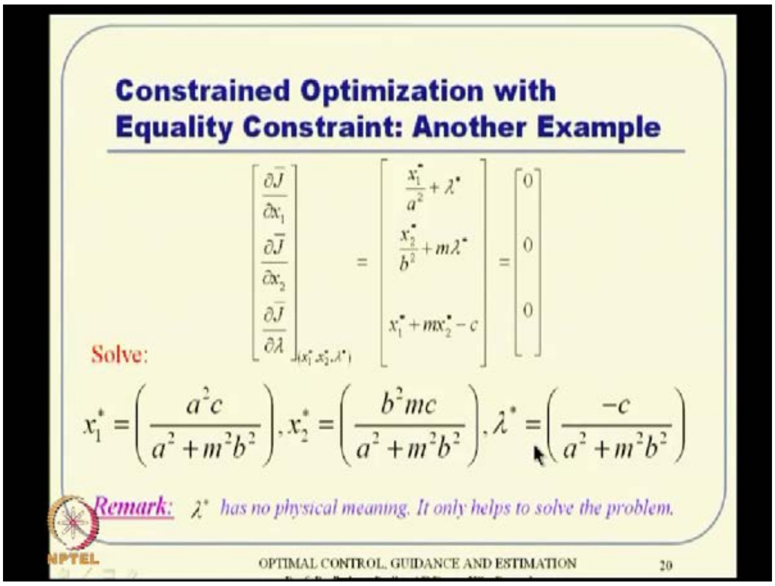

$$\begin{bmatrix} \frac{\partial \bar{J}}{\partial x_1} \\ \frac{\partial \bar{J}}{\partial x_2} \\ \frac{\partial \bar{J}}{\partial \lambda} \end{bmatrix}_{(x_1^*, x_2^*, \lambda^*)} = \begin{bmatrix} \frac{x_1^*}{a^2} + \lambda^* \\ \frac{x_2^*}{b^2} + m\lambda^* \\ x_1^* + m x_2^* - c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1^* = \left(\frac{a^2 c}{a^2 + m^2 b^2} \right), x_2^* = \left(\frac{b^2 mc}{a^2 + m^2 b^2} \right), \lambda^* = \left(\frac{-c}{a^2 + m^2 b^2} \right)$$

Same algebra del j bar by del x 1 that turns out to be like that equal to 0 del j bar del x 2 turns out to be like that equal to 0, and del j bar by del lambda we will give us the same constraint equation; there is also same equal to 0. So, you solve it in generic since in it to the generic turns out to be like that.

And also small comment here that lambda star has no physical meaning, and it helps to told to solve the problems actually; that means, you can think of if it is something like a get let actually, and also remember without this cons this lambda we may not able to solve this problem in very good way; the of course, we can talk about numerical solution and all that, but if whenever you have a close form expression and obviously, you do not want numerical solution see the actually.

But anyway numerical ideas philosophy little bit of that you can see the in the next lecture actually that is not go to that part of state here.

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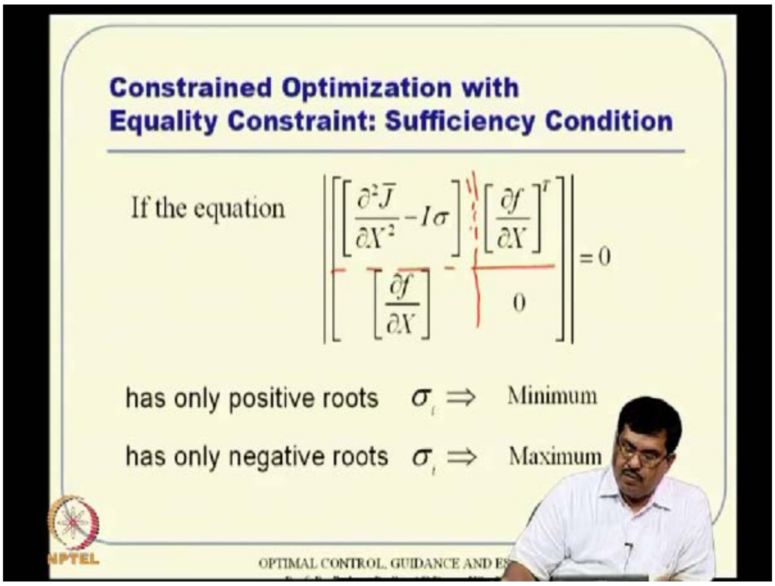

