Optimal Control, Guidance and Estimation

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Module No. # 15
Lecture No. # 36
Constrained Optimal Control – III

Hello everybody let us continue our lectures series on Optimal Control, Guidance and Estimation.

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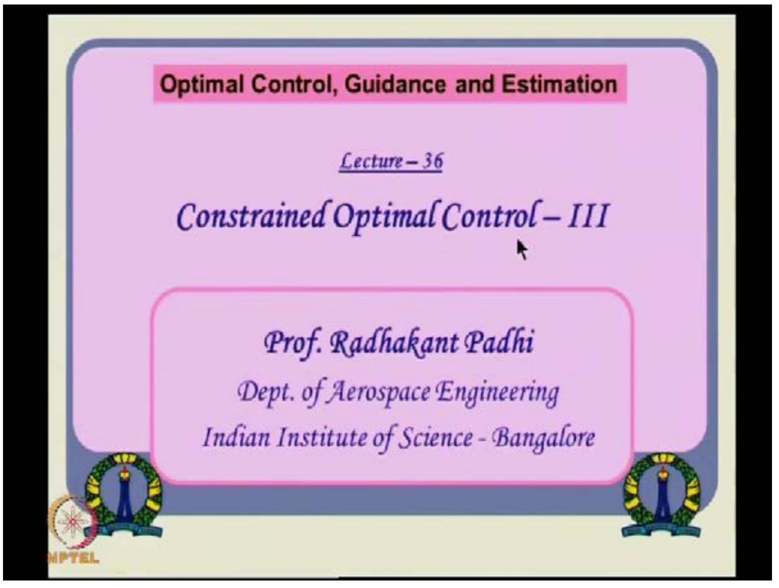

We are in the third lecture of this Constrained Optimal Control, which which is going to be the last lecture in this constrained optimal control series.

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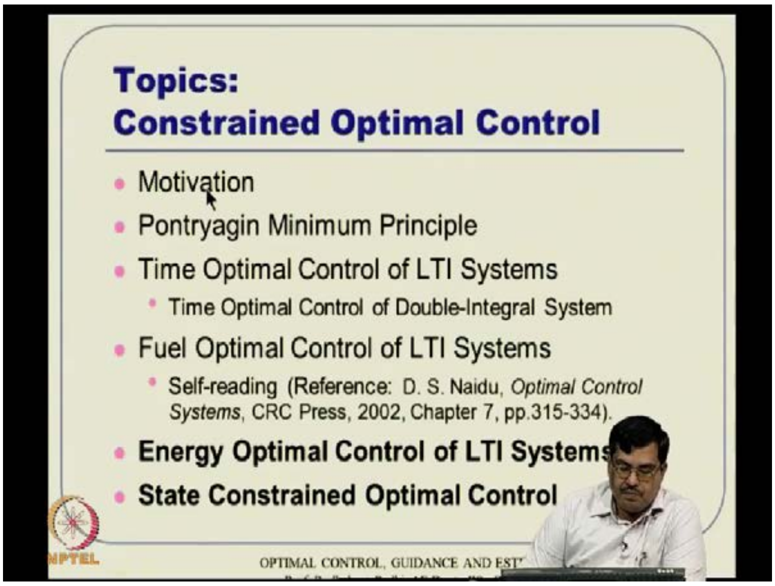

So, far we have seen, little bit motivation of why we need and especially lot of practical problems demand that we explicitly account for that. Then, we studied this control constrained optimal control problems in a generic framework in the Pontryagin's minimum principle framework actually.

Then we use that in time optimal control of LTI system, essentially the minimum time problem sort of thing; and then demonstrated in detail about this double integral system. And fuel optimal control happens to be one of the topics in book, but because of lack of time I will rather leave it for self reading actually. You can read this this topic from this chapter 7 in this pages actually; it is not that difficult it is very similar to time optimal control problem; just that there will be a similar arc in between and then the function happens to be slightly different then just the I mean signal function actually.

But, largely today and this particular lecture we will talk about energy optimal control, which is which is even more practically significant especially in the framework of again linear time invariant system. And then and toward the towards end of this lecture, we will also talk about straight state constrained optimal control, which is again quite critical actually in in many applications all right.

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So, just a little bit round up of summary of Pontryagin's minimum principle.

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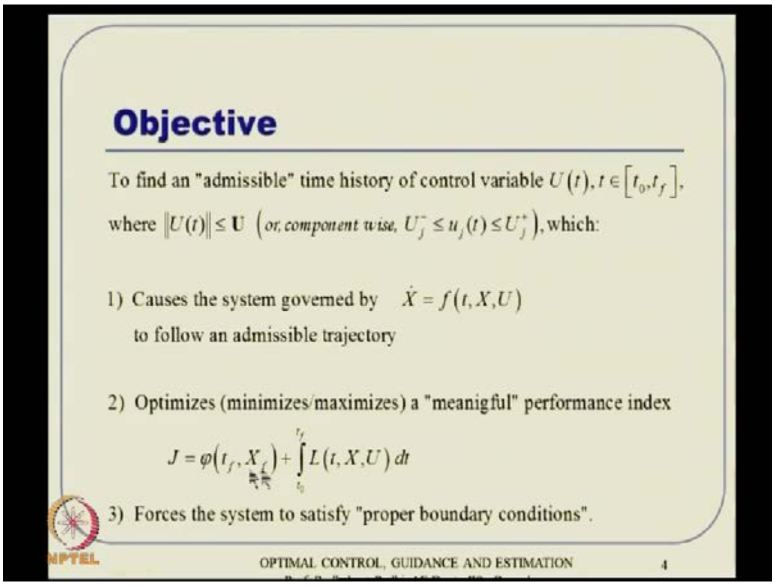

The problem that we have been talking about is is fairly generic; we have this non-linear system dynamics and then we want optimize this generic cost function. And it also satisfies these proper boundary conditions. We are interested in finding some sort of an admissible time history of control variable from t naught to t f. As compared to pre optimal control or pre optimization problems; so, the only difference is control is constrained to lie on an admissible set actually or in or component wise it lies each of the component of control variable lies between some minimum and maximum value ok.

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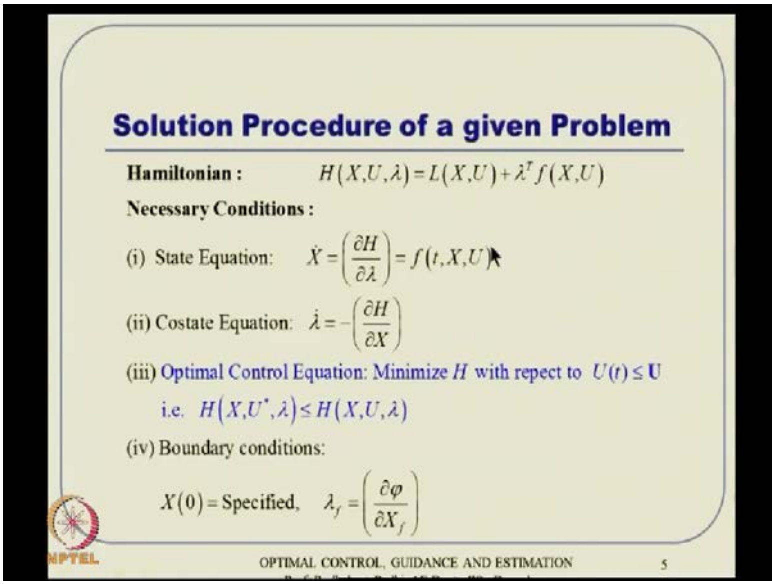

So, in that setting we have already seen that the necessary conditions of optimality are are very very close to I mean unconstrained case, but the only difference is the optimal control equation should satisfy this in equality. So, we can put it, del H by del U equal to 0 and then try to solve a control actually. So, this inequality has to be carefully analyzed. And you have to actually see what what U star minimizes this Hamiltonian function actually. So, the that that analysis is to be carried out in a careful manner.

And even even if you see about kind of numerical procedures, this this constrained has to be accounted for, rather than solving the control from that del H by del U equal to 0. That is that is the only difference actually, but that is a huge difference just because of that difference, we will end up with many different cases different analysis and all that actually all right. So, that is the that is the trick there.

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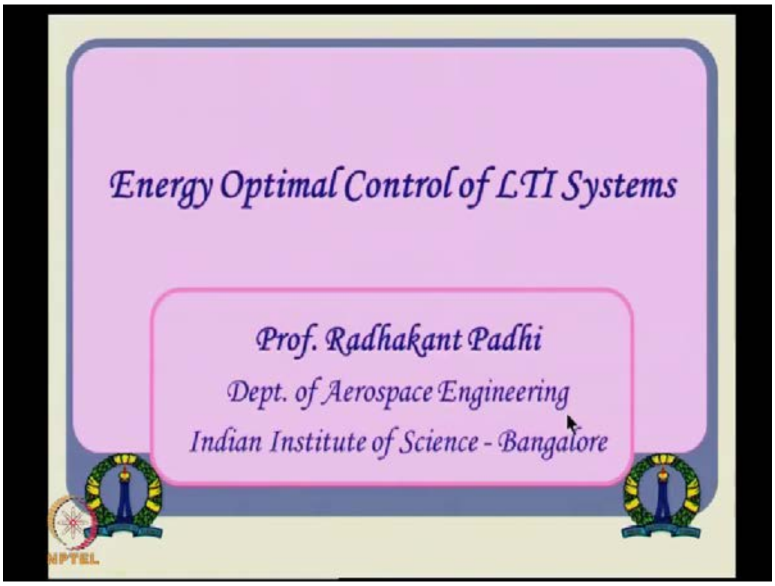

So, now moving on to the energy optimal control of especially linear time invariant systems.

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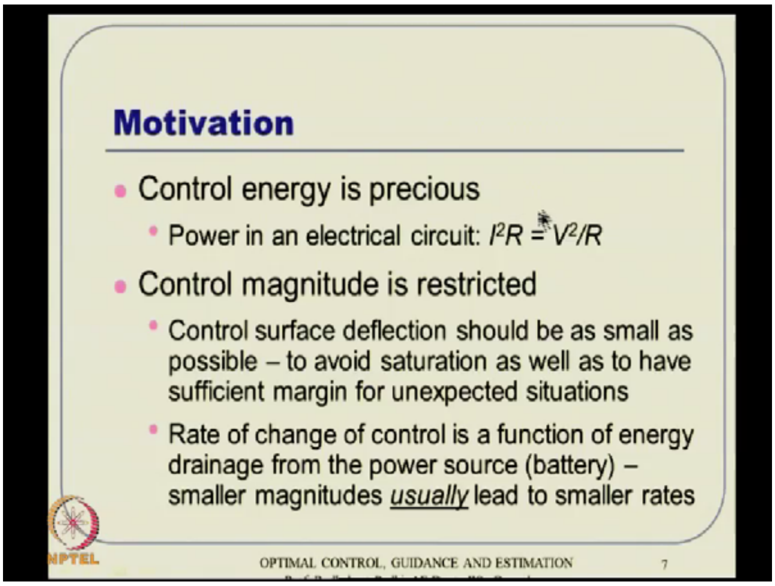

The problem is natural in many many cases. For example, if you have the control energy is is typically it happens to be precious. And if you see electrical circuit, then the power conjunction is I square R or V square by R. And if we consider this V as some sort of a control variable and obviously we want to minimize that the V square actually ok.

And then control may I mean in aerospace problems, control magnitude is is restricted. So, control surface deflection should not should be as small as possible, especially to avoid saturation as well as we typically want to minimize that to have sufficient margin for unexpected situations as well. For example, win cost we have not taken into account, but suppose you wanted to want to get a for win cost in while the plant is in operation then it is better that we have some control margin left out, some bounds are there. So, that we can still excessive that ok.

And also as a small remark, rate of change of control which happens to be a strong function of energy drainage from the power source, essentially some sort of a battery source in a specifically that is small means it actually leads to some sort of smaller rates also. So, it is not really kind of obvious, I mean it is not mathematical rigorous actually; in other words, somebody can also argue that magnitude is small, but the the control control can chatter within that actually (Refer Slide Time: 04:58). So, the rate can be infinity actually ok.

So, those situations there side, typically when the control happens to be a small it does not I mean what happens is, if you if you have bounded small and then there is right it rate I mean it does not change that much actually. So, rate happens to be is I mean small also. Because, if you if you see this finite difference sort of things, rate is nothing but, like x k plus 1 minus x k by del t sort of things. If you fix del t, when the magnitude as smaller than the rate happens to be smaller as well actually. So, usually it is smaller even though mathematically it cannot claim actually all right. So, these are the small motivation why we want to talk about energy optimal control.

