Optimal Control, Guidance and Estimation

Prof. Radhakant Padhi
Department of Aerospace Engineering
Indian Institute of Science, Bangalore

Lecture No. # 35

Constrained Optimal Control – II

Let us, continue our lectures on optimal control guidance estimation course, and we last class, we have started this Constrained Optimal Control. And derived some basic philosophy, so basic principles about this constrained optimal control have largely inspired from pontryagin of minimum principle.

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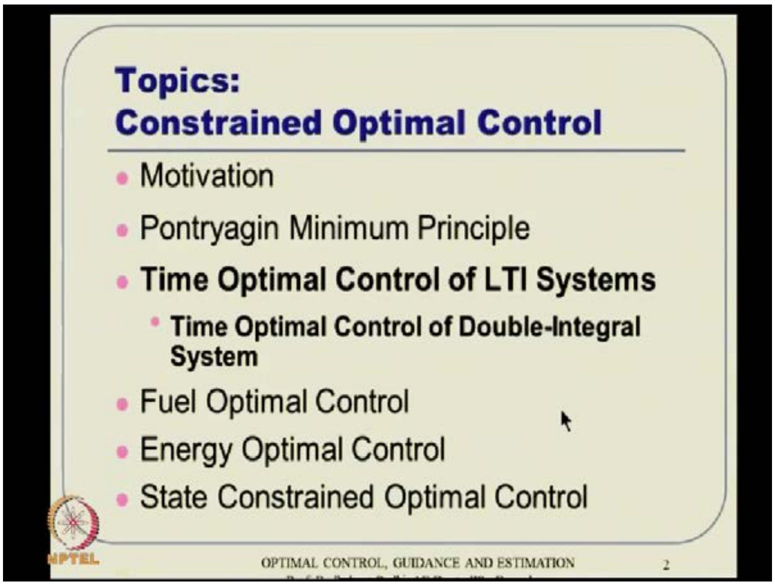

So, let us continue our discussion on those lines, and this particular lecture, I will talk about little more little more at on time optimal control rather actually. Anyway the topics on constrained optimal control, that we are talking about is first little bit motivation; I have given that in the last class. Then pontryagin principle, we have derived that in the last class as well, and then this particular lecture, I will talk in detail about time optimal control of linear time invariance system.

Especially, we will towards end of the lecture, we will take time optimal control of double-integral system, which is very standard kind of bench mark problem. And then, we will demonstrate how do you kind of get it get the control for I mean control for this double-integral system, in a closed form sense actually, and that too in a state feedback sense basically.

So, that is we will see that, then the rest of the topics will contain little bit on fuel optimal control possibly an energy optimal control, if time permits. And then a state constrained optimal control little bit hints of that actually, and more on that we can reading several text books as well actually.

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Now, let to start with, let us go through just one slide about summary of pontryagin minimum principle; that we discussed in the last class.

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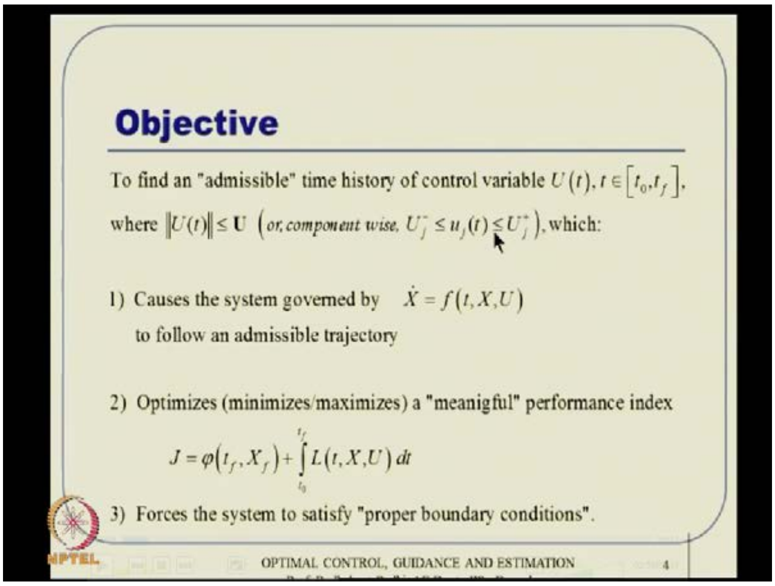

So, the objective something like this to find an admissible time history of the control variable, from t naught to t f such that, all of there objective that have been studying about throughout the course is met, but the only condition is the control is constraint actually. Either in a total norm sense or component wise, which is constraint between plus, I mean plus certain value and minus certain value actually, so it is constraint between the two allowable limits actually.

All other things as it is, the control history subject to this control bound should cause the system govern by this non-linear system dynamics to follow an admissible trajectory, should also optimize certain meaningful performance index. And also force the system to satisfy appropriate boundary condition other thing is there, but only thing is this control is, now constrained actually.

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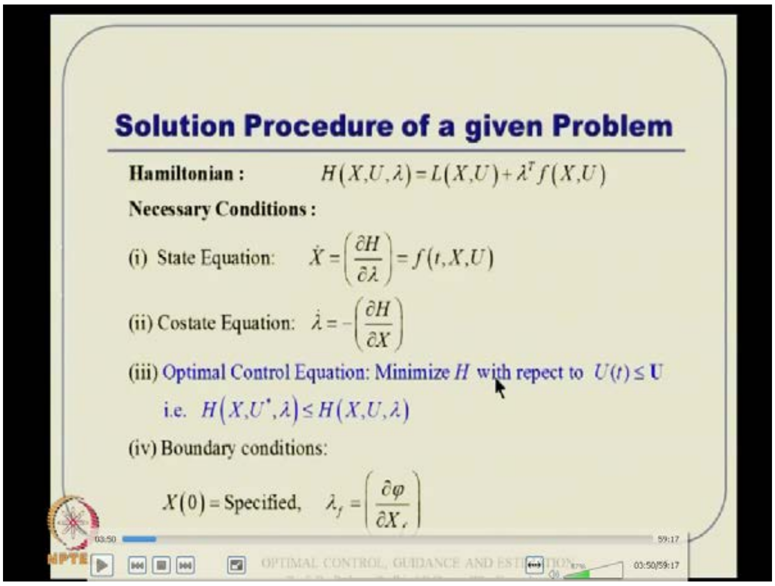

So, what was the outcome of the analysis last class is something like this, you can still go ahead, and define Hamiltonian the regular way, and still most of the conditions are satisfied, I mean state equation is satisfied, costate equation is satisfied, as well as boundary condition is also satisfied. The only difference is the about the optimal control equation, we can write del H by del x equal to 0, and try to solve the control.

And I mean instead of that, we have to really analyze this inequality, Hamiltonian of X U star lambda is less than equal to Hamiltonian of X U lambda, so this inequality should give us some U star actually. So, how do get it, and thing like that we will discussed in this particular lecture, about LTI system, and all that alright.

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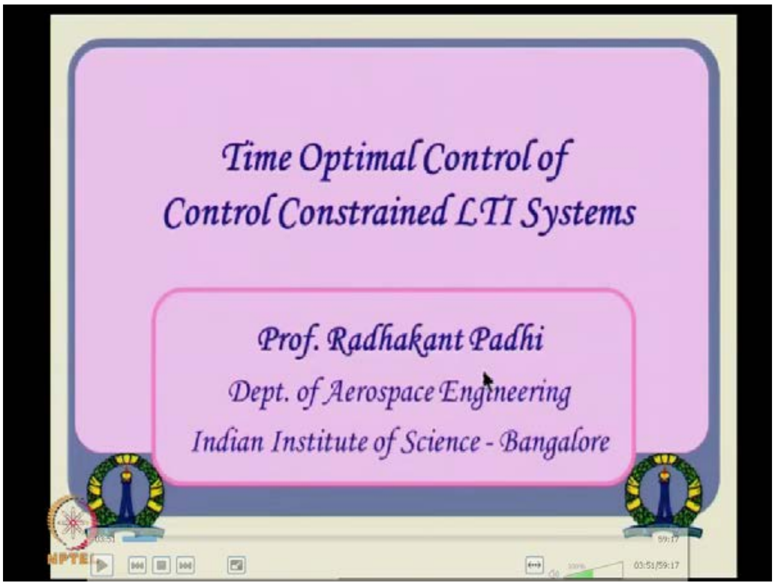

So, let us go ahead and start the topic, and rest of the course largely about control constraint problems, so will be interested in LTI systems only that is linear time invariance systems, where things are easier to see, easier to analyze, and visualize, also basically. And interestingly it will also turn out, that even through the system is linear, the control structure will be eventually non-linear actually, so (()) towards the end of this lecture will see actually alright.