Now, what about sufficiency condition, I mean this is all necessary condition and any way and I will not going into the derivation part of it, but it terms out that you formulate this matrix what you saw here, and inside the determinant we talk about a second order matrix, I mean of j bar remember that this session matrix for j bar in terms of only x then minus I sigma this is matrix itself. And essentially what we are doing here is creating a matrix out of this partition matrices all that you can think putting a partition matrix here like that, and I constitute this I mean this components indusial this matrix or other sub matrix this is sub matrix, and a lot of thing defending or matrix defending on number of f and x remember x k x is always vector, but f is also vector if f is a scalars and vector there the different meaning, and sometimes this transpose you will see here defending on the situation actually.

But essentially you constitute create this components, and put them together like this take a determine make it equal 0, and interfluent everything else is known, because already have a solution known to you by that time x star lambda star all that you know to you the only thing that is non sigma actually.

So, interfluent this equation as a equation for sigma, and then if sigma has all positive roots when the also sigma are positive, then it leads to minimum and also take a roots negative, then it turns out to be maximum actually and again the proof part and all I will not going to that actually. So, we will just take it in phase values sort of thing. But example we will see…

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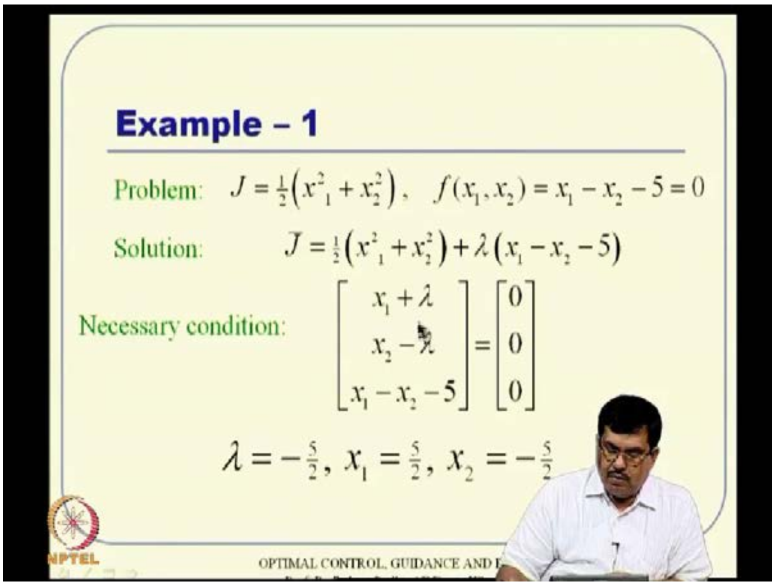

And we will go back to that that example. So, very similar example actually we will take this again, this the quadratic function followed by this constrain the constrained actually nothing but equation of a plane solution approach is again very similar to what you done before to constitute j bar. And then talk about this equation which come from del j bar by del x 1 is equal to 0 del j bar by del x 2 is equal to 0 del j bar by del lambda equal to 0.

So, once you have this equation solve it and you can just carry out the solution yourself probably because x 1 has to be minus lambda x 2 plus lambda is substitute here minus lambda, and then it is plus lambda. So, it is minus two lambda minus five sort of thing. So, lambda happens to be minus 5 by 2 and x 1 happens to be minus lambda. So, five by two next happens to be 5 lambda which is minus five by two. This is how we get a solution, now the question is how do go at tells us for sufficient condition basically...