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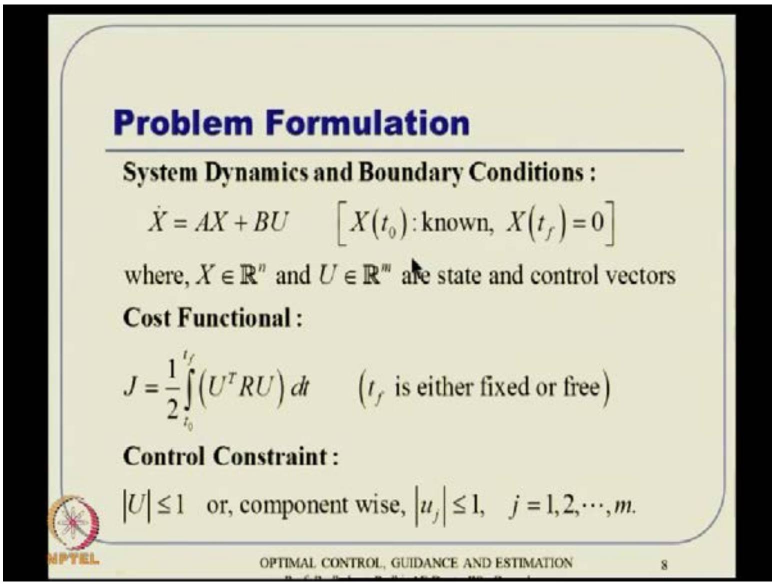

So, what are you what is the problem here? We have a system dynamics in this form linear form, X dot is A X plus B U, initial condition is known, final time at t equal to t f, X of t f should go to 0, especially when t f is free. If t f is fixed then, I mean it depends on where it wants to go I mean how the system ways to that and think like that actually Ok.

But, ultimately if this X of t f has to be 0, because the otherwise what will happens is, you have to account for that in the cost function actually, some some quadratic terms will be there in the cost function. If that is not there, what you are interested in is X of t f should to be 0 with with this minimization actually for control the cost I mean cost functional to be minimize is a quadratic function of the control variable ok.

So, t f is either fixed or free depending on the situation actually. Anyway the control constraint typically it in a normalized sense it can be written something like that; in a magnitude of U is less than equal to 1; or component wise, if you analyze then each of the component u j is magnitude bound with with 1. So, control variable is normalized and the normalizing quantity is observed in the B matrix actually. So, the that is that is what we have been talking about actually all right.

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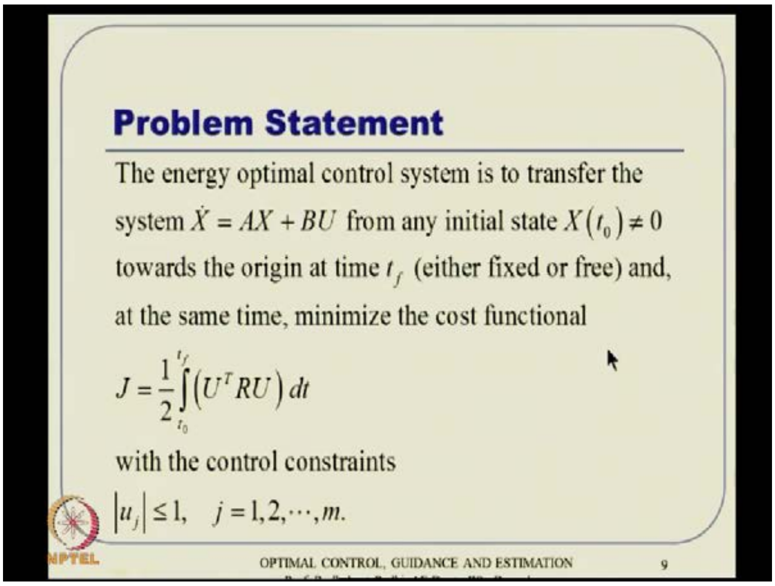

So, moving further the energy optimal control system, formal statement happens to like that. Energy optimal control system is to transfer the system dynamics from any initial state towards the origin at time t f, either which with time t f can be either fixed or free and, at the same time, it has to minimize this this cost function with the control constrains in place actually all right. So, that is the formal definition of a energy optimal control problem.

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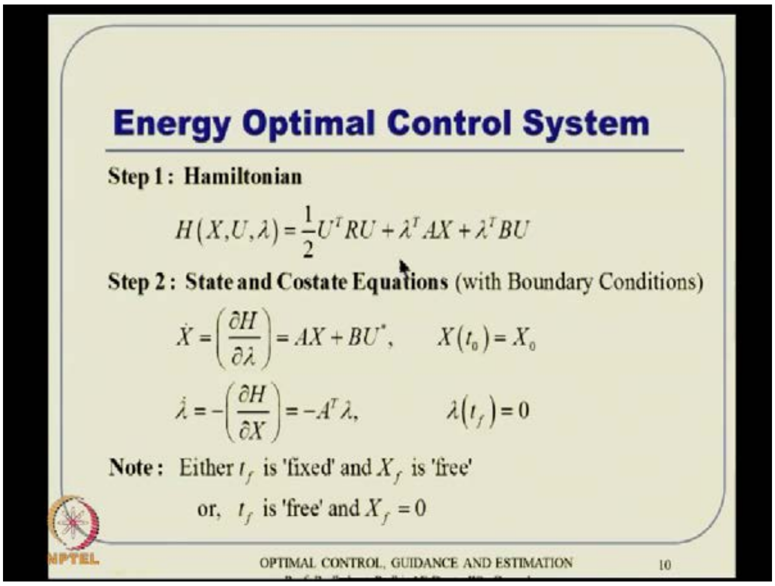

Now, how do we go ahead and solve it? We have the this regular technique of of applying necessary conditions of optimality. So, Hamiltonian we defined first, l plus lambda transpose f. So, l happens to be half of U transpose R U; and lambda transpose, f happen to be A X plus B U. So, this is what is written, l half of U transfer R U plus lambda transpose A X plus B U; but I can expand that, lambda transpose A X plus lambda transpose B U.

Now, the state and costate equations after that is a is like this X dot is del H by del lambda. So, again the same costate equation appears with optimal control function there. And lambda dot is minus del H by del X is equal to minus A transpose lambda; boundary conditions X of t naught is X naught, and lambda of t f is equal to 0. Anyway, so these are certain details there actually to it does not does not matter to us so much about this this formulation and all very standard actually.

And also let me put a small remark which I have being telling that, if you see the book everywhere you will see the star notation X X star, lambda star like that actually. Here you are purposefully taking out, because the inequality that you are interested in is control inequality only (Refer Slide Time: 09:05). So, what you purposefully I have U star here and not any star anywhere else actually, but that does not mean that the state and lambda I mean state and costate are non optimal; they are also optimal trajectories actually all right.

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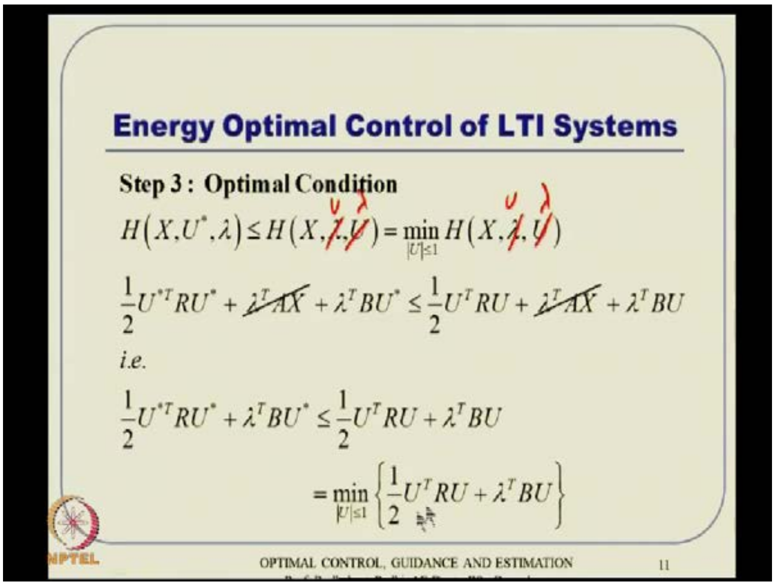

So, this is a critical thing, optimal control condition we want to analyze actually. So, when you want to when you want to analyze that Hamiltonian of (X, U star, lambda) should be less than equal to Hamiltonian of (X, lambda, U) basically; (X, U, lambda) actually neither way actually well. We have been following (X, U, lambda). So, probably you can put that as (X, U, lambda) everywhere actually it is U is to be lambda and this will be sorry is it is just a notation sort of thing.

Anyway, so this coming back to Hamiltonian definition; so, as this expression has to be put there. And one side is the optimal control function, one side is a now any other control function actually. So, this same expression with U star in the left hand side and the same expression without U star is just U in the right hand side. You can see one term cancels out, lambda transpose A X and lambda transpose A X cancels out.

And what you are interested in is something like this actually this (Refer Slide Time: 10:38). This inequality results in something like this, you will have half of U transpose R U with with U star and this expression with lambda transpose B U star has to be less than equal to any other U basically. So, equivalent of telling that, this expression is nothing but, minimization of this expression with this constrained in place actually.

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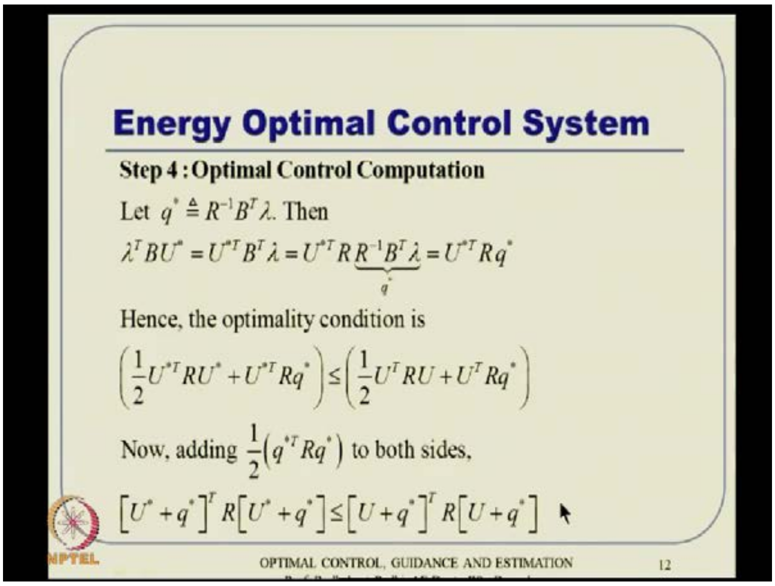

Now, here is a little bit piece of analysis. First, we define q star is something like R inverse B transpose lambda. And then this expression lambda transpose times B U star which appears here can be actually as I have been analyzed a little further tell a lambda transpose B U star is something like its own transpose, because ultimately it is a scalar quantity. So, take it own transpose.

So, U star transpose B transpose lambda and then we introduce R times R inverse identity, it R time R inverse is identity, so we can we can introduce here; and then observe that this quantity is nothing but q star. So, then we we note that lambda transpose B U star can be written as something like this also basically ok.

So, hence this this inequality what I see here, I can write it something like these instead of that expression I will rather prefer to use that expression there is a reason for that actually. Now, after doing that we can add this quantity to both sides and then you can you can see that it essentially leads to some sort of a quadratic expression in the left hand side, and quadric expression in the right hand side ok.

If you if you add this quantity here and this this split into two parts, half of that plus half of the half of other one. And one time if you use this expression and one time you use that expression and an and then combine in in matrix vector form. Then it turns out to be something like this (Refer Slide Time: 12:33), this is actually a quadratic term in the left hand side, and quadric term in the right hand side also. The difference is left hand side has used U U star function, and right hand side has simply U function actually all right.