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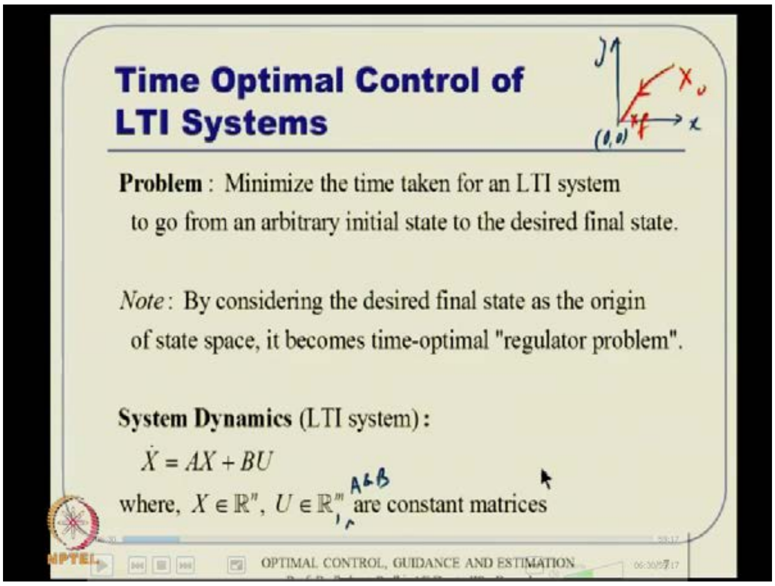

So, the problem is something like this means, I mean what you are interested in is to minimize the time taken for an LTI system, to go from an arbitrary initial state to the desired final state. So, we start with certain point in states space, then you want to go somewhere, and then let us pictorially saw that, we start in certain initial condition X naught, and we want to set go certain point X f, and that should happen in minimum time actually.

So, what we what we next we visualize is something like will put the, then will put an X i system, let us say we will put an X i system; let us say like this at the desired final stage. So, then it turns out that the desired system, I mean desired final point is nothing but, the origin actually, so that is (0, 0). So, that is without loss of generality, we can do that so, the analysis becomes easier actually.

So, essentially if you if you drive certain initial condition, and all possible initial condition in the states rather, with other towards the origin, then that is that is the problem, which is called as regulator problem; we are we are interested in regulating the state about (0, 0) actually. So, essentially it becomes time optimal regulator problem, we want to drive any initial condition towards the final state which is nothing but, the origin in the state space actually. Because, we are talking about system dynamics been linear or other LTI system dynamics takes in this form X dot is AX plus BU where X is n dimensional vector, and U is in m dimensional vector and this, and A, B are actually constant matrices, and we are constant matrices actually.

So that means, we satisfies the linear structure as well as time invariant thing, because A, and B, are constant matrices actually alright. The key point here is the control magnitude satisfies this constrained or rather actually; we can think about putting it in a norm sense and all that something like norm of few is less than equal to certain capital U, and all that actually or more structure wise more easier to visualize is something like this, component wise, if I can take U 1, U 2, U 3, like that component wise, they are actually bounded between certain values, and without loss of generality, again we are taking the symmetric values sort of things.

So, that one condition you can describe actually, u j is less than equal to U j sort of thing, so this u minus is a lower bound or u j means is the lower bound of this small u j, and U j plus will be the upper bound of this small u j basically.

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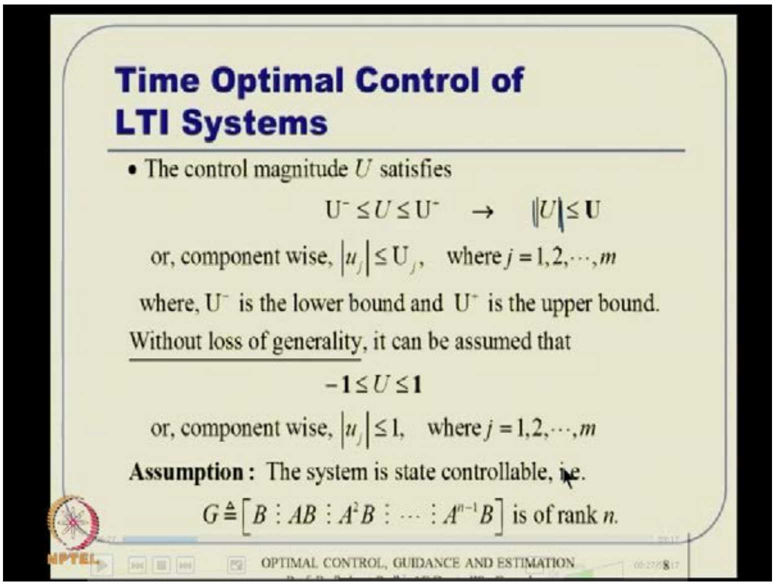

So, again without loss of generality, we can we can assume that this bounds nothing but, minus 1, and plus 1 and we can do that by observing the magnitude into the B matrix actually, so if you if you normalize this U component wise, and whatever this normalize invariable; we can observe that in the B matrix. And then, whatever the normalized variable will contain in the control components and hence, we can write the control is constrained between minus 1 vector, and plus 1 vector or rather component wise; this is constrained u j magnitude is constrained to be less that equal to 1 alright.

So, the assumption here is, the system is state controllable, and it will turn out later; that is turns out to be kind of a necessary condition for something called normal time optimal control problems. If you if you does not satisfy, it leaves it leaves to well abnormal situation that is something like singular control, and all that actually, we will we will see that is, we go along actually.

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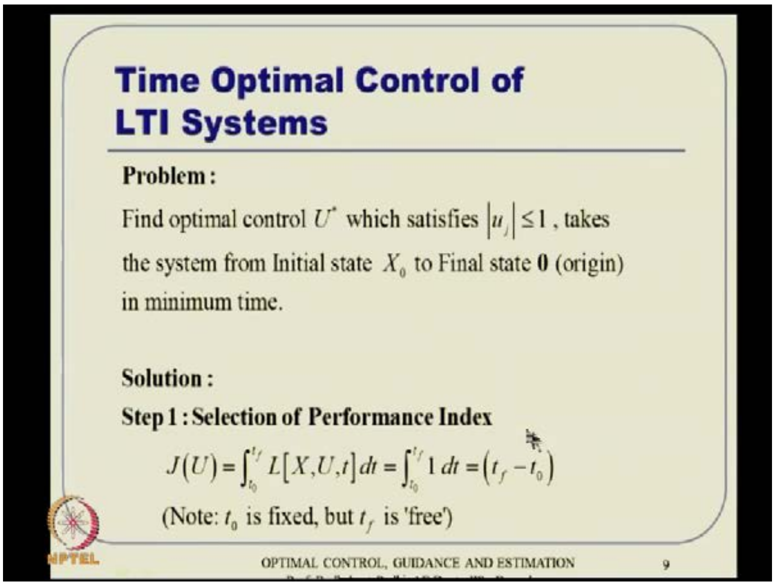

The problem here is again to restate formally to find in an optimal control U star, which satisfies these components wise constraint u j mod less than equal to 1, and which takes the system from any initial state X naught to the final state 0, and that is origin in minimum time. So, that that goes on to that kind of a statement, and subject to the still the system dynamics being this one actually (Refer Slide Time: 08:59). So, step 1, we will start the solution processes first thing, and first we want to put our objective in the in mathematical form.

So, what you are interested here is time optimization, and in other words minimum time problem, so going back to the control structure in the very first, this objective function is to be defined (Refer Slide Time: 09:23).

So, here you can take phi equal to 0, and L equal to 1, and then we will end up with this kind of a cross function t 0 to t f is nothing but, t f minus t naught, so this is the quantity; that means, to be minimized. And also remember, that t naught is fixed, but t f needs to be free here the t f is also fixed there nothing to be minimized really actually, because that is a constant number t f minus t naught also becomes a constant number; we can talk about minimizing their quantity and all that. So, by default t f happens to be a free variable here actually.

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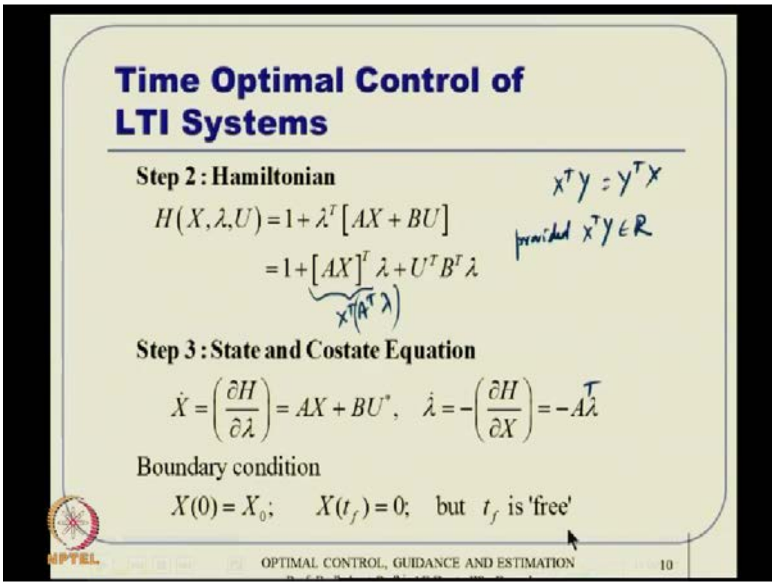

Now step 2, we have to start the solution procedure, because now we have a cost function l is 1, and phi is 0, and we have a state equation X dot is [A X plus B U] and the we have are ready to write the Hamiltonian actually, so H of X lambda U is something like 1 plus 1 is l plus lambda transpose f and f is [A X plus B U[. So, that turns out to be 1 plus, if I expand this quantity, and also observe that, I can take this transpose, I mean I can alter the transpose, because we know that a linear algebra something like X transpose Y is equal to Y transpose X provided this resulting, X transpose Y happens to be a scalar quantity.