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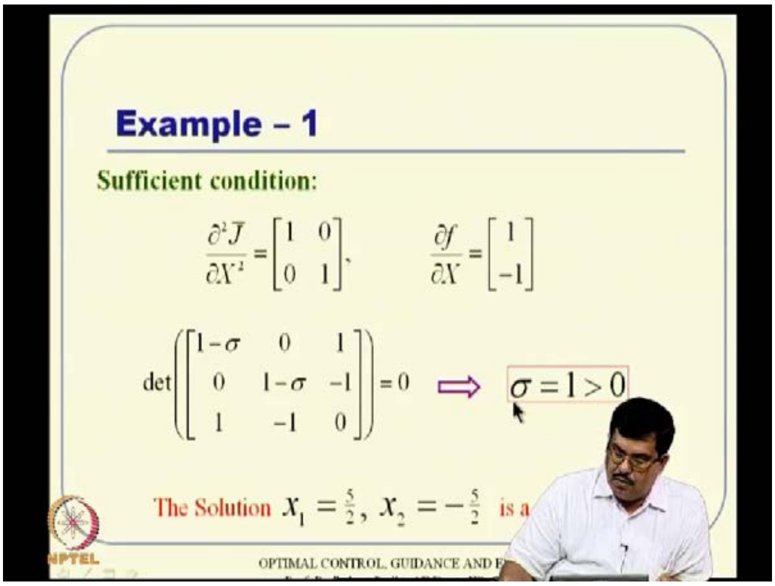

So, this is we will try to into this formula whatever you have here and this turns out to be like. So, this is del square j bar del x square happens to identity matrix, we seen that before and del j by del x is equal to nothing but one minus 1 because del f is here del f by del x plus 1 del f by del x is minus 1. So, put them together is like this

So, I go back to this formula tell this is I will put it here, and this is one and remember this formula what I gave in general it is a when f is a f is vector, and x is vector what here we have a scalar. So, this transpose is not necessary I mean infect this transpose coming here actually this will this will become a column, and this will become rows sort of thing actually.

And that essentially come from the definition if you go back to the this Jacobian, and gradient vector and all that gradient vector it is column vector, if Jacobian vector Jacobian matrix it the similar partial derivatives are a line in the rows; since basically that is the from that definition the typically comes actually in away.

But remember if it is both are I mean f is a scalar then the transpose gradient shifted here. So, with that essence we will just we just see that whatever these components are available. So, this will come here and that transpose that will come here one minus 1 here and one minus 1 come here that. So, now we have determine equation here and if you carry out the determine, and then turns out that it is quantity expression one minus sigma whole square equal to kind of 0; it will turn out to be actually.

So, then sigma is nothing but one, one sort of thing actually and both are reference to be positive. So, hence this the theorem tells you if that all the roots are positive, and then it is minimum and hence whatever solution you got it here, this five by two minus five by two happens to be minimum point actually also this small comment if some this is a actually the point that are getting actually in a in picture since this is what I shown here what we getting here is the point if somebody ask you what is the value of that particular function and that point and then you have to go back and evaluate this of x and get a value for that.