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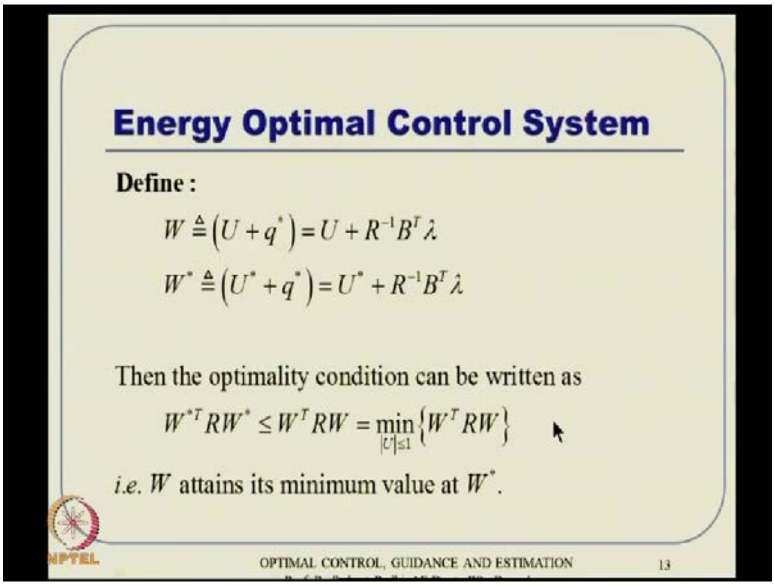

Now, we define for algebra simplicity this term whatever term I mean U plus q star sort of things. And W star is actually U star plus q star. W is U plus q star, and W star is U star plus q star. So, what you see is U plus q star here, and U star plus q star here that is the reason, if you want to kind of simplify the algebra little bit.

Then this same expression what you see here same inequality can be expressed as something like this inequality now (Refer Slide Time: 13:27). And that is nothing but, the this this minimization of this quantity with respect to this constrained basically all right. So, this is this is the simplified notation form.

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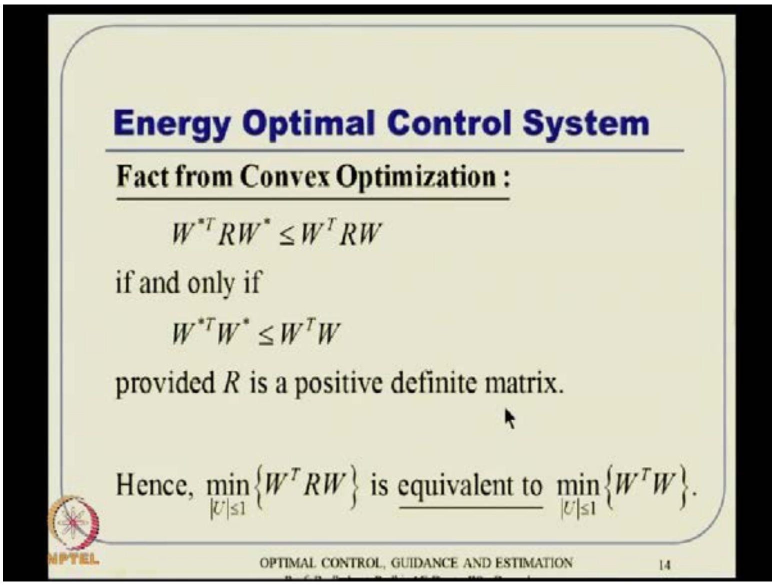

Now, here is a trick actually. That is a fact from convex optimization that, this expression I mean this inequality is true, if an only if this inequality is also true. There is cursory proof in the book, it is not very rigorous or some of you want to see you can you can see that actually. So, they if this inequality is true, if an only if this inequality is also true, provided R is positive definite matrix. So, R does not does not matter to so much actually, as long as R becomes R is a positives definite matrix actually ok.

And utilizing this fact what you are telling is, this is this minimization of this quantity subject to this constrained is equivalent to minimization of that quantity subject to this constraint. So, this is a critical observation actually, we will without worrying so much on that proof part, we will will continue our analysis on assuming that this is always true.

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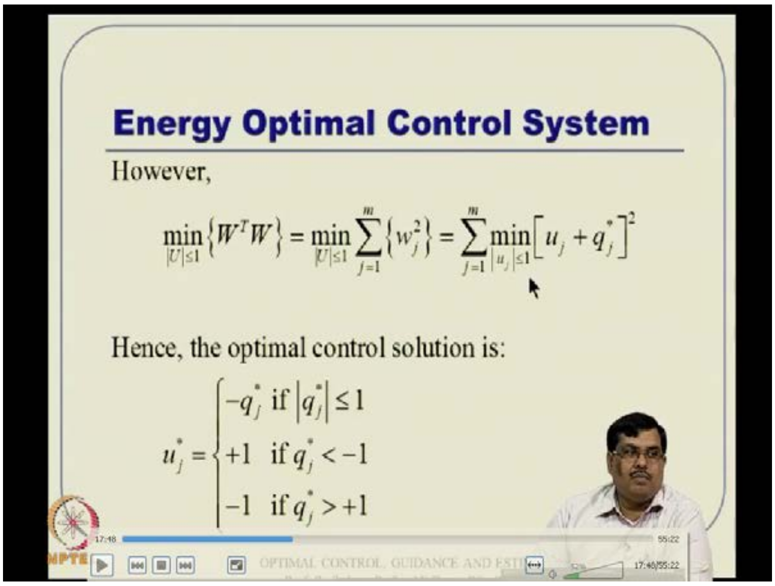

So, now now we will end up with a case, where you have to see that we have to minimize this quantity W transpose W with respect to this constrained actually. But, this W transpose W with with norm of I mean and norm of U or modulus of U will be less than 1, this is nothing but, this is a just quadratic expression. So, this W transpose W, W is a vector remember that. So, because definition of W is like this, W is vector. So, W transpose W is nothing but, summation of W j squares it is W 1 square plus W 2 squares like that actually.

Now, but W j by definition if I just go through that and put that definition there in the W j happens to be u j plus q j basically right; this is definition here, this is a vector, this is vector. So, I can collect any component of W as component of that plus component of that basically. So, that we write it here actually u j plus q j star whole square. Hence, the optimal control solution now, here we we are almost here almost there actually. Now, what you are telling is, we will end up with a case where you have to minimize this quantity subject to this and this is nothing but that basically.

Now, u j depends strongly on q j now; and q j happens to be this definition R inverse B transpose lambda. And and if you recall your I mean your knowledge about unconstrained optimal control then, this expression pops up naturally right U star equal to minus R inverse B transpose lambda actually in that in that situation actually. So, anyway, so that is we will end up with some some situation like this. Now, let us see what value of u j I will select, so that this quantity becomes minimum quantity basically.

And remember this is a whole squares term there actually, so each of the term is to be minimum this this I mean u j plus q j star whole square; that means u 1 plus q 1 whole square plus u 2 plus q 2 whole square like that actually. So, to be minimize the entire quantity has we should have minimize of individual quantities actually ok.

Now, if you analysis this, if you see this expression little carefully all that it tells is, if q j q j star is within the bound that means it is it is inside that then all that is I can comfortably make it negative of that and make it 0 basically; this expression can the minimum value is 0. So, if q j star modulus is less than equal to 1 than u j can we comfortably written as minus q j star, because that will satisfy the control bound actually ok.

However, if if q j star is something like less than minus 1 let us say then whatever quantities I will have I should have to have a minimum quantizes there. So, if it is less than minus 1 then the best I can do is probably by putting plus 1 there that is the magnitude bound actually ok.

So, even if it is something like minus 3 it is still going to be minus I mean the summation will be minus 2; if it is minus 4 then it is summation will be minus 3 that is the maximum we can do actually. So, we just leave it there. And similarly, if q j star happens to be greater than plus 1 we just we just take it as minus 1 actually ok. So, that is the that control formula that will end up with; and also again as a as a remark what you see here is for even for linear system, we will end up with a non-linear control actually. So, that is how it is.

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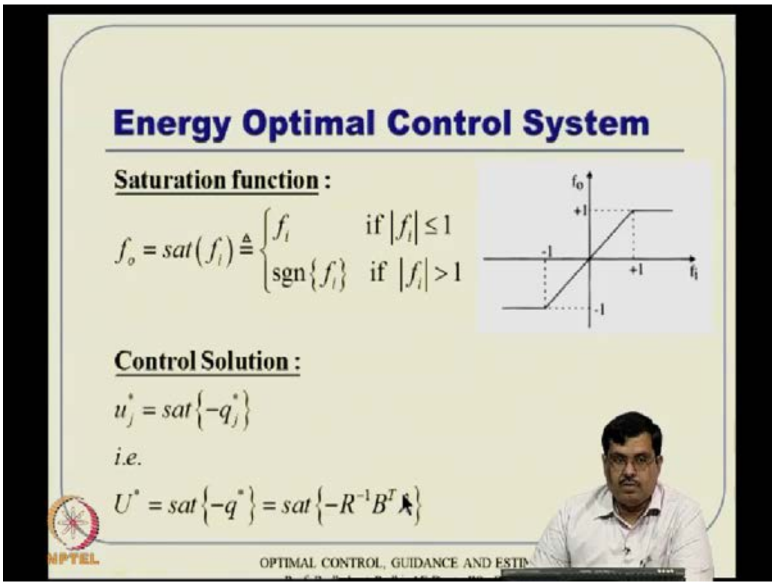

Anyway, so now to kind of write it in a compact form, we notice that they we can we can make I mean take help of this saturation function, which is defined as something like this (Refer Slide Time: 18:24), this input versus output. And saturation function as defined like this, if the magnitude by if the magnitude is constrained within plus or minus 1 then this is simply the same function; output function is nothing but, the input one, but if it is beyond that then it is restricted to the bound value basically ok.

So, if you if you take take help of that then u star u j star happens to be the saturation function of negative q j star that all actually. And and combining all the components, so we can write it in vector form as something like this (Refer Slide Time: 18:58).

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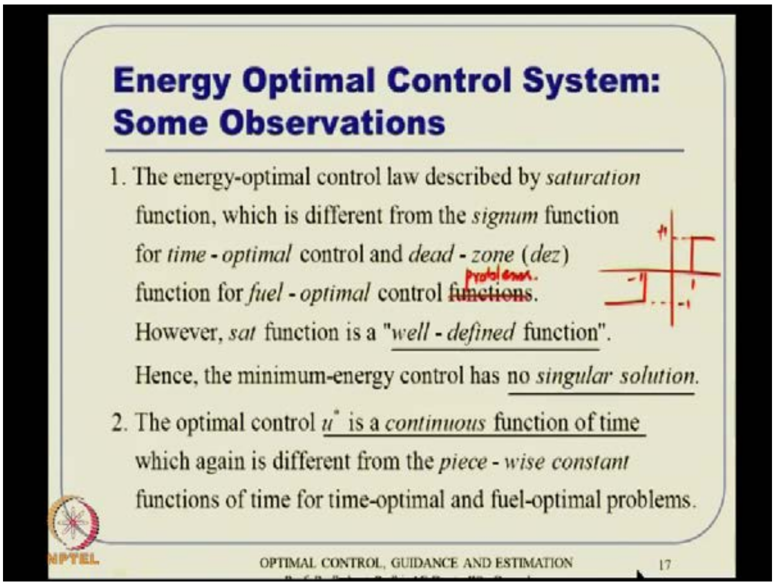

Now, some commands or some observation rather. First of all, the energy optimal control law is described by saturation function, which is different from the signum function for time optimal control that is the bang-bang control sort of thing, is defined by signum function; and it also like a dead - zone function for fuel optimal control control problems actually, this fuel optimal control problems.

So, is there we will end up of something like this actually, we we have this this kind of thing actually sorry it is not slight mistake there. Their we want to minimize the magnitude of control and if you want to do that, this some sort of dead – zone, here it will be minus 1, this this value is minus 1, this value is is plus 1, and this value is plus 1, and this value is minus 1 like that actually. We will end up with this this this dead zone sort of control actually. So, bang of bang control basically that I leave it for for I leave it to you for self reading all sort of thing.

So, you just read it in the book and see. So, now what happens here is, in the in the singular arc between minus 1 to plus 1, control is really not defined; if it is not defined, people typically 0 control. So, it is but still if it is theoretically if we take 0 controls then it becomes define control basically. So, the control is simply not defined, so we switch of switch of control basically in a way basically. So, these are singular problems.

But, but I mean if you talk about energy optimal control it is very well defined function; it is actually continuous function. There is a minus 1 value, it is starts varying to plus 1 and then goes their actually. So, this singular arc sort of thing does not happen actually. Now, u star is actually continues function of time this is this kind of function is certainly a continuous function; there are points at which, the derivative is not defined, it is derivative discontinuous sort of thing, but then function sense it is continuous actually ok.

And here is a critical observation (Refer Slide Time: 21:18), if the control is not constrained then u n star is such that minus R inverse B transpose lambda which is negative q star. And hence, if q star this this constrained is in place q star modules is less then equal to 1 then, the constrained and unconstrained optimal control are are typically the same thing actually all right.