That means, this is a multiplication of two vector happens to be a scalar, so I can do that in a reverse sense also, I mean in other wards it is x 1 y 1 plus x 2 y 2 plus x 3 y 3 like that, and this is nothing but, y 1 x 1 plus y 2 x 2 like that, so both are similar quantity. So, that operation has been carried out here, so we can we can write it something like A X transpose times lambda also, and B U transpose times lambda also, and B U whole transpose is nothing but, U transpose B transpose, you know that actually, so is written that way.

Then the state, and costate equations we can now write something like this, also let me, kind of put a comment here that, if you see the text book and largely am following this material from Naidu book everywhere he will find stars notation, I mean the X star lambda star and think like that, I thought I will simplify that by keeping only U star notation for optimal control and X and lambda, I will not put stars actually. So, but that still means, that we are talking about optimal control in other words, optimal state trajectory, and optimal costate trajectory as well actually. Anyway so this state, and costate equation can be derived very quickly as like this X dot is del X by del lambda, which is nothing but, the same state equation with optimal control function of course, and then lambda dot is minus (del H by del X), and then H being like this. We can compute (del H by del X) is nothing but, coming from here lambda dot is del H del X. So, X transpose A transpose lambda sort of thing is, this is B this quantity.

Let me, show that this quantity is nothing but, X transpose A transpose lambda, and if I take a derivative with that, with respect to X I think this only, this part will remain actually, so this is a transpose (()) probably, so this is how it is actually. So, X dot is A X plus B U and lambda dot is minus A transpose lambda, so the boundary conditions still remain same X of 0 is X naught, and X of t f is 0, because that is our final objective, but t f is free, remember that actually.

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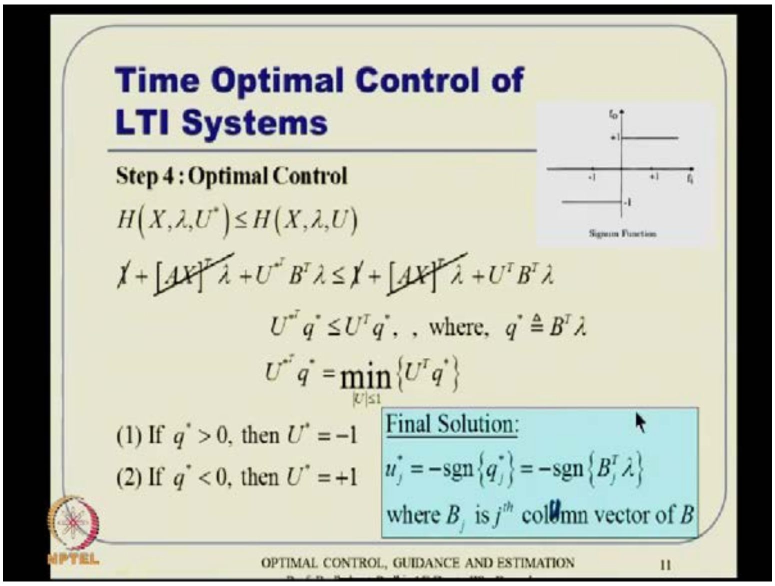

Now, step 4 is the critical thing we want to see, what kind of optimal control it that leads to actually, and remember we have to find the control U star, which is optimal in such a way that this Hamiltonian, what you are seeing here with respect to U star is less than equal to any other Hamiltonian with respect to a non-optimal U really. So, that means, if I substitute this this Hamiltonian expression here, and try to kind of see what is going on then y naught, but I observed here is 1 and 1 will cancel out from both sides, and this quantity also remains similar, so same rather that will also get cancelled out. So, we land up with some expression like this, U star transpose times q star is less than equal to U transpose times q star, where q star is defined something like this. And look for simplicity we just define this B transpose lambda is to be q star, so we can write it that way.

Now, you have to find the U say I mean, so what does it tell us I mean this inequality tells us, that this q star I mean this U star transpose q star is nothing but, minimization of this U star q star with respect with this constraints influence actually. So, now the question is what kind of minimum value, it can really take this U really now, q star happens to be you can think of it is something like a number basically. So, q star is already known to us, now we have to find this selective U such that, this quantity will become minimum.

So obviously, we no need to see two conditions here, one case is, if q star happens to be positive then obviously, this is positive, so I can I can drag it down by putting a minus sign here minus 1, so that is the best, I can do as long as q star is positive. And if q star is negative, I can do in the reverse, and I can still, we have brought in the minimum by putting U star U equal to plus 1 actually.

So, all that it based on is as long as q star is positive I simply apply minus 1, and if q star happens to be negative; then suddenly, I will have to apply U star is plus 1 actually. So, this can be written something like this final solution u j star component y again, the signum of q j star with a negative sign, this happens to be positive, then I will apply negative, if it happens to be negative, I will apply positive actually.

So, if a very compact sense, you can write u j star is negative of signum of this q j star q j star by definition is B transpose lambda is nothing but, b j transpose lambda b j happens to be, I mean the corresponding column vector, actually a small spelling mistake actually. So, that is that is what it actually, now signum function, I think is very clear for that you do not know really this is, this is how it is pictorially represented as long as some quantity is less than 0. Then the output is minus 1 as long as positive, the output is plus 1, so if I just it is a negative sign, then this objective is met actually.

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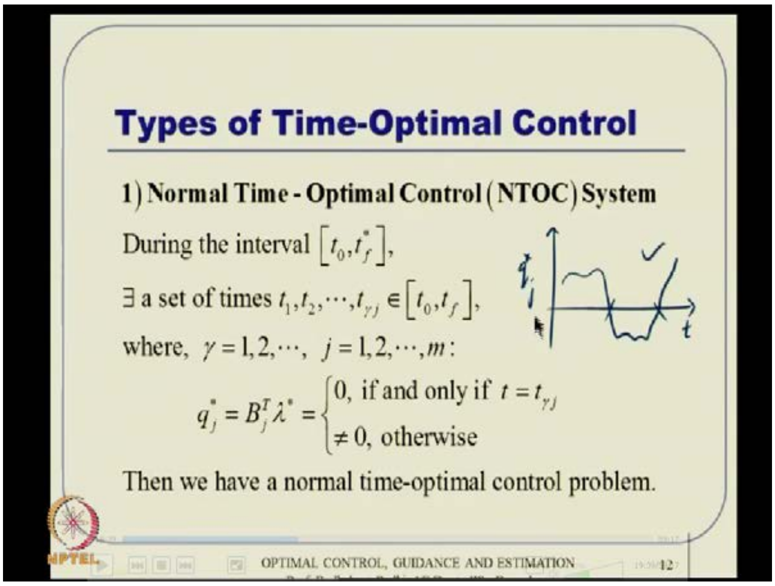

Now, here is case now, this will result in a situation, where it is the question is, whether this happens to be either strictly greater than 0 or strictly less than 0 for all the time, and it turns out, that it may not be the case, and if it can actually it switch sign from 0, I mean positive to negative and think like that. Now, if that happens, and does it switch times, I mean switch from positive to negative side in finite number of cases or does it happen like for a segment it remains 0, and all that actually so that also becomes (()).

So, what So, what we are asking here is something like that, the normal time optimal control system the condition happens to be like this, that during this interval something like [t naught to t f] or t f star rather, because we have to find an optimum final time also. So, within this within this time zone, there exists a set of times t 1 t 2 up to t gamma j, where gamma varies with 1, 2, 3 whatever number of situation, all that; where j varies from 1 to m that means, j stands for the components of the control and gamma varies, gamma stands for like number of cosines actually.

So, ultimately it will happen to across something like t 1 t 2 up to t gamma t times basically at the maximum, so there exists a set of times this these times actually t 1, t 2, excreta, where this q j star satisfies this condition that means, it is 0, if and only if t is exactly equal to this gamma, I mean this gamma j quantity sort of thing actually. If and only if it is continuous I mean, if it is if and only if t equal to this discrete point; then only it is 0 otherwise, not 0 actually, so that means, pictorially speaking you have this time axis. And we are plotting this quantity q star j, we are all asking is, if it is positive is positive what about it changes, it will change one time in a discrete point only basically. So, this is allowed, but if I talk about something like this one, and it remains 0, and then it goes positive, and remains 0 comes down. These kinds of things, this finite segment it becomes 0, and those kinds of things are not allowed actually.