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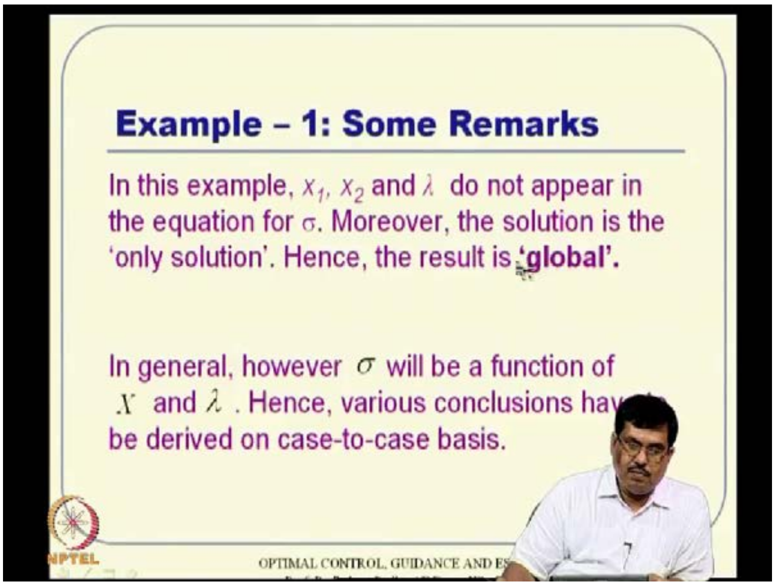

So, that is we implicitly assume actually that is the possible, because this just simply evaluation of the function that way some remarks, we sort of with this example; in this example this variables x 1, x 2, and lambda do not appear in the equation for sigma whatever sigma is there it is independent of that. And hence it is really does not defend on with operator or with candidate solution, and we are talking about and hence this result happens to be global actually.

This solution is the only solution in that since also it happens to be global actually in general; however, sigma will be a function of x, and lambda, and hence various conclusions that that we have seen here in this conclusion this. Since it can defend on which candidate solution, we are talking about around each candidate solution we have a certain sigma, and this certain sigma is has to satisfy sudden condition like that actually.

So, that analysis goes that way and we will also give an example to demonstrate that actually. So, this is what the the example is this in the constraint optimization; since in sufficient condition since we will go this another example like that.

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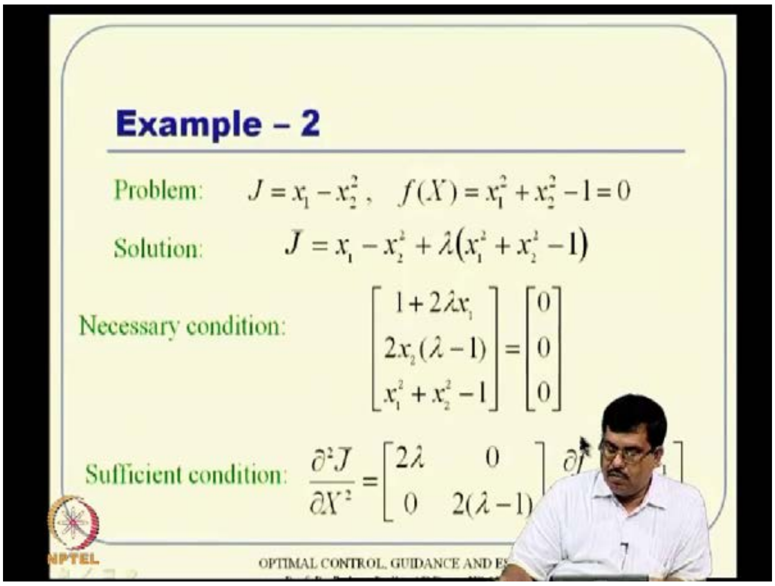

So, if it is like this and subject to this is a non-linear equation remembers that it is not a equation of apply anymore, but probably something like a equation of a kind of sphere actually.

So, this particular of value I mean what we are talking about I mean what we are talking about is a analyzing this particular problem. So, we will go back and formulate j bar; j bar is a nothing but this plus lambda to that and carry out to the necessary algebra, and then like land of some candidate solution what going ahead we need this del square j bar by del x square, and all that it turn not be like that. So, del f by del x also, if you if you see this turns out to be two x 1 by two x 2 actually.