So, this is the now another observation is the closed loop closed loop system is dictated by this this function X dot is A X plus B U. So, X dot is A X, but U happens to be something like this situation of minus R inverse B transpose lambda. So, we can put that, and sometimes people write it as big letter sometimes people I means write small letter like that actually ok.

So, any way, so that is X dot is A X star minus saturation B times saturation function of that. And then this system dynamic happens to be non-linear actually. So, there is no close form solution for that, you cannot write in exponential solution sort of thing. So, it if you really want solution of that, then you has to do this numerical solution way actually all right. So, this is this is what it is.

Now, the critical observation here is this this particular thing actually what we observed there, the constrained and unconstrained problems happens to be very very close to each other basically. That means, if you as long as it is within the bound I take the I be it be of edge if its unconstrained problem actually; and only when it crosses that limit, then it simply the same value the bound value basically; and which is very intuitive I mean even if you do not know optimal control theory sort of thing then this is how do we we I mean we implement basically. But, naturally it turns out to be the case when you when we talk about quadratic function of U are being minimized actually; that means energy optimal control these kind of cost function actually all right.

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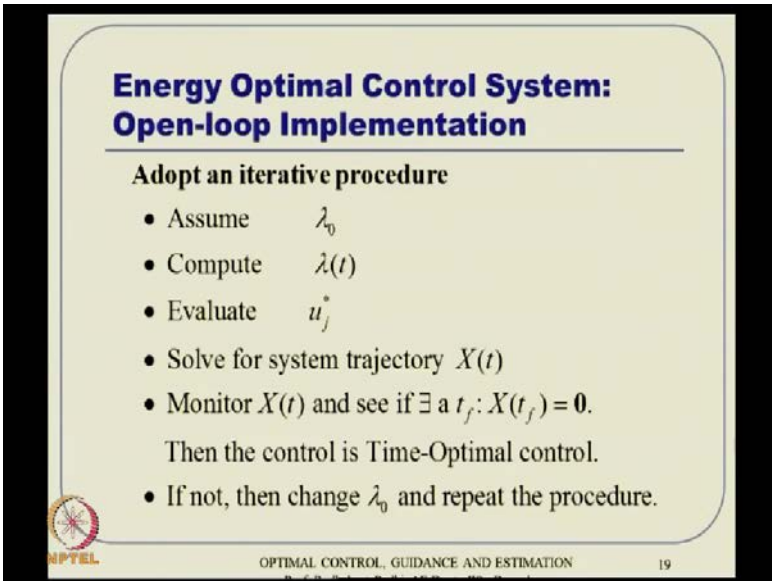

So, that is an interesting observation. Now, to to implement it we can have again two ways of implementing, one is an open loop implementation sense. So, that means you can assumes something like lambda 0, and compute lambda t, because this expression is available with us we can we can solve it once lambda 0 is available.

And then we can evaluate u j star and then then you can solve the system trajectory; and monitor whether were X of t f is 0 or not; if it is 0 fine, otherwise you come back and revise the lambda 0; that is not very good, because there are infinite I mean trials you have to do to get it there. And hence, this is not a very good procedure basically; and that should be numerical procedure of how to adjust this lambda 0 in a proper direction and all that, to like shooting methods sort of thing, but we are not talking about that actually ok.

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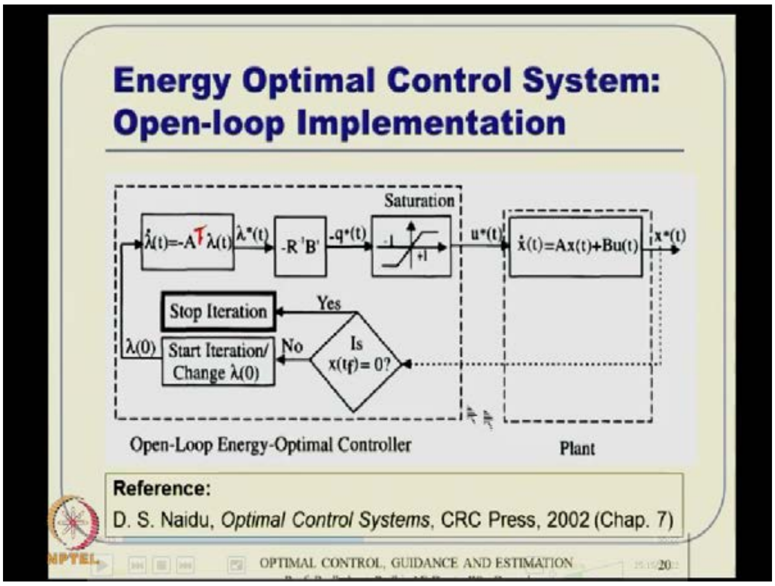

So, we are what we are interested in especially, because it is linear to linear problem sort of thing, we are interested in having some sort of a closed loop or state feed back control actually. But, anyway somebody wants to be implement this way, this this how it is. You start with some some guess value of lambda 0 then solve the system, this A transpose and then get your lambda of t and then get your q star, plus passing it through the saturation function here, and then u star is available. But, monitor x star how it develops and let t equal to t f is it is 0 or not, if it is 0 then stop it otherwise continue actually all right.

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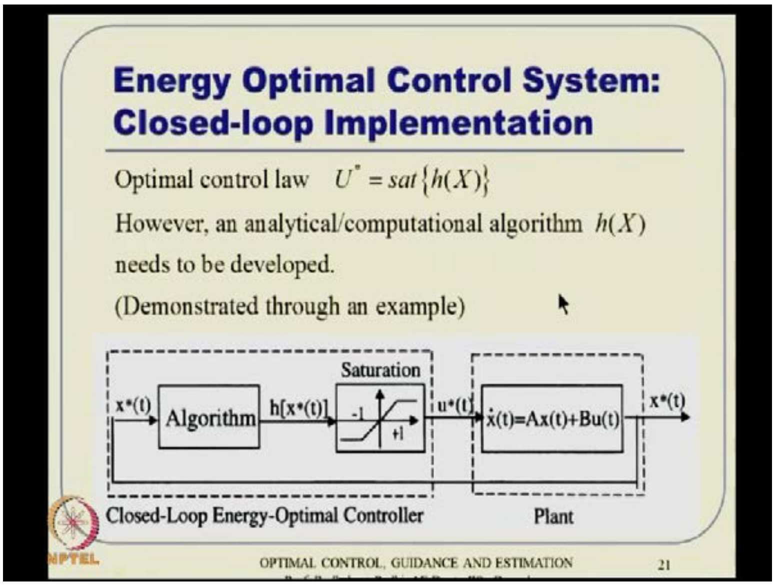

But, what you are interested in is closed loop implementation as a state feedback form. So, we are interested in some sort of a function and we can still pass through the saturation function sort of thing, so that we will get optimal control. So, can we do that? And that is demonstrated again through a through a small example it depends on case to case actually all right.

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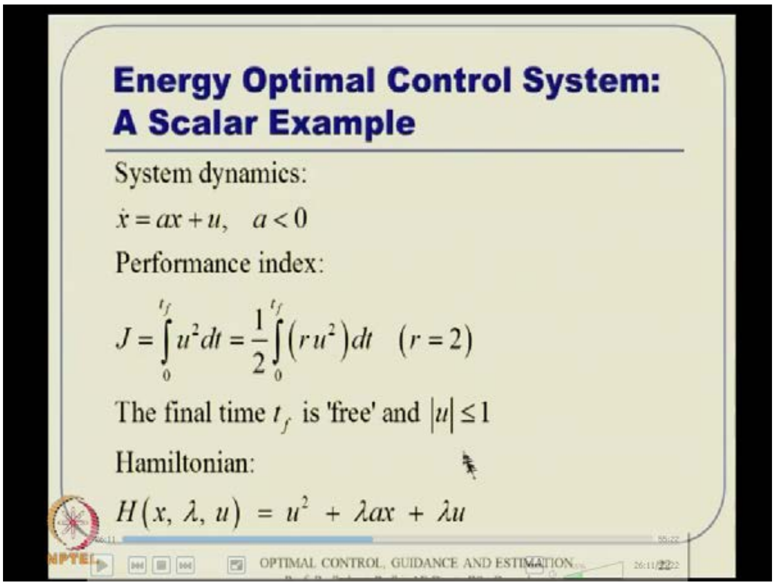

Let us start that example, we have x dot is a x plus u, where a is negative actually. So, it is a kind of a stable system. And we also know that for stable system typically the costate dynamics is unstable. So, just remember that actually we will need that actually. Anyway performance index is like this, so if once you have u square d t then r half and you can put r, r equal to 2 actually ok.

So, we have a is small a, and u is 1 and q is not there of course, and r is 2 basically; t f is free and control is bound with with absolute value of u being less than equal to 1; thus the standard problem actually. So, we have this Hamiltonian u square I mean u square coming from coming from this cost function l, l is u square plus lambda transpose f lambda times a x plus lambda times u basically ok.

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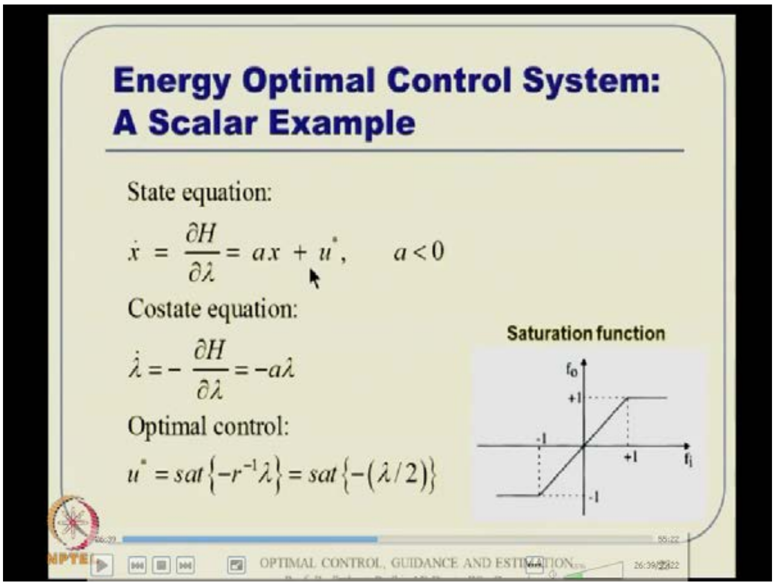

Now, state equation x dot is del H by del lambda. So, a x plus u star, but remember a is negative quantity again I emphasis that actually. So, costate equation happens to be negative of del H by del x actually. So, that happens to be minus a lambda basically. So, that means minus a is positive remember; so so lambda dot is now now unstable equation actually. So, up to optimal control we already derive that u has to be a saturation function of minus r inverse b transpose, b b is not there b is I mean b is 1. So, that is not written here.

So, minus r inverse lambda that means r is nothing but 2, so r inverse is half actually. So, saturation function is nothing but negative of lambda by 2 basically, a saturation function defined this way again.

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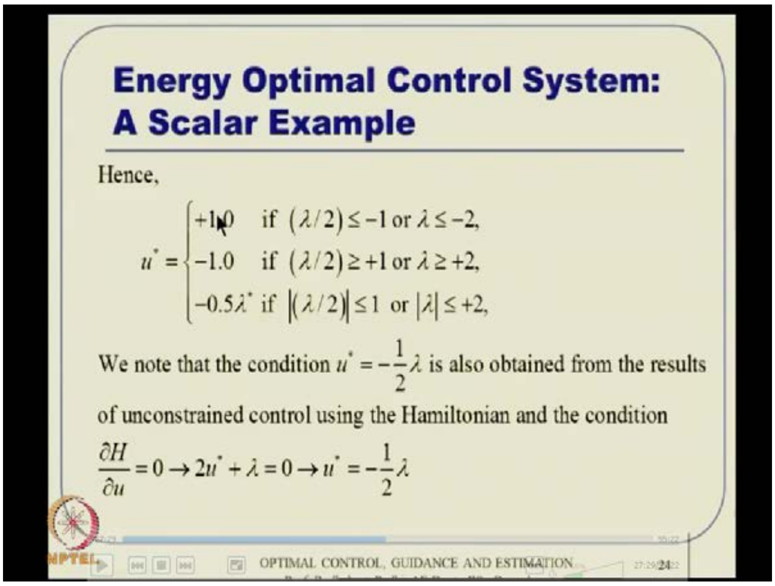

So, u star happens to be plus 1, if lambda by 2 is less than minus 1; it happens to be minus 1, if lambda by 2 is greater than plus 1 or it happens to be just negative of that value lambda is I mean minus lambda by 2, if this is within the bound actually all right. So, this may not be needed we are not using lambda star anywhere actually all right. Anyway, so that is all right actually ok.