If it, if this kind of things are not allowed, but this is this is there, then that is something called normal time optimal control problem actually (No audio from 19:23 to 19:32), otherwise it will be a kind of, it will be called a single optimal control. So, normal time optimal controls have some little more elegant solution, and think like that (()) that actually.

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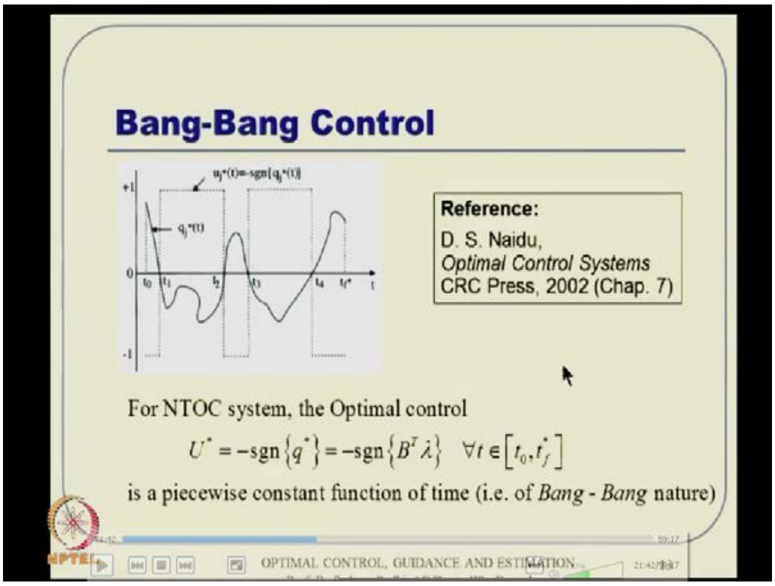

So, this is pictorially written, I mean given also in the book, we can see that there as long as q q j c a flip over can change sign, but it is to change time at discrete point of times only. And then, as per our result says as long as q star is positive, then U star is minus 1, and q star is negative, then U star is to be plus 1. So, this is, what it tells us q star happens to be positive here, so U star happens to be minus 1, when q j star is negative here. So, u j star is positive 1 and then again, this is positive here q j star, so u j as to be minus 1, and think like that.

So, now can very quickly see also that this either minus 1, then suddenly goes to plus 1 then states there, and again comes to minus 1, and states were and instantaneously changes to close one states that per like that actually. So, this I mean intuitively, and also logically, it is called bang control, it actually just goes to minus 1 the maximum bound, then bangs to the plus 1, then other side then goes to, and then again bangs to other side like that, actually is called bang control actually.

So, this normal time optimal control system, the optimal control is finally, given by this expression and essentially, it is a piece wise constant function of time; I mean this is constant for a piece and then, again it is a constant for a piece, and think like that. And you can clearly see that the solution nature happens to be non-linear actually, as does not does not stay linear even with even, if you plot with respect to X actually, will see that little later, it does not remain linear actually, this is what is plotted with respect to time, but even if you plotted with respect to state it does not remain linearly, so this is this is bang-bang control actually.

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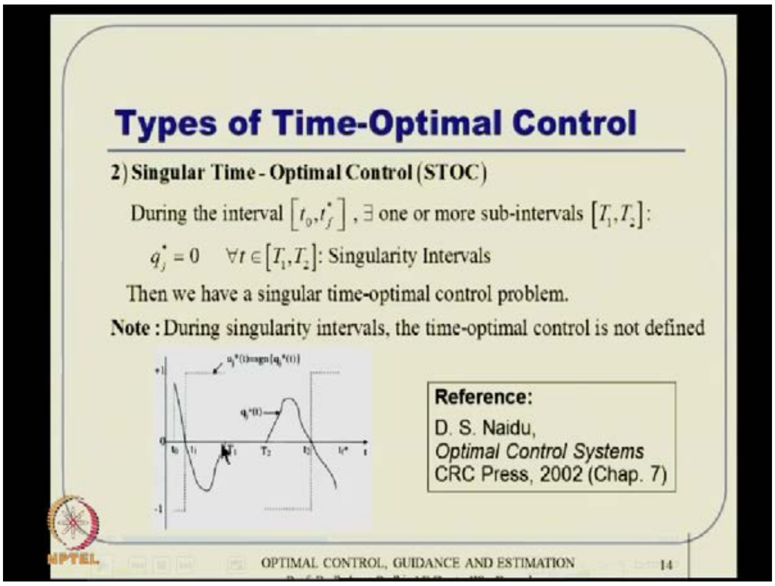

Now, that does not happen, so that now onwards q j not only remains not only remains positive or changes sign it finite point out time, this changes I mean signal at discrete point here and discrete point here, but it also remains 0 for finite interval of time T 1 to T 2, and then that is called singular singularity interval and the resulting control is called singular control sort of things actually. In other words, the problems is called singular time optimal control were in this segment T 1 to T 1 to T 2 control is really not defined and while implementing, you can we can put it 0 and think like that, because it is not defined. So, I will continue to apply 0. But, does not the (()) the mathematically is speaking within T 1 to T 2 the control is really not defined, so that is why I mention here, leading singularity intervals that time optimal control is not really defined. So, in that case it is called singular time optimal control, as not very elegant, because of this undefined nature of the control structure and all that.

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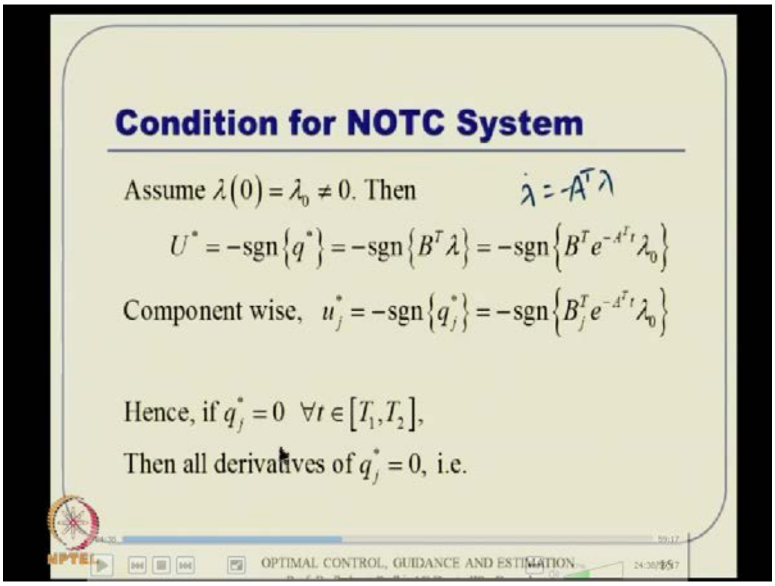

Now, the condition is suppose it happens, like that then very natural question is when does it happen, so we like to know that actually. So, for that we need to find a solution of lambda and all that, so let us start doing that actually now, we assume that lambda 0 initial condition of lambda is naught, I mean does not 0, that is that some value lambda naught, and then U star is negative signum of q star, but q star is B stars plus lambda, but lambda is time varying lambda of t really.

But, lambda dot, remember lambda dot will derive sometimes back here is minus A plus transpose lambda, so that is, what is written here minus signum B transpose this B transpose coming from here, and lambda is nothing but, e to the power minus A transpose t times lambda naught (Refer Slide Time: 23:40). Let us, that is because, lambda dot is A transpose lambda minus A transpose lambda; the solution of lambda happens to be e to the power minus A transpose t times lambda naught from linear system theory, it is very clear.

Now, component wise, if I try to analyze that; then all that, I try doing is instead of B transpose, I will put B j transpose really, that is will to my u j star actually. So, if q j star happens to be equal to 0 throughout this interval, identically equal to 0 throughout this interval then obviously, one observation is and this interval all its derivative are also 0. It is not only the function is 0, but we call its 0 and constant 0, throughout out this interval, then all its derivative needs to be 0 as well actually.