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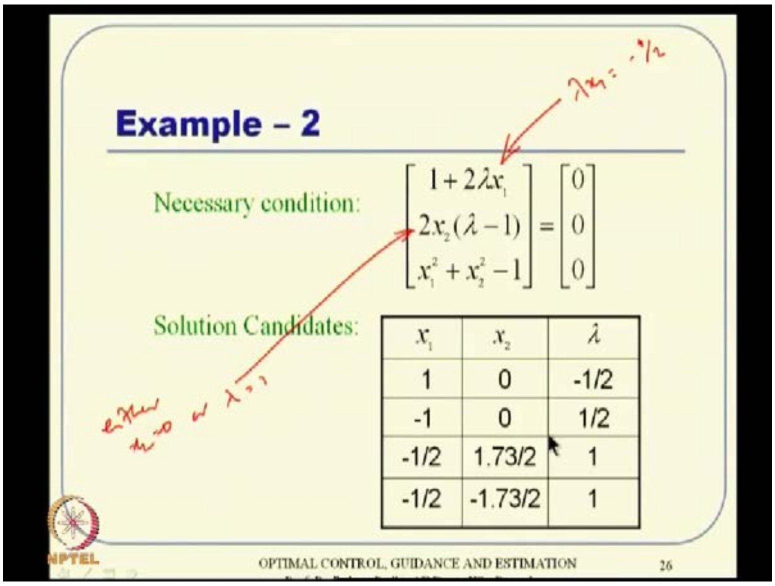

So, now we put it back here all necessary condition will set will be like this and because it is a non-linear equation setting here and also this is also is a non-linear term, this is also non-linear term that we will suddenly have multiple candidate solution, and this candidate solution turns out to be four candidates.

And this four candidates happens, you can quickly verify also, you take a take this set up values, and then put it satisfy the all equation you take a this one put it this one actually. So, that way, but in solution since we really need to solve it, and get some gradient solution all possible solutions actually.

Now, it turns out that each point to evaluate what the sigma values are; and for that, you have to do this; for that, we to take one particular thing, and then go back to this, what I showed this go back to this, and then put it here, and then try to analyze what sigma values are those. So, that since actually it happens to be. So, let us talk little bit that part of it alright just before well, we talk about just demonstrate your confidence and all let us talk about verification of this sort of solution probably. So, we talk about one 0 and half.

If I take this is like one. So, this one, but this is minus half actually here, so minus half into two basically. So, this is happens to be minus half into two minus 1 multiplied with one; so it is still minus 1. So, minus 1 is 0 basically now what about this equation this is the solution anyway. So, we are talking about x 2 being 0; so x 2 0 here. So, this equation is 0. I do not have to look at all the things. Now, what about this equation here we have the... This x 1 square plus x 2 square x 1 square is one plus x 2 square is 0. So, one plus 0 minus 1 is 0 this certain equation satisfy this equation actually again similar thing you can do it for this, for this and all that. But in a in solution since again we need to very careful for example, if demonstrate little bit, I think most of any way, but still this solution since you have see this particular for example, a either a to write something like a x 2 equal to 0 or lambda equal to one.

So, you have to see this kind of candidate solution sort of thing actually. Similarly, from that, you can talk about this; one this will take you either lambda is 0 or x 1 equal to not lambda is 0 either this no we cannot talk about that what we are talking about here sorry we just below that here for this equation tells us that that lambda sorry this actually, it this equations tells that lambda times x 1 is nothing but minus half actually.

So, we have to analysis this case little bit carefully and then put it you probably can have some cases case by case solutions and all that ready I suggest that all of you do this exercise for I mean, in longer algebra since. So, that you will get again lot of confluence actually, what we are doing this and again not recommend at you go to this numerical solvers actually nothing it a bit can be done in final paper it is a final algebra also to carry out actually.

So, suggest that you do it yourself actually that way anyway. So, you carry out this, then the turns out to be that these are the candidate solution each other candidate solution. Now we have to see what is going on; now you go back to that whatever formula had here analysis. So, in that setting way to put it here what is going on here. So, you take a first candidate solution probably, then think about putting into all this whatever variables are there, we will substitute the corresponding value actually...