So, we can also see that, u star is equal to minus lambda by 2 we can also be obtained from the unconstrained optimization. Because, once you have del H by del u equal to 0 and H is that, del H by del u is 2 u plus lambda equal to 0; so, which is u equal to minus lambda by 2 basically. So, that comment is still there basically all right.

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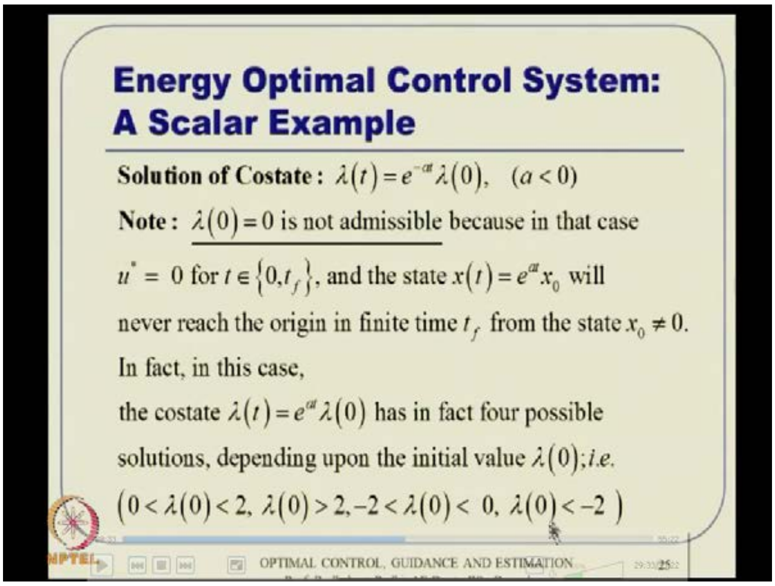

So, now coming to the analysis part of it, we want to have some some closed loop or state feedback sort of solution actually. So, we go back to the costate equation first. And costate equation solution is this way, remember a is negative, so minus a is positive actually, so this the solution.

And also note that lambda 0 is 0, it is not admissible, because once lambda 0 is 0, and lambda t happens to be 0 it will remain at 0. And if lambda t happens to be 0 then then u star happens to be 0 all the time actually. And then x of t has to be just the homogenous system and this homogenous system will never reach to the origin in finite time actually. So, that is typically ruled out actually.

Now, what you see is, now this solution is still valid lambda t equal to e to the power a t into lambda of 0, but it has now four regions actually. We have to talk in something like lambda of 0 is 0 to 2 or it is greater than 2 or it is minus 2 to 0 or it is less than minus 2.

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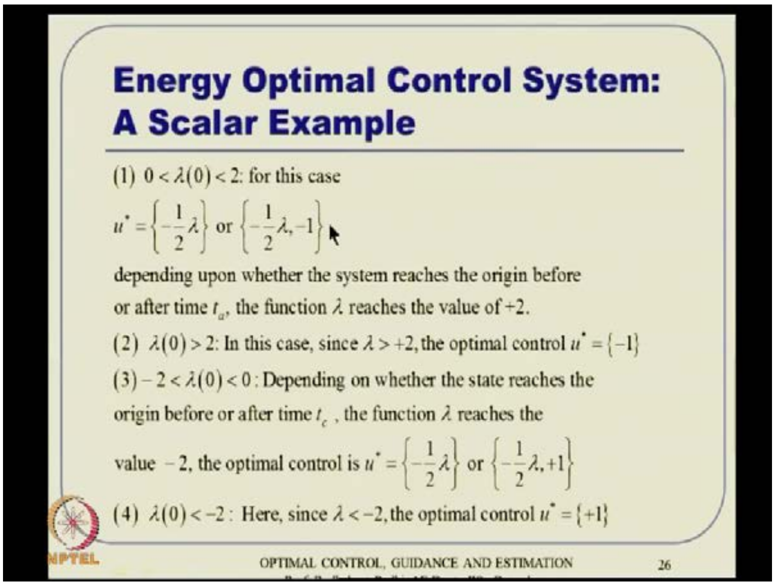

The region being, if if is if it if it lies between 0 to 2 ok.

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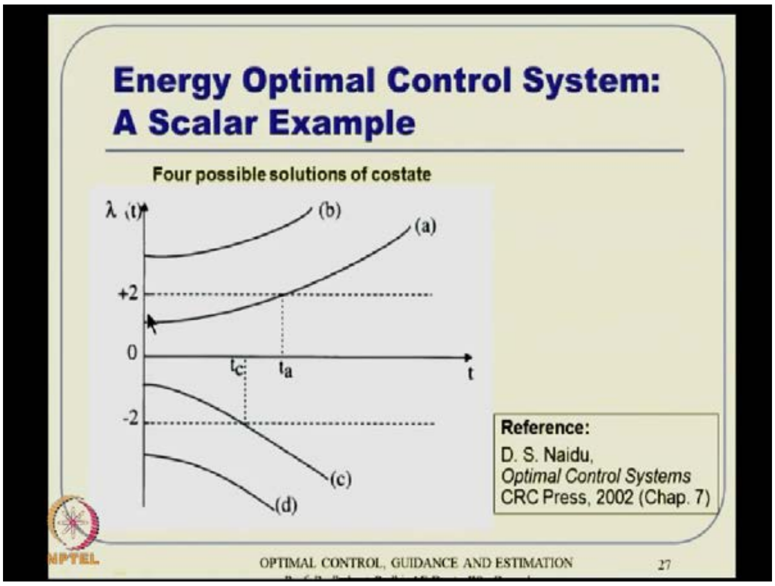

Now, I I will come back to this slide. The solution nature happens to be something like this actually. If we starts between something something there then it remains there; if we starts somewhere inside this bound minus 2 to plus 2, if you starts in a positive direction it will sometime t a it will cross that and go away; if it happens to start within this 0 to minus 2, it is sometime t c if will it will cross and go away actually ok. Otherwise, it will remain outside the bound actually. These are the four cases that here we are talking actually.

So, if it is if it is there, why it is important? Because, once this one crosses this bound of plus 2 then the saturation is in place actually, control saturation is in place within that it is is is in not in place sort of thing actually. So, that is why we are talking about that. So, if it is that, the u star is either that either minus lambda by 2 or either I mean you take this value minus lambda by 2 a minimum of that value actually; that means initially it will like that and later it will stabilize it that. That is the meaning of this curly bracket actually ok.

If it is always greater than 2 in that in that case lambda remains to be greater than 2 and the optimal solution happens to be minus 1 just the saturation values sort of things. And similar analysis is there, if it is within within this bound then at sometime t c you will it will cross that, so until that we apply minus lambda by 2 and then then state plus 1 actually. And if it happens to be this side, then it is always saturated. So, it happens to be like that actually all right.

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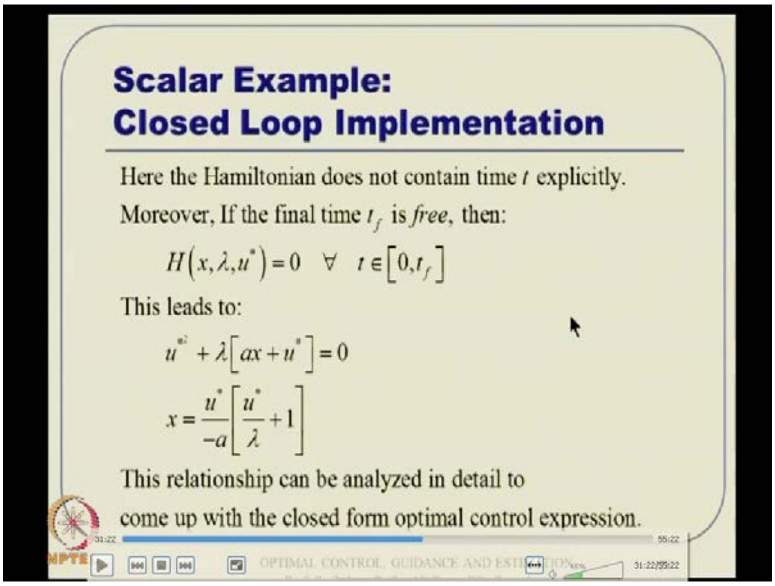

So, this states this diagram I already explained actually. Now, coming back to the closed form solution that is we are hunting out actually, how do we get that? So, here is a critical observation we know that, on optimal path Hamiltonian is a constant function. And if time t f is free, Hamiltonian happens to be 0 actually because, t at I mean finally, the Hamiltonian go to 0. Now, Hamiltonian if you just go back and see, what the expression for Hamiltonian is, this is expression for Hamiltonian (Refer Slide Time: 31:44).

So, here we can substitute that and tell this expression happens to be 0 and there I can solve for u as a function of x and lambda basically; or I can solve for x as a function of u and lambda either way basically. So, this is solving for x as u and lambda. This this constrained equation will be analyzed in detail basically.

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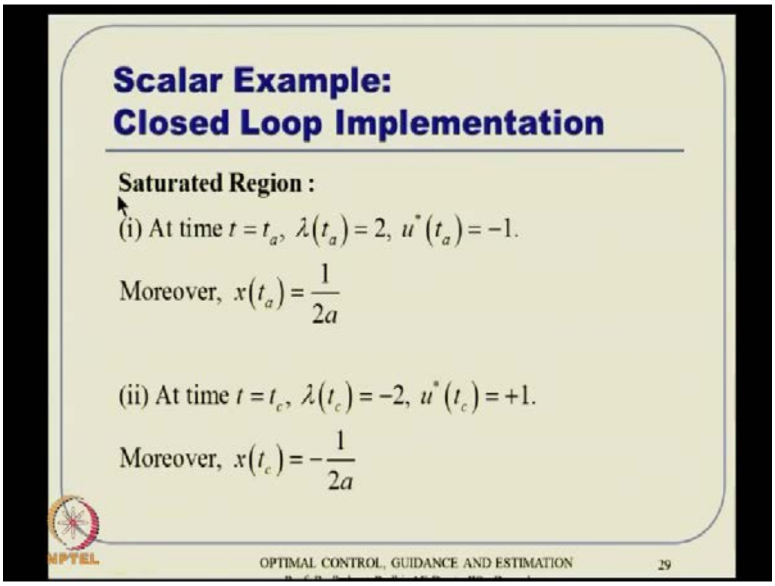

Now, we talk about a saturated region and unsaturated region separately basically. If it happens to be saturated region at t equal to t a, because this is remember this is t a or t c either way. So, the what is that particular what is that that particular value of x at that particular time basically that is were we are interested in.

So, at time t equal to t a, lambda of t a happens to be 2 and u star happens to be minus 1. So, we plug it back here, this expression whatever expression we are having u star is minus 1, and lambda expand to be 2. So, if plug it that you get 1 1 over 2 basically. And similarly, at t equal to t c, lambda of t c is minus 2 the other case actually, what we are telling; and x of t c we get a value now basically.

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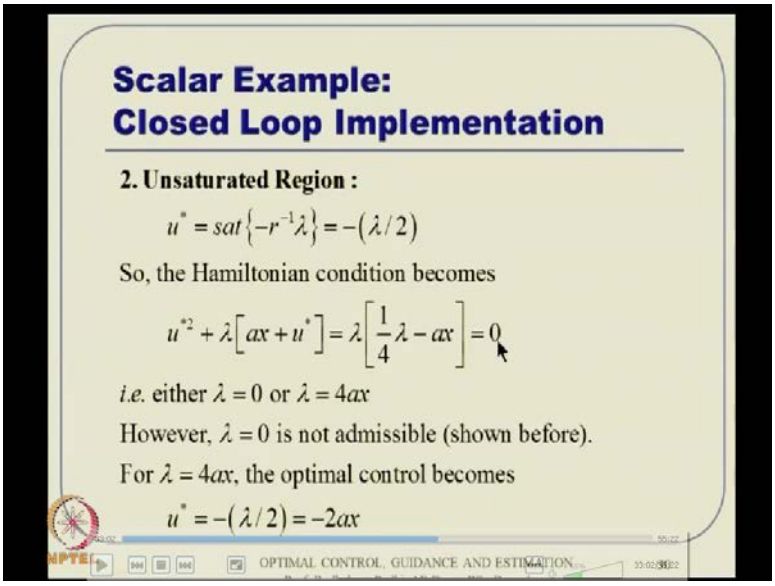

In unsaturated region, u star is saturation function of this quantity. So, this happens to be negative lambda by 2. So, the Hamiltonian condition, if you again go back and substitute that u has a function of lambda, if you substitute that we will end up with some some quadratic expression in terms of lambda now basically. So, either lambda is 0 or lambda is 4 a x actually, but lambda 0 is ruled out we just saw that little little before actually.