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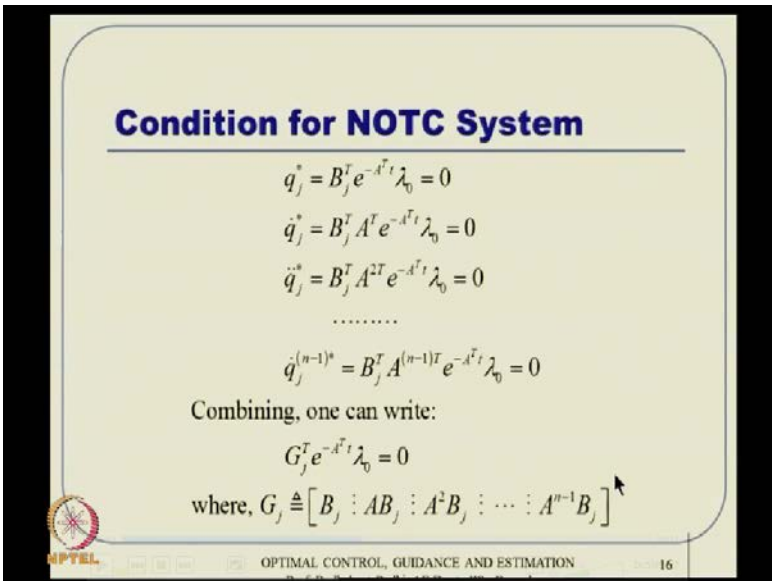

That means, q j star is 0 q j star dot now, that also can be computed from this expression now, with a transpose will come here that will become 0, but all these negative sign will keep changing in all that actually. So, that is ignored here, but identical, if you take derivative that, then negative sign will come here, and this will be positive again the other one, second I mean, third derivative will be negative like that actually.

So, then does not matter the question here is, if negative sign is 0, then positive sign is also 0 actually, so that can be able to read also basically, anyway so coming back q j star is 0 q j star dot is 0 q j star double dot is 0 you keep on doing that. And then in the process this resulting matrix will turn out to be all 0 actually. So, for vectorially this is will, these take positive quality basically actually.

However, otherwise this can be forgotten actually, after q n minus 1th derivative, if you take that happens to be like that, and hence combining all the results, we can we can really right that g j transpose is e transpose, e to the power minus A transpose t lambda 0.

Now, g j represent to be these matrix us, if you if you see this carefully, this represent to be g j transpose is collection of all the transposition all that. So, g j will transpose on everything then it will (()), so this is will become A times B this will become A square time b j like that actually, so all these are collected here (()) and g j happens to be B j AB j like that actually there.

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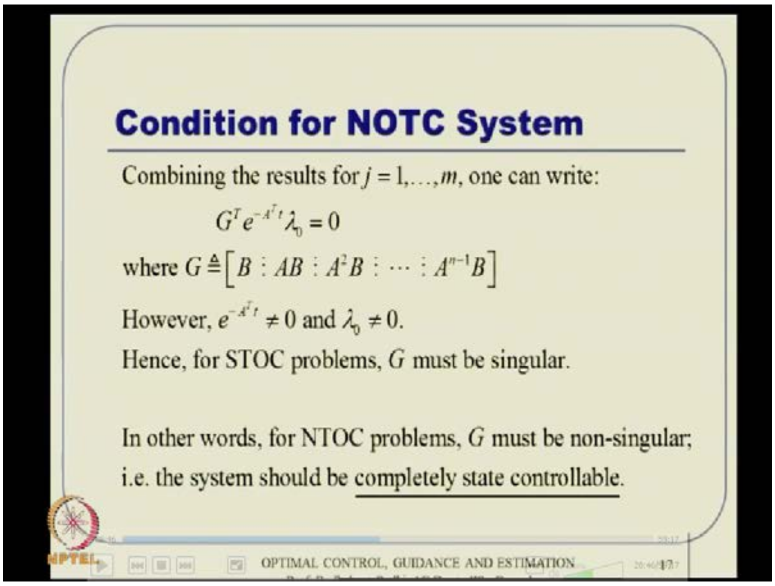

Now, if you do that repeat that exercise for equal to one to remain all that, so if will turn out to be this expression, all these B j will can we combined, and then analyzed appropriately. And then, put it together then turns out to be the G transpose e transpose e to the power minus lambda A minus A transpose t times lambda not equal to 0.

Now, the thing that is, that can be observe is any exponential function of time is never 0, it happens to be 0 only at equal to minus infinity provided A transpose positive or plus infinity A transpose is negative; but that does not I mean sorry here e to power minus A transpose t never 0 that is more important actually for all finite time intervals on all that. And by nature lambda naught is not 0 also, we started doing that and then the only possibility of having; I mean this only possibility that this solution the equation will have solution is when this matrix is actually is singular.

So, what you are telling here, this situation this situation arises only one these matrices become a singular actually. And we can very clearly see that is this is the nothing but, the controllability matrix actually. In other words, for normal time optimal control problems, these must be non singular; that means, the system should be completely state controllable. So, if the system happens to be straight controllable and then, this condition this G, I mean this singular optimal control condition is not satisfied, and hence will end up with normal time optimal control problems.

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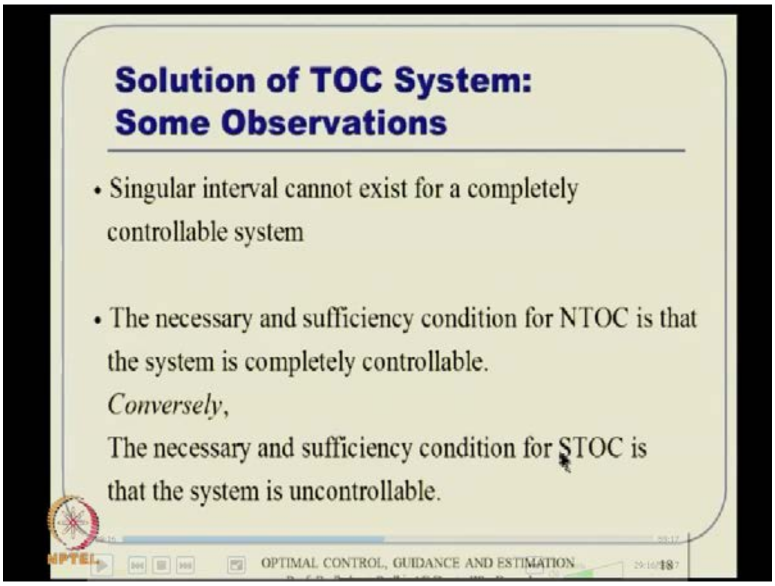

So, let us assume that our subsequent system are actually controllable, and then carry out for their thing actually before doing that, some observation here, a singular intervals cannot exist for a complete controllable system, because, this condition will not be satisfied actually. The second one is the necessary, and sufficient conditions for normal time optimal control is that, the system is completely state controllable or conversely you can also till like this, the necessary, and sufficiency condition for the singular time optimal control is at the system is on controllable actually.

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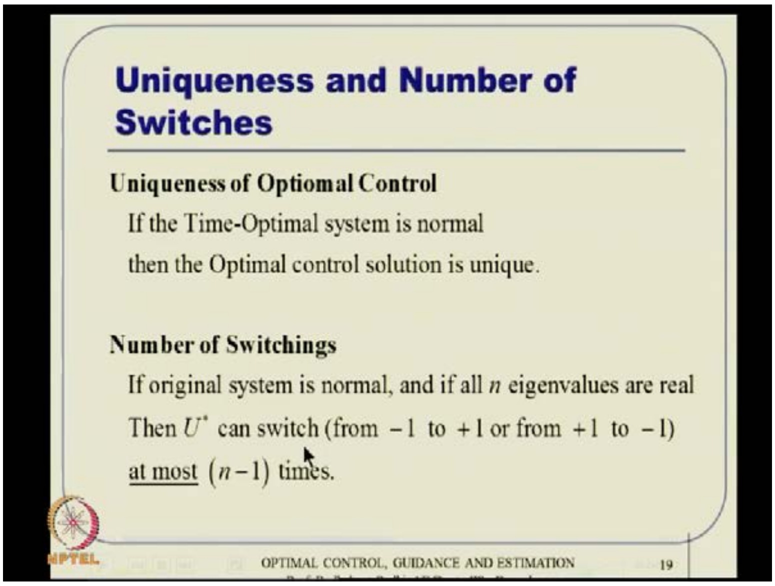

Now, they also interesting theorems, which tell us some uniqueness nature of the problem and tell us that, when it when is the solution unique, the next question is that actually, so this turns out that, if the time optimal control system is normal; then the optimal control solution is also unique, it cannot a multiple solution, and all that. And in that situation these also an interesting observation in the in the form of a theorem, which tell us the normally maximum number of switching's (()).

And it turns the theorem tells us the, if the original system is normal that means, state controllable, and if all n eigen values are real, then U star can switch at most n minus 1 times actually. That means, all switching from minus 1 to plus 1 and from plus 1 to minus 1 counted, if you count all that, then maximum number of switches can be at the most n minus 1 actually.