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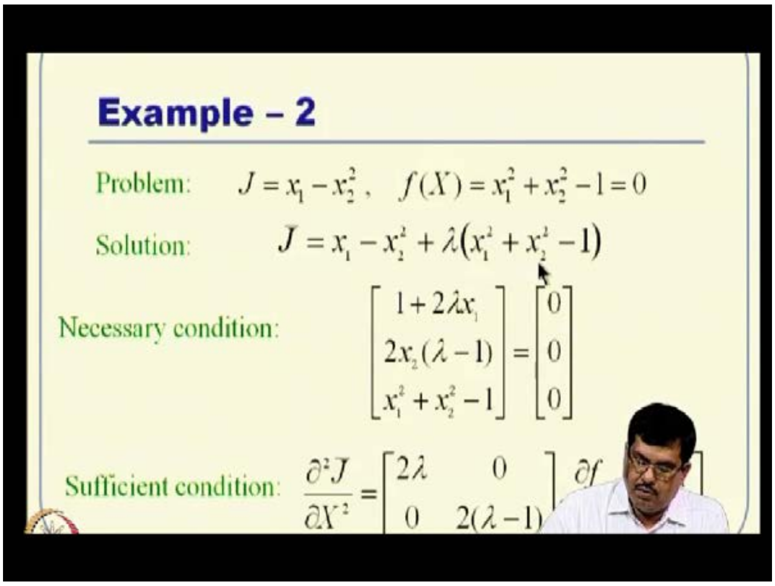

So, if I take the first case, actually this particular case, then turns out to be like this, then evaluate this determinate and evaluate using whatever either first row first column actually, I evaluating that using first row, then it turns out to be like this into this 0 minus 0 this term minus 1 minus sigma into this term into 0, which is 0 minus this term into 0 0. So, this is this term into 0 minus 0 0 into this determinate, what you see here sorry this one this determine 0 0 two 0. So obviously, one row 0, so this is 0 also typically so, 0 into 0 sort of thing that is what it is.

Whatever the last term, last term two into this is 0 anyway, but minus two into this term actually, so that that is a candidate which is non 0 really. So, that equation tells you that everything is I mean this is 0 this is 0, but this term gives me something like four into three plus sigma this is there two into two one minus and I will take it from here. So, that is four into three plus sigma is 0 so; that means sigma equal to minus three.

So, what we are getting one sigma one candidates sigma only and that sigma happens to be a negative quantity. So obviously, we will lead of land of with a case, which is maximum case actually. So, at this point x 1 1and 0, we have a local maximum basically. Similarly, I will carry out to this analysis for also points, if you do that, then you land of with this table basically.

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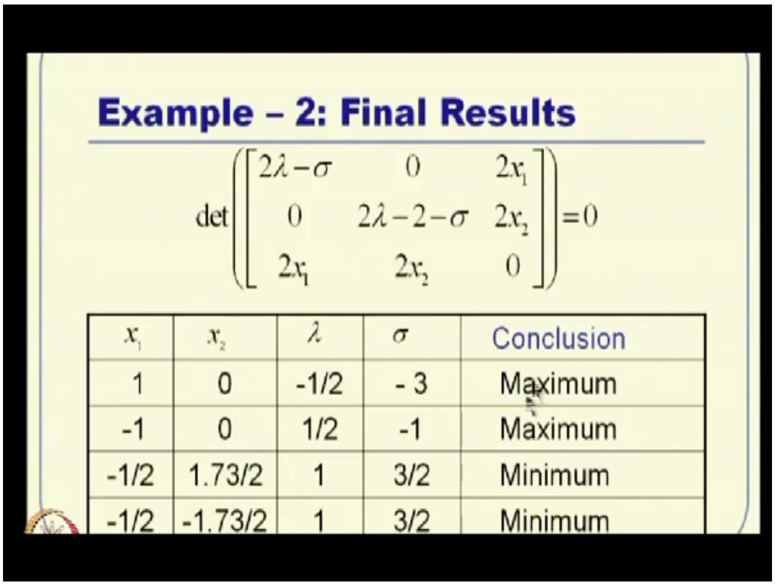