So, lambda 0 is not possible is not admissible. So, the only condition is where lambda has to be equal to 4 a x; and hence u star equal to minus lambda by 2 and that is nothing but, minus 2 a x.

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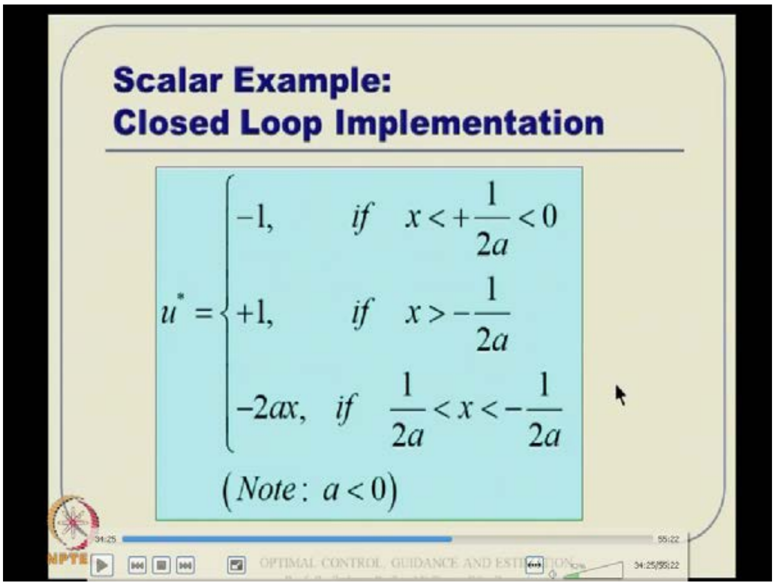

So, summarizing all these what we can see is, if x is less then that then instead of condition on lambda this part of time we got a value for x also; and remember x is also the kind of monotonic actually in a way actually. So, you got the value for x at this particular time or this particular time that will kind of give us a feedback sort of formula and which will tell I can summarize this something like a condition on x not condition on lambda really ok.

So, I will get if if x is less than this 1 remember a is negative again. So, this is a negative quantity if x if x is less than that, but it is within that I mean sorry outside that basically x is less than that then I will have minus 1; x is greater than that quantity then I will have plus 1; and otherwise, if it is within the bound then I can I can operate that way actually. So, this is a state feedback formula basically all right.

So, this is what I want to do, talk in terms of this this energy optimal control problem; and also it depends on case to case, if you have non-linear problem and or different sort of little more complex cost function and all, analysis has to be slightly different. The steps are layout and you have to analyze in those steps actually basically ok. So, with that I think will migrate to the state constant problem now, before before we wind up this serious of lectures, unconstrained optimum control.

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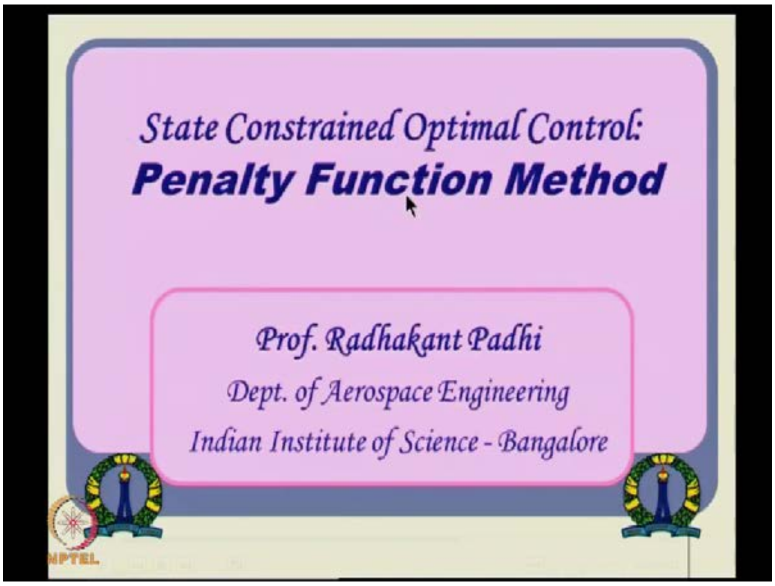

So, state constant problem as well study from a variety of I mean analysis tools essentially. And obviously, everything not possible to possible for me to talk, so I will confine into two methods two elegant methods are there basically. So, first thing we talks about is penalty function method sort of things. So, let us see what it is actually.

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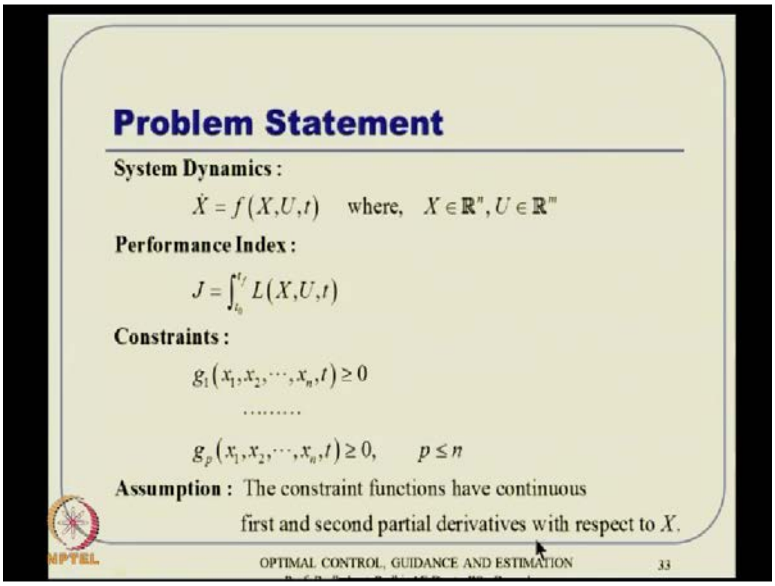

So, suddenly you come out of this this this Pontryagin’s principles and thing like that, we are it we are talking about state constrained features here actually. So, it thus the different tricks has to be applied here, unless it is a big state and control constrained that

that has to be I mean that is again the topic in general actually. You can have problem with state constrains as well as control constrains. So, that is that is what is, but here we are typically confine to primarily list state constrained actually.

Anyway system dynamic will go back to non-linear systems. System dynamics is non-linear, performance index is generic. And then the constraints are there, several constraint in fact 1, 2, 3, up to p constrained are there; some sort of algebraic functions which has to be greater than equal to 0. It is a very generic way of looking at it, only condition is p has to be less than equal to n, n is the dimension of the state actually ok.

However, it may lead to come sort of a over constrained problem and all that actually. So, we do not want to learned of there actually anyway. So, the also there is another assumption which tells that the constraints functions whatever function we are talking here, they have continuous first and second order partial derivatives; in other words they are kind of smooth functions in all variables actually ok.

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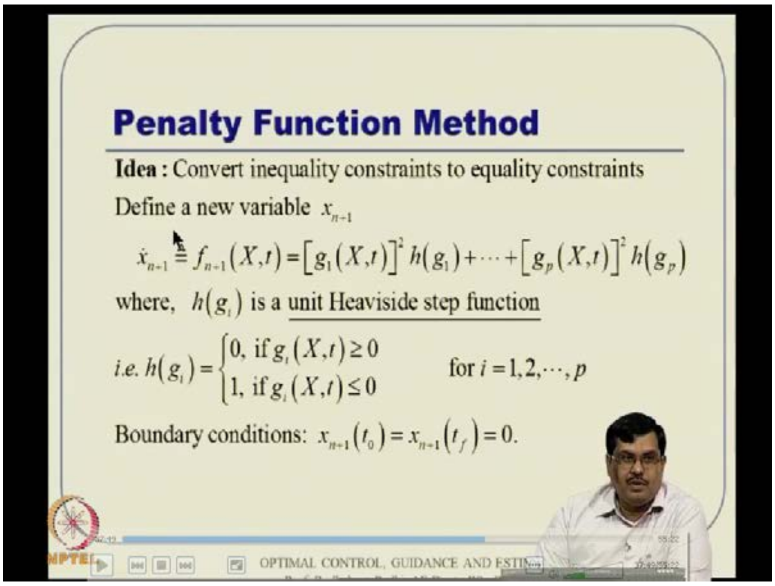

So, how do handle that? So, one idea is like this. So, all this constraint function I will collect whatever whatever constraint functions are there; and I will write some sort of a state equation, additional auxiliary state equation x n plus 1 dot as f n plus 1 (X, t), where this f n plus 1 I will define it this way; g 1 square times h of g 1 plus g 2 square g 2 square times h of g 2 like that. And what are this h actually? h h is nothing but, something called unit Heaviside step function defined as something like this (Refer Slide Time: 37:05).

In other words, if the constraint is satisfied then it is 0, if it is not satisfied then it is 1 then then is implies actually; it if it is if it is constraint is satisfied is not implies, if satisfy it implies actually. But, critical observation is remember that, this is either 0 or 1, so it cannot be negative each of the quantity. And each of the coefficients are also positive quantity; that means, x n plus 1 dot is always guaranteed to a positive quantity actually I mean greater than equal to 0 either it can be 0 or it can be it is guaranteed to be greater than 0 basically.

But, the interesting phenomenon is this equation also demands a boundary condition in fact optimal control means we want two boundary conditions really for each state equation. And these two boundary condition that we want to put purposefully is something like this, x n plus 1 at t 0 is 0, and x n plus 1 t f is also 0. Remember, it starts with some value and it has to end up with that same value only and both are 0's actually ok.

Now, it is possible only when all this only when this Heaviside functions is 0 actually; that means all state equation I mean all constraints are satisfied. If 1 is not satisfied then this is guaranteed to be positive and even if it is start with some 0 value, it is going to go away actually it is not going to come back to 0; because no point of time, the derivative is really negative actually that is the whole idea there. So, essentially what it does is? This formulation makes it some sort of infeasible problem, unless all constraints are satisfied that is the critical observation actually all right.

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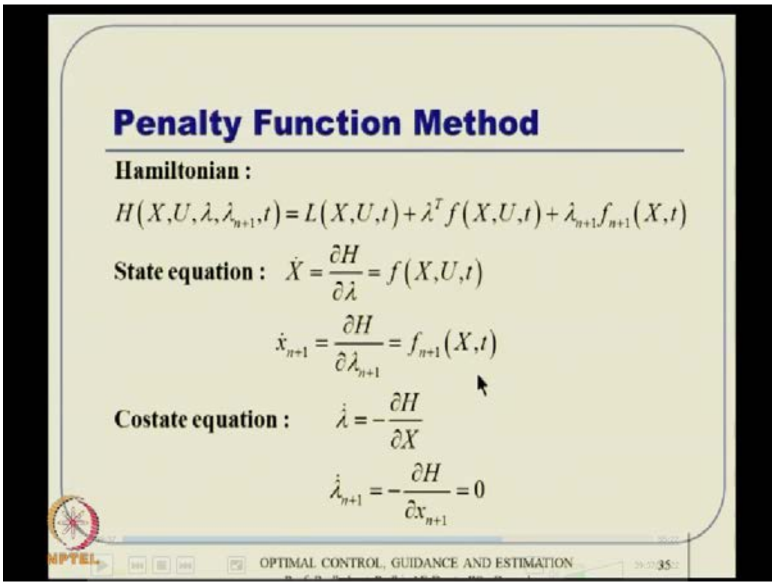

Now, now what is happening? We have an additional state equation additional two boundary condition that is all we have actually. So, we have this original problem, but in addition to that we are imposing this round this state I mean additional auxiliary state equation along with these two boundary condition, which pulls forces the system to satisfy the constraints actually ok.

So, now the rest of the things are plain algebra sort of thing, we have Hamiltonian which is the L plus lambda transpose f, but we have a one more equations. So, lambda n plus 1 times f n plus 1 actually right; this this terminal of lambda n plus 1 times f n plus 1 is coming in addition to what you are already have; and state equation obviously we have one more state equation and costate equation also we have one more costate equation. So, it will follow like that actually.