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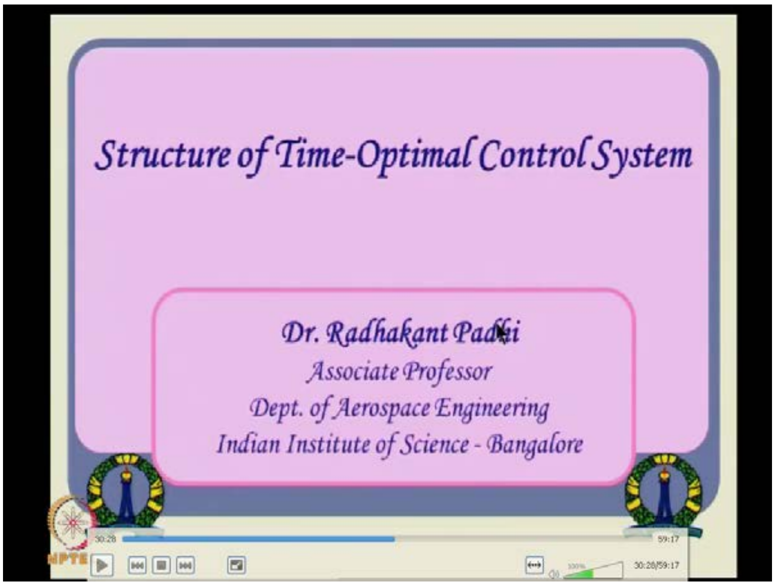

Let us see, that I mean these are the observations, which will help us a little bit also basically, the structure of the time optimal control now, we got all these observations, and all we can derive certain structural form of that control solution really.

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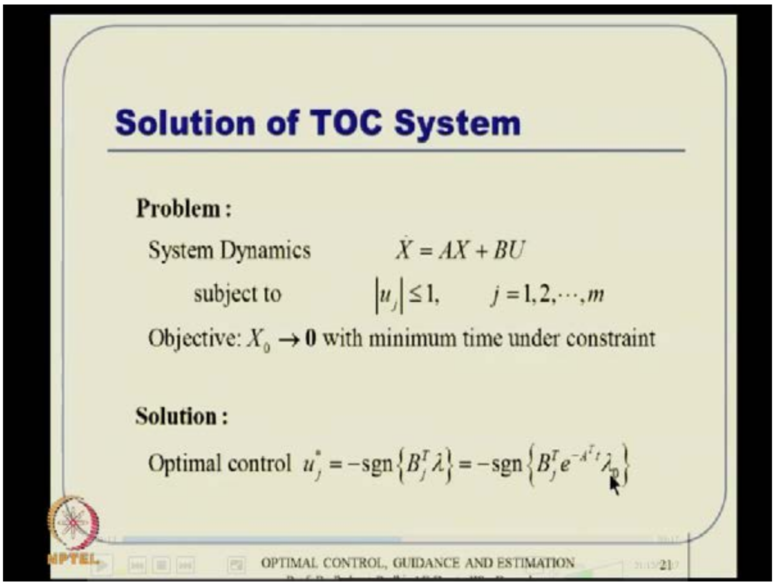

So, this the problem is again just to summarize, we have a state equation something like this, linear time invariant system subject to this control constrained objective is X naught should be driven towards 0 with minimum time under these constraints actually. And the solutions, that we derived is something like this, u j star is negative signum of u j transpose times lambda or something like this, lambda solution is this one.

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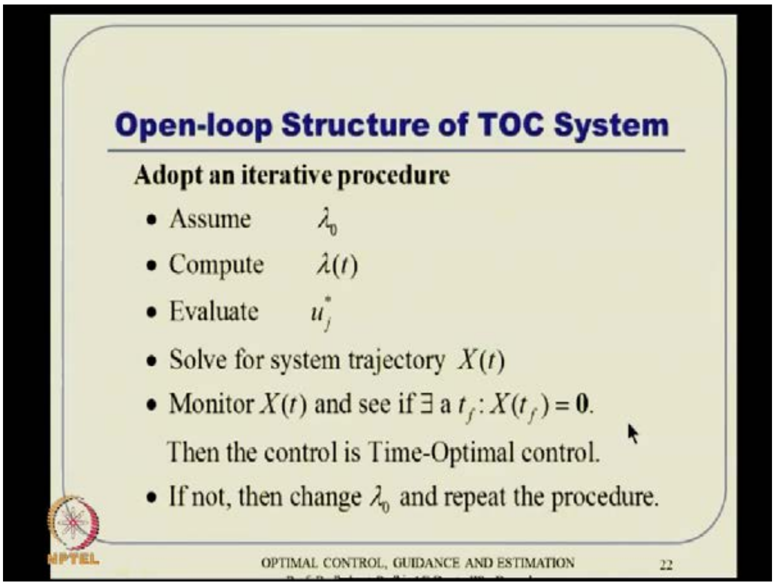

So, then somebody can think about a easy way of doing that well, if this is the case, then the only thing, that is not known to me is really lambda naught, the initial condition of lambda naught then well, I can I can start with some guess value, I will I will guess some lambda naught; then everything else will be self driven this expression, then evolve once lambda naught is known to us actually. So, it keeps computing this expression, and evaluates the signum the negative signum that expression actually, so that is my u j star that is all.

So, we assume lambda naught compute lambda of to evaluate u j star u j star is like this, and hence solve for the system trajectory in parallel as well, because once u j star is known everything is known for the system actually. Now, this becomes at kind of an explicit function of time, so this is known to us so starting from n initial condition, that which is known to us anyway, which start an integrated system dynamics.

And then, observe that for some finite time t f where t f is going to be 0 or naught, and if t f becomes sufficiently loss; we are not happy about that, we know that it is not going to happen, then we terminate that procedure or their integration process, and come by you can, and guess a different lambda naught actually. So, we change this lambda naught and repeat the procedure this is actually an open loop structure well lambda not you guess and hope for the best, if it does not happen we change the lambda naught, and again repeat the procedure again hope for the best now obviously, is not a very good strategy, and we can lend up with this infinite loop sort of thing, unless there is a mechanism, where lambda naught can be existed in an appropriate direction our appropriate increment. So, that the subsequent iteration can, you can lead to be a better result, and remember t f is pre variable, and it even more difficult to find out, what t f is there, and no there actually, so this procedure even though, it is very simple; it is it actually not really very useful. So, this open loop structure is not very lenient, but still somebody wants to implement it; this is the way to do that.

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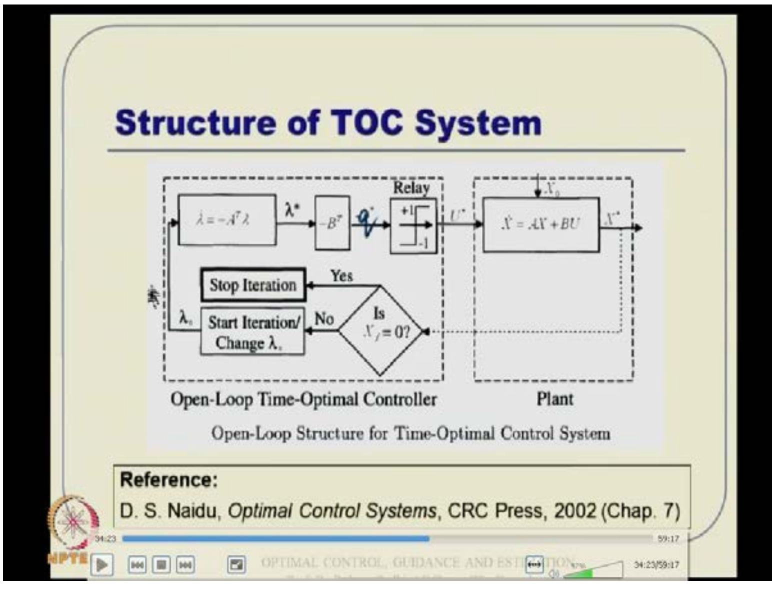

We know this initial condition of this state, now we can start this iteration guess a lambda naught and go to this costate equation find out lambda star, and then multiplying that and all that will give q star well, the book notation is like q, but what we see here is 1 q. So, this negative q star sort of thing so, then you can you can you can pass through the relay function and all that, you will get U star and then continue the iteration, and finally, we are assuming that after sometime, whether X f is 0 or not. If t f is sufficiently loss and X f is still not 0 well my guess itself are not good, so we will change this lambda naught, and repeat the procedures actually.