So, this happens to be maximum; this happens to be also maximum, this and this two happens to be minimum actually. Now if you see this, if these are two maximum, there are two maximum two minimum, we can always evaluate the function j; again go back and see which is global maximum and which is global minimum actually, because it the maximum of this two will lead us to the that the point, where the function takes a global maximum and minimum of two lead us to the point, where the function takes a global minimum sort of thing.

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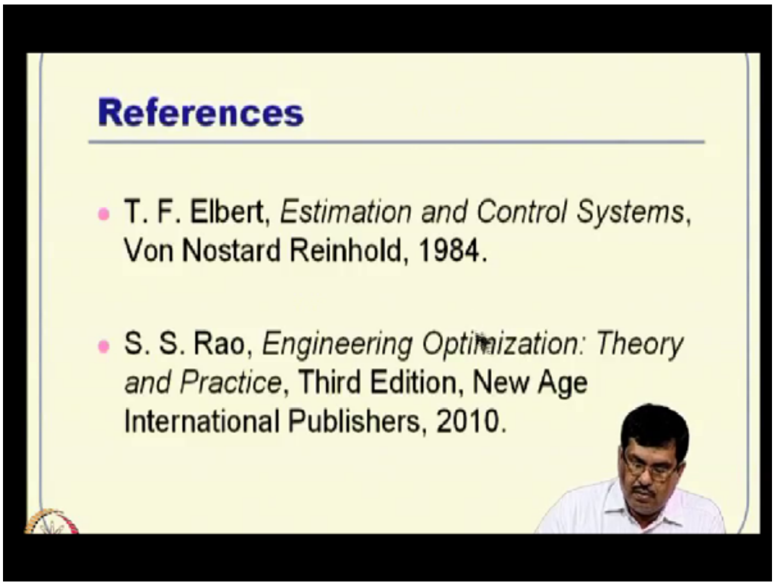

With this probably, I will stop here, but also before stopping, I will suggest that some of the concepts, I have taken from these two books actually. And these I do not know whether it is a available in market or not you can think about if it is available somewhere I think Mozilla dot com something like that or there is a good website also that I typically use this is called book find out dot com. So, it is does not at the does not necessarily search for only amazon, but it searches all over place we are part of the network and gives some possibility if something is telling a old book or something you can also buy from there actually.

That typically there are available use book and all that also available in the from those things and international shipping and all those are possible in really do not know whether and Indian print is available for this, and also old books are lot of thing. But this size suddenly knows that that is a Indian book, in Indian edition book available. This a Indian edition I mean, it is a global addition there is a Indian edition and also available and it is gone through some edition; that means, some error correction edition and almost be happens some news after some coming all actually.

So, this particular book, you can always think of buying is not very expensive, the basically now this book latest studies as per as I remember, I know this is two 2010 and also remember these books are written for static optimization only. That means, what you see here is a all this I mean, longer algebra and all that typically not reality, because many cases the objective function and constraints all large dimension there are some non-linear function all this, we cannot keep on doing this. So, it will go to this numerical idea solving actually.

And all the static optimization books, we will talk a lot about that a numerical optimization schemes, it talk about various doing things and number of follow by or in efficient way or in less iterations I mean, that kind of ideas will be there the we talk about line optimize lines are algorithm for example, they talk about line gradient there talk about they talk about algorithm think like that actually. So, we will see little bit of that ideas especially in the next lecture, but this particular book is very useful for those of you want kind of double of your knowledge is in the static optimization again all many other books are also available. I am not recommending only this if you happen to like something some other book, it is you are also welcome to see that actually alright. So, with this, I will probably stop this lecture here. Thanks a lot for your attention bye.