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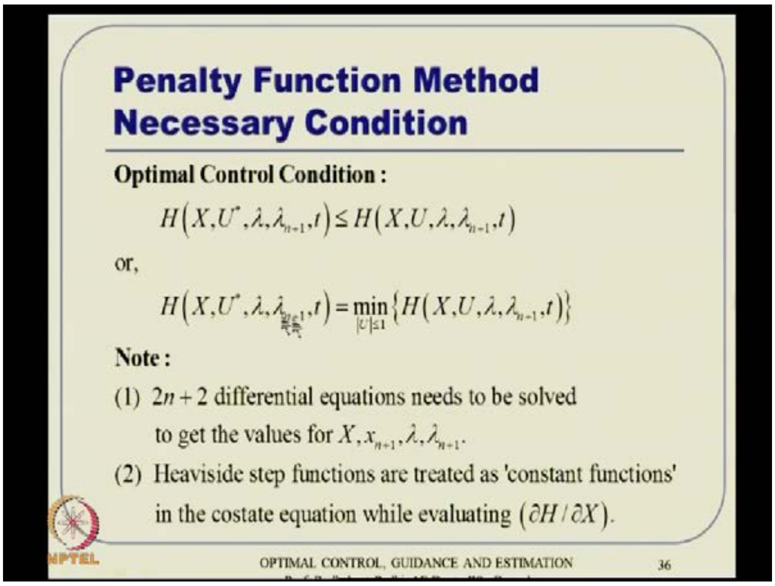

But, optimal control equation in general, when if you an account for control constraints also in the on the way then we have talk about this way; this Hamiltonian has to be less than that and you have to analyze there actually carefully. Even I am not talking take they are I mean these require numerical procedure to solve it also basically. And there are there are efficient ways of doing that as well. I am not going to layout any numerical procedure in this lecture I am just telling this ideas here and rest of the things you can you can study in some reference lecture also like thing actually.

So, whole idea here is you have this set of constraints and how to handle with that constraints? The constraints you can one way of doing that is you putting this things and then caring out rest of the things actually; in auxiliary state equation along with two boundary conditions, which is guaranteed to I mean if you if they are satisfy then the problem is guaranteed to satisfy this this constrained actually whatever solution you get. Anyway, so this is what it is, rest of the things is algebra all right.

So, observation is here we end up with 2 n not only 2 n, but 2 n plus 2 differential equations, one for X n dimensional state, and one more x n plus 1, one for lambda n n dimensional costate; and then the corresponding n plus 1, this this is a vector, this is scalar actually. And also remember when you do this algebra this del H by del X all that. Now, you should remember that, this these are also I mean this Hamiltonian contain this f n plus 1, and f n plus 1 contains this Heaviside function.

And Heaviside function is kind of discontinuous function actually. But, having said that it is discontinuous only at one one point that x is equal to 0 that too. So, we can ignore that, but irrespective I mean other that it is actually a constant, the value of the constant is different, but actually it is a constant value.

So, do not get confuse with that that the derivative has to be taken and all that. As far as, derivative are concerned del H by del X and all, this function what you see here this Heaviside functions can be treated as constants actually. So, that is that is how it is actually, that is what it is written here. Heaviside step functions are treated as constant functions in the costate equation while evaluating this this expression actually. And in fact, if you are interested curious to example you can see a typical small text book example in in the (( )) book actually the best that small example actually all right.

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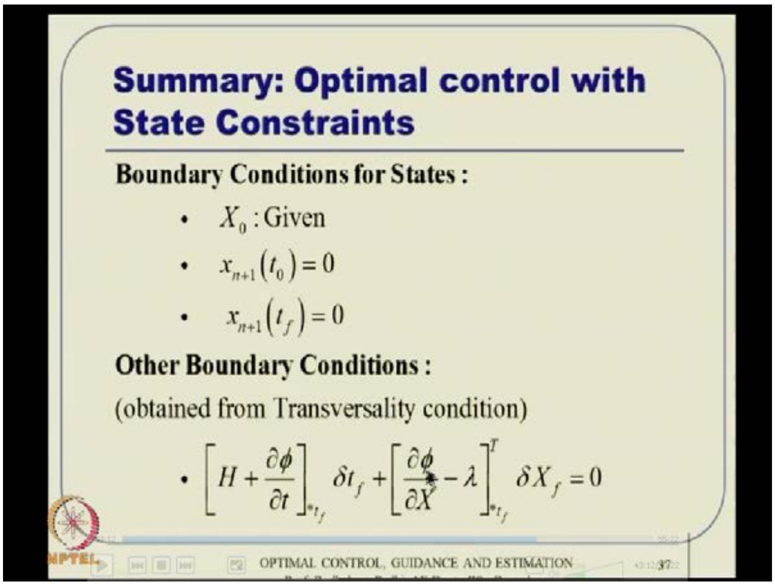

So, what are the boundary conditions that you are talking here, one is these boundary conditions which is already available; these two are impose these two boundary conditions also available. But, we still we are still not down, because we still need to have boundary conditions some lambda f, t f if it free like that actually. So, these boundary conditions can still be obtained from this general transversality conditions.

And suppose t f is fixed than then this is 0, so this is gone and X f is free. So, I mean if you you consider that way then it ultimately what happens is this has to be 0 that means lambda f is del phi by del X, which is known to us actually ok. So, like that actually. So, the boundary condition has to be 2 n plus 2, n is here, 2 are here and n more will come from here. And if t f is also free, one more will come depending on the situation Hamiltonian final times has to be equal to 0 it and depending on situation like that actually. We have talked about that in in early classes actually, when you talk to about that calculus of variations value. Anyway, so this is the summary of it.

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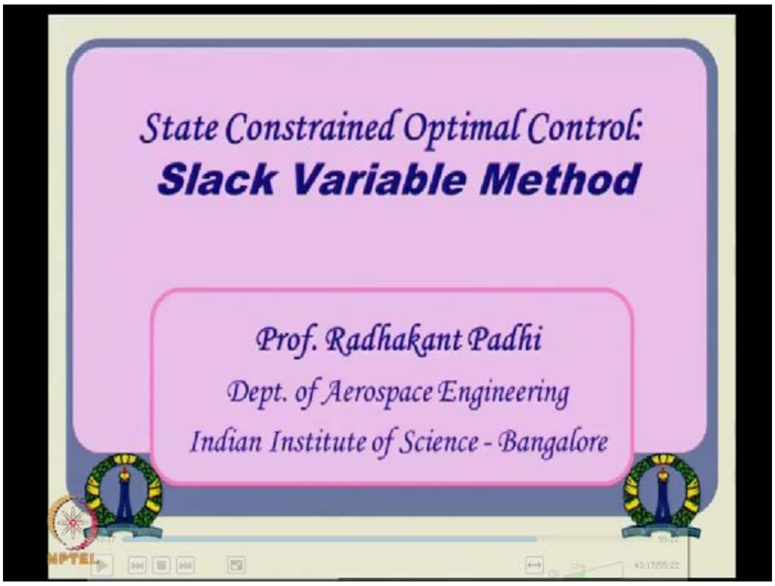

The next next idea other the last idea that I want to talk in this class is something called slack variable method. It is a different idea all together, which is a very elegant method actually.

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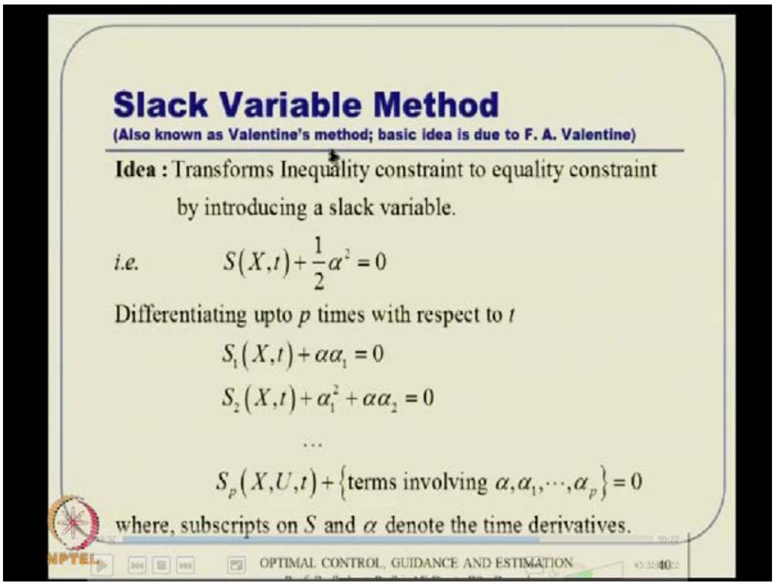

And in fact, it is its also called Valentine's method. So, it is because of this method I mean, because of if you Valentine, who first proposed this idea in some sort of static optimization sense actually.

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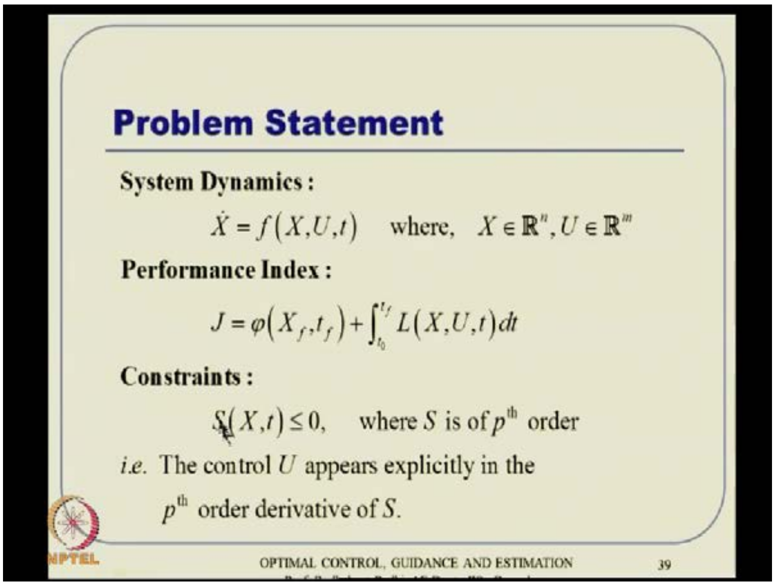

But anyway, so coming back this is the same problem. Now, for more generality we are having the penalty function also, the terminal penalty also here. And then we have this constraint in place and for simplicity we will take that as something like scalar constraints. So, that is the reason why S is taken actually, but in general it can be vector also, it does not (( )) actually.

The formulation is without loss of generality, you will end up with more number of state equations actually that is how it is. But, assume that S is, this is a scalar function which is one constraint in place, but S is actually p th order. That means what you what you mean by a function being p th order the definition is like this. If you take p times derivative p times tan derivatives then U will appears explicitly very close to what you know a dynamic inversion actually though if you know that.

So, if I take p th time derivative of this function then the control appears explicitly in the p th order derivative. And normally we have this either first or second order derivative with this control start appearing I mean most of the cases. We have this let us say height constraints in in a vehicle then S dot is v sin gamma and then if you have double dot of that then gamma dot appears and then control start appearing actually ok.

So, like that actually either that way actually, but in general it can be p th order actually. Now, how do you handle that? How do How do you analyze that actually? The idea is something very very elegant. So, we actually introduce some sort of a slack variable alpha and then impose this constrained rather in equality sense.

So, whatever what happen to be inequality here, we want to impose this this alpha actually, but remember this is also slack variable; the it is actually a slightly different in the sense of the this K K T condition, this is called Karush–Kuhn–Tucker conditions static optimization. That is the way I mean if you recall that, the at that point of time we had the slack variable, but the slack variable computation was not done and that was an mainly analyze mainly for analysis only basically. It led to some elegant results of course basically.

But, here we are interested in computing this alpha actually. So, that is the critical difference actually. So, we we have this inequality constrained, but we introducing this slack variable alpha we make it equality constrained first and depending on the problem, we want to really compute alpha basically.

Now, we differentiate this this expression start differentiate start differentiating this expression. So, obviously we have this S dot is written is S 1, S double dot is written as S

2 like that actually. So, then if you take first time derivative, because it is identically 0, all right hand side will be 0; and the first one will be S 1 plus alpha times alpha 1; alpha times alpha dot equal to 0. Then take second derivative it will happen to be like that, you continuous up to p th derivative actually ok.

Now, here terms will involve also alpha, alpha dot, alpha double dot or alpha 1, alpha 2 like that actually up to alpha p. So, it will contain alpha 0 th order derivative up to p th order derivative terms it will contain actually all right. So, now what actually.