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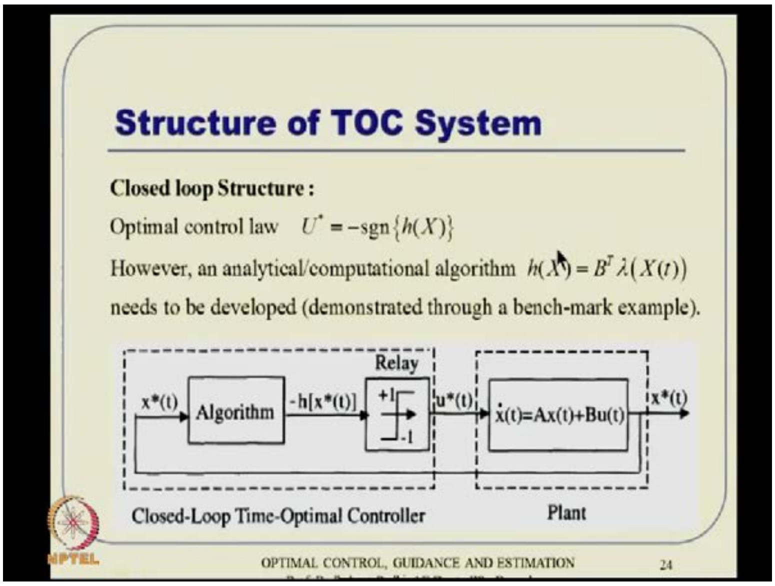

So, again as I told that is not very elegant, what we are looking for is something like a closed loop structure in other words, control should be some form of a state feedback form, and remember that in a regular LQR system, but the objective was also fairly similar, but we did not help this control constraint implies actually. Then, what we had we had this U transpose equal to minus r inverse v transpose p times x basically. So, essentially we are having a closed formula basically, in state feedback form.

So, here also we are looking for something like that is it feasible or not, while in other words, U star can be written as negative signum of this h of X and because, we also see that this is lambda, and lambda in regular LQR it will nothing but, p times x; we cannot write it; we know that, but still getting inspired from that, we can observe that the moment somebody writes p times x here. That is essentially, becomes a state feedback form similarly, you are hunting out for an expression for h of X which will solve the similar purpose actually.

So, any analytical and computational algorithm, essentially we are talking about this kind of a structure where we know that h of X is nothing but, B times lambda, but what we are looking for lambda is a function of X actually, which is where X is X itself piece time varying depending on the system dynamics, and all that actually. Essentially this function needs to be develop, what you are what you are seeing there, and that depend some problem to problem actually. So, essentially in a next few lectures, next few slides, I will take it some this idea of how to develop this; through this standard bench mark example problem this is nothing but, the double integrated system actually. But, once we had this implies this let us say, you have this h of X in place, then the operation becomes much easier; this is you take this X star out of this, and then pass it through this algorithm of whatever algorithm is that to compute this h of X star rather negative h of X star, and all that in then pass it through a Euler function and then, we get it U star actually, then it will implement.

It essentially closed loop time optimal controller implementation, much desirable compared to this structurally, so let us see how will do that, at least in a small example, sense actually, which is a very standard example in text books of course.

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So, here we are talking about time optimal control of double integral systems really now, these double integral systems appear natural in many applications. And the very simplest form, I can think about is just a spring mass application actually; we will not even no (()) spring mass actually, if we see that.

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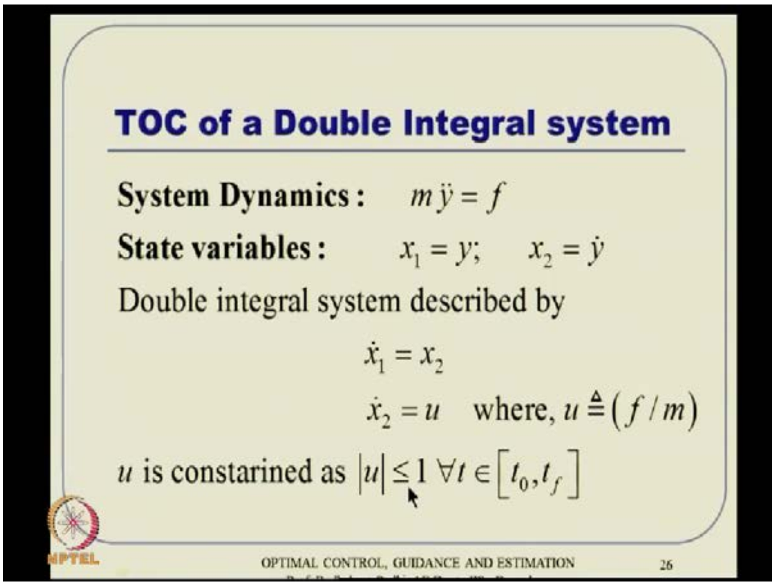

So, here the here the variation is something like m y double dot is f actually, the same thing can be express, you can visualize a frictionless star moving on the roll, the moment you apply a force there is a resolution, and thing like that draggles airplanes actually. So, then it will happen like this equation will be observe like the, I mean this equation will be observe, clearly (()) it happens similar thing can be written also basically.

So, the state variable, we can define x 1 is y, and x 2 is y dot, then the double integral system can be written very easily at that x 1 dot is x 2, where as x 2 dot is u is nothing but, f over m. Essentially, you are writing y double dot is nothing but, f by m, and then define in this first dynamical form, you can first companion form, you can write this way very easily; the different series u is constrained again between minus 1, and plus 1 that is what it is, it here.

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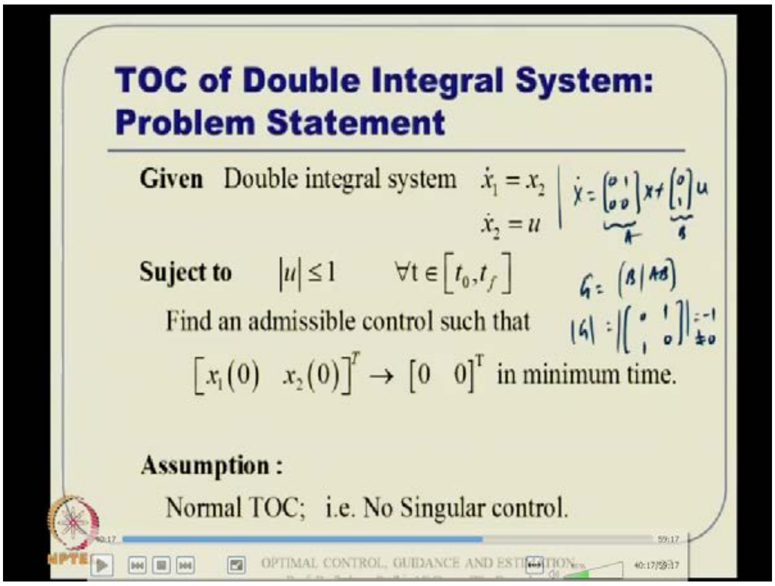

So, once this is this is implies tell, what is happening (()) we will see in a much more details sense actually, so we come out this problem and tell system dynamics is x 1 dot is x 2, x 1 dot is x 2 where as x 2 dot is u, and this is subject to this constraint that, I mean magnitude of u is formed between minus 1 and plus 1. And then, that constraint is valid for all time between t naught to t f, the objective is to find an admissible control history such that, this condition x 1 (0), x 2 (0) is driven towards origin in minimum time.

And we are listening in normal optimal control, normal time optimal control sense that means, no singular control, and you can very clearly see that this controllability is actually define in other words, controllable is this actually costate controllable problem actually. So, you can see that, this can be written as x dot equal to A X plus B U, and U is one here of course, so this is nothing but there, and x 1 dot is x 2, and that is what it is actually, so that is your A matrix, and that is your B matrix.

So, controllability matrix, if you if you formulate sometimes it equalize as G, sometimes n like that, so this is B and A B, so that is nothing but, something like (0 1) and if you multiply that it still becomes (1 0), this is (0 0 plus 1 times 1), which is 1, and then obviously, second one is 0. So, this if you take determinant of G, and obviously, that determinant is happens to be minus 1 which is not 0 so obviously, state controllable problem actually here, so normal time optimal control condition is valid here.

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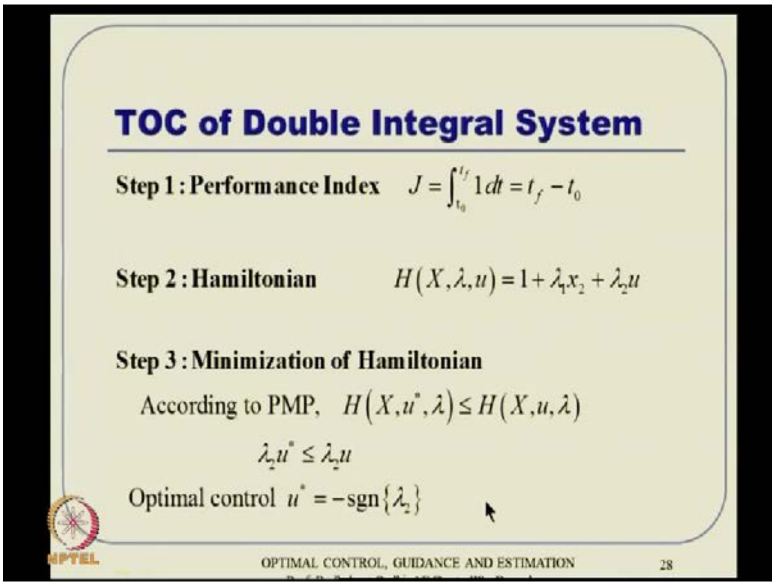

Performance index as usual we select t f minus t naught minimum time problem actually, Hamiltonian is also very straight forward 1 plus lambda transpose x actually, so lambda transpose A X plus B U so essentially, lambda 1 time x 2 plus lambda 2 times u so, we write that way. And according to this pontryagin minimum principle, this condition has to be satisfied, that is the critical observation.