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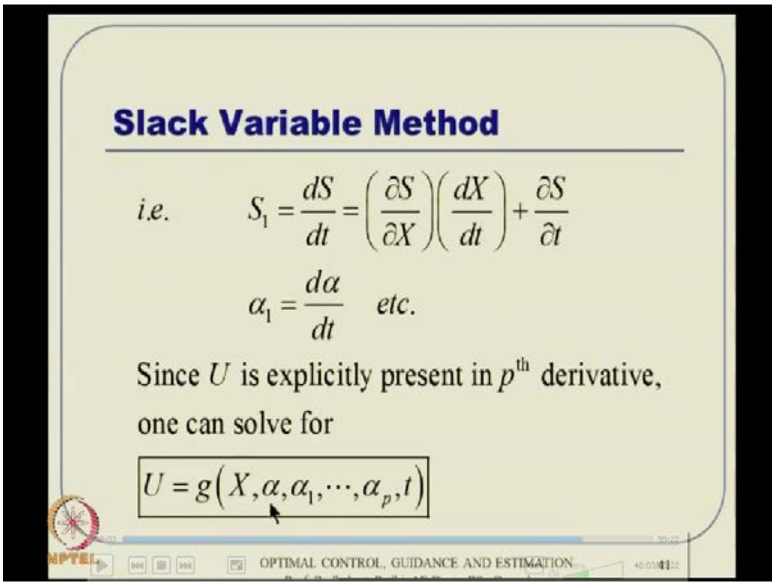

So, this is the definition of S 1, what you mean by S 1, and what you mean by alpha 1 like that actually. So, do not forget that d S by d t means we can compute this del S by del X, where S is a function of states. So, del S by del X into X dot and like that we can continue actually. Similarly, alpha 1 by definition is d alpha by d t actually. Any way now coming back, what happens is we are assuming that it this function is p th order; that means, U appears explicitly, if I take p th order derivative here.

So, from this p th order derivative constraint this this this equation I can actually solve for the control in terms of other variables. This this utilizing this expression I can actually solve this control in terms of other variables actually. And that that is where I symbolically you can write something like this U equal to some function of X, alpha and its derivatives ok.

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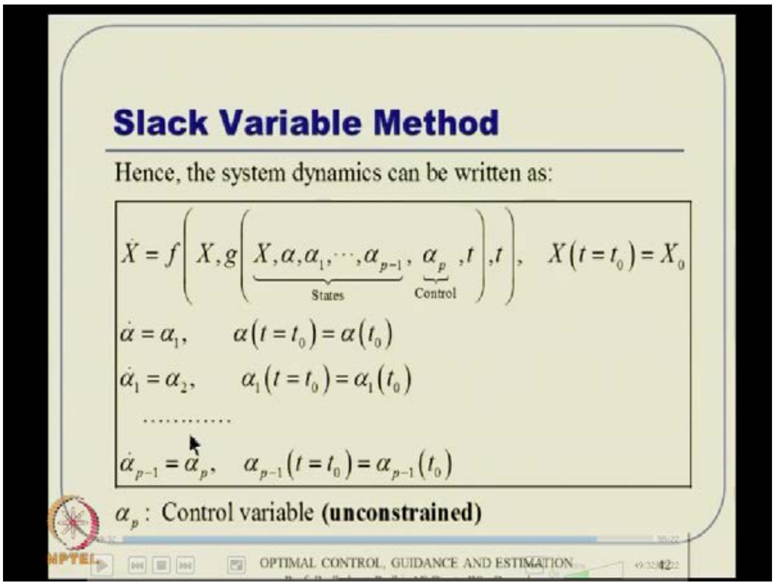

So, this is the critical observation really actually. Then what happens? I will go back and try to substitute my my control in terms of that, remember f X dot is f of X U and t of course. So, X of f of X U, U I can substitute in that that way, because this is U right; X dot is f of (X, U, t) basically. So, whatever U is there U is that, so I will substitute that is that. Then here I will I will treat this alpha p as control variable new control variable, all other things are states actually.

So, that means I will introduce one new states alpha, alpha 1 thing like that up to alpha p minus 1 actually. So, p more states are coming basically. And alpha p is treated as control variable. So, by first companion I mean in this first canonical form and all that if you if you see that, first companion form and things like that; this this one you can you can write it alpha dot equal to alpha 1, alpha 1 dot is alpha 2 all sort of the things definition. And alpha p plus would be alpha p minus 1 dot is nothing but, alpha p; and this alpha p happens to be control variable basically ok.

Now, what are the conditions? We also need initial conditions for that remember that actually. So, alpha t equal to t 0 is alpha t 0, these are definitions alpha 1 t 0 and think like that. So, we need values for that, then only the differential equation are complete actually.

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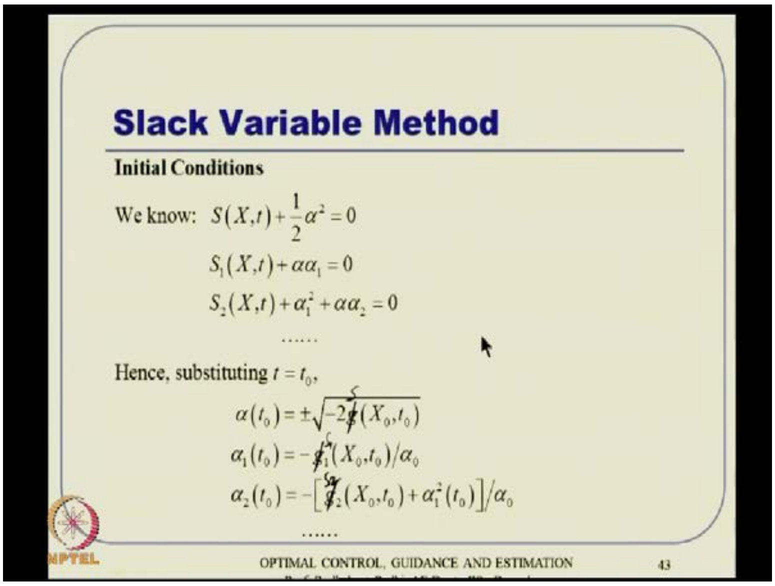

Now, how do you do that, this this is also not that bad, because we know this expression now right, we started with that this expression actually. So, if you if you open to I mean why cannot we solve alpha, alpha 1 like that actually; sequentially we can solve for that. And we utilizing this expression, we can first get alpha t naught like that, remember x naught is available to us that is the initial condition. So, it happens to be just a function of x naught only basically; x naught and the constraint form whatever the constraint form actually. I think this is here, I think there is a small print mistake here let me correct that, these are these is like S basically, this is S 1, this is S 2 like that actually ok (Refer Slide Time: 50:26).

So, from there you can see that actually. So, from the constraint equation expression and knowing the initial condition we can know the initial conditions for t naught this (( )) also. The small elevate here what, but typically I think I will try for minus first and then go for plus actually, because all other happens to be minus. So, to retain that same thing, but either way it it is ok, because as far as constraint is concerned here both the values will satisfy that that is not a problem ok all right.

So, that is that is how I will select actually all right. Now all right what you are telling here is we have this additional state equation and we have we know we want to know this initial conditions. And this initial conditions are available that is that is the critical observation actually. So, it makes the problem complete really basically.

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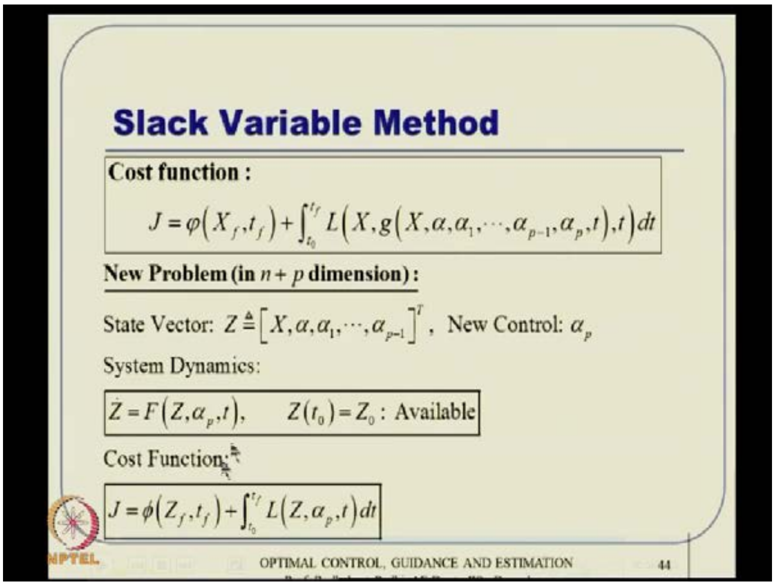

So, the cost function is matter of writing rewriting the cost function also. Instead of L instead of u, wherever u appears I will substitute that and then treat it as something like n plus p dimension problem actually ok. In other words, if I define a state vector as X and n all these p vectors alpha, alpha 1 up to alpha p minus 1. Then I can rewrite the system dynamics as Z dot is some some of the function F of Z alpha p time's t; alpha p, t where alpha p is my control variables.

And similarly the cost function can be written also like that. So, in the in this I mean in this expanded state vector the high dimensional state vector, the problem gets redefined actually, where everything else is known to one; and we know the differential equations that it is this are very simple differential equation anyway. And the associated initial condition are also available to us, which happens to be derive from the same state equation same kind of constraint equation function ok. So, it is elegant actually in that sense.

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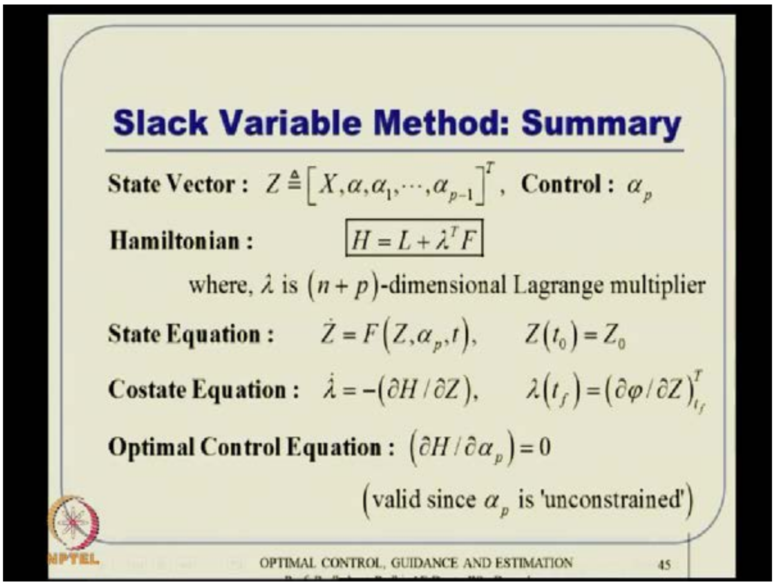

So, from there onwards it is algebra then some reasons we can we can always talk like we redefined the state vector as something like this, loser dimensional state vector, control we treat that it as alpha p; and then rest of the things we know how to do, define a Hamiltonian that way. And then the state equation happens to be this one in in Z. And lambda dot happens to be like that and hence it is thing like that. Remember part of this del phi by del Z will happen to be simply 0 actually, phi is a function of X not alphas. So, when this alpha part comes there that part will become 0 actually. The optimal control equation happens to be that way ok.

So and remember, we have actually what is the trick here by doing all these, the constraints are accounted for; but absolutely I mean this alpha p when it comes to that, there is no constraint on that, it is actually unconstraint control problem. Because, this right I mean when when you come back come to here, there is no constraint equations alpha p is still free basically that way. So, because alpha p is free, we can go it straight away use that actually del H by del alpha p equal to 0.

So, this I thing in my ways kind of elegant methods, slack variable methods and then if you end will up this some problem like that, we can give a try in this method actually. Because, it is not Heaviside, the other approach with using this Heaviside function and all is is mathematically some extends, but you may struggle numerically actually. It may not happen, it then you may have to struggle hard to tune the design in a good way.

But, I think the slack variable approach is a is kind of natural way of handling the problem basically; you have this inequality constraint and you make it equality by introducing this this function here right there. And then rest of the things is algebra and it elegantly turns out that you can not only derive the state equations, but you can also get thus is stated in initial equation for that actually all right. So, this is all about what I wanted to talk in this constraint optimal control series.

And certainly I will encourage you to to read many things including thorough reading of this this chapter 7 of of this made up of Naidu book actually. There are several elegant I mean this kind of review papers are also available, which you can see many things. But, after knowing some of these things, things may be easier for you know actually. So, I think with that I will I will stop this lecture, thank you.