So, if I substitute these, this expression, once with respect to u star, the other one is at with respect to any other u, these two quantities will cancel out, what will remain is, only this quantity, lambda 2 of lambda 2 times u star has to be less than equal to lambda 2 times u. So, the optimal control should be u star is nothing but, negative signum of lambda 2, that is the very clear part observation actually, so all that we need to know is lambda 2 really.

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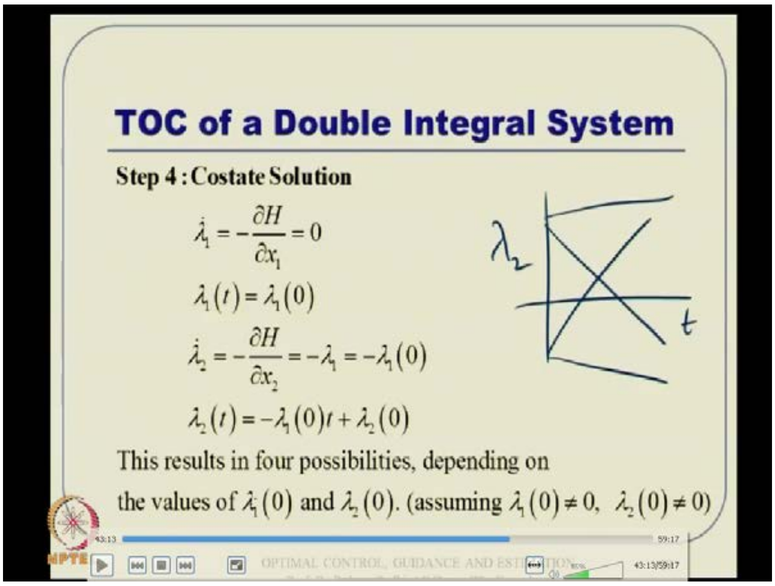

Now, how to how do find out we go back to the costate equation now, remember lambda dot is minus A transpose lambda really, but you can go back and then look among lambda 1 dot is nothing but, del H by del x minus del H by del x. So, well H is here, so del H by del x 1 is nothing but, 0 whereas, del H by del x 2 the is nothing but, lambda 1, this can be clearly seen there.

So, lambda 1 dot is negative of del H by del x 1 0, lambda 2 dot is negative of del H by del x 2 which is lambda 1, then this del H by del x 2 is lambda 1, so it is negative lambda 1. Now, if it 0 lambda 1 dot is 0, then lambda 1 of any lambda 1 of t is nothing but, the same initial condition lambda 1 (0), it does not change from that value. Now, if lambda 2 dot is negative lambda 1 that means, minus lambda 1 of 0, then lambda 2 t is nothing bu, this integral of this equation. So, lambda 1 of 0 times t plus lambda 2 of 0.

So, this result in essential I mean essentially this results in four possibilities, because you this is a straight line equation. We can see that is this something like y equal to m x plus c sort of thing, this is lambda 2 is nothing but, some coefficient negative of that times t plus some of the coefficient.

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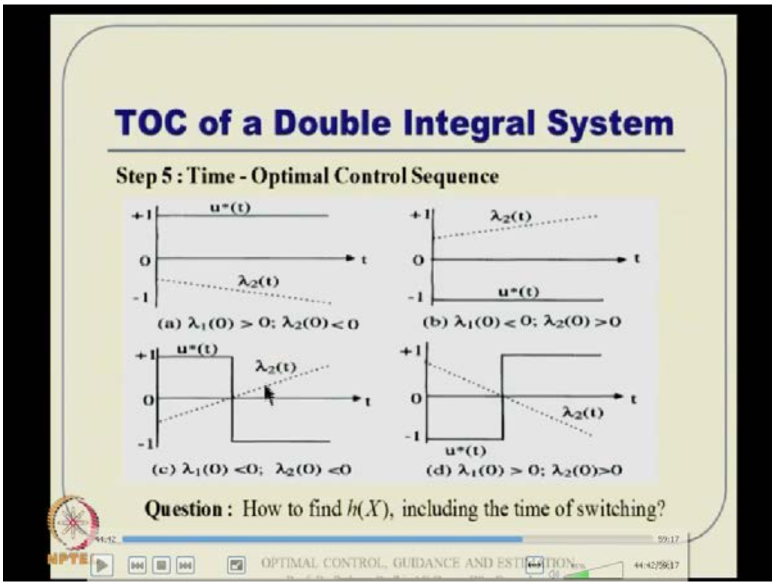

So, in time and this one in time versus lambda 2 plot, if I if I see that is guaranteed to be a straight line, now depending on the situation, it can be like that, it can be this, it can be just a constant, it can be like this, like that actually. So, here the four possibilities like that depending on the constants this lambda 1 naught, and lambda 2 naught actually. So, all the four cases, we need to see what is happening there.

This is pictorially written here, either lambda 2 can be like this, in which case lambda 2 is always negative remains negative, starts negative and remains negative this building in the negative direction; then according to our law control has negative signum of lambda 2, so control has to be plus 1. Similarly, if lambda 2 happens to be all positive, then the control has to be minus 1 throughout actually, that is very clear. Now, what about the case like this, it starts from negative, but because of the sloping because, the slope is positive it goes like this.

So, at some instant it actually crosses the 0 line that means, after that this value is negative hence my control is positive, and after that this value is negative, I mean sorry this lambda 2 value is positive and hence my control is negative. So, I start with the positive value off control and I go negative value, and here it is exactly opposite, I start with the positive value of lambda 2, and hence my control was negative; and later on my lambda 2 happens to be negative hence my control becomes positive. So, you can see the switching very clearly here. Now, the question is how to find this h of (X), which are hunting out; here this is in time domain, but how do you find out a function of state, and also we are interested in which time it will switch, the time of switching is also important actually.

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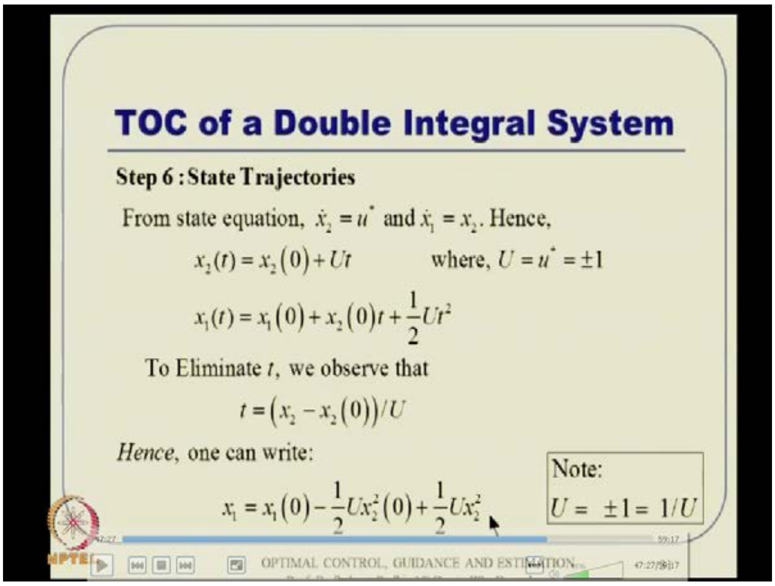

So, for that we need to develop the state trajectories and state trajectories in other words, we are interested in something like a face plane diagram sort of thing. So, from state equation, we have to go back now and tell x 1 dot is x 2 whereas, x 2 is nothing but, U and now we have u star basically. So, if you redefine this u star is something like capital U, which can be either plus 1 or minus 1 depending on the situation.

So, in general we can write that x 2 dot is u star, so that means x 2 of t is nothing but, some initial condition times this u star times t, but u star is defined as capital U, so x 2 dot sorry x 2 of t is nothing but, x 2 of 0 plus U times t. So, then x 1 dot is x 2, that means, x 1 dot is nothing but, this expression, so x 1 of t is nothing but, x 1 of 0 plus integral of these that means, x 2 of 0 times t plus half U t square actually. So, this expression is coming for the x 1 actually.

And also these are very compatible actually if you for our very very early physics and all that x 2 minus x 2 of t minus x 2 is nothing but, the velocity and the velocity is nothing but, U times t resolution time will lost actually. I mean delta v is nothing but, a t and delta x this is difference between that is nothing but, a kind of U t plus half half a t square and all that what we know earlier actually anyway, these are the same similar expressions actually. So, x 2 of t is nothing but, these and x 1 t is nothing but, that so here is a possibility of eliminating this time variable thus right. So, we can do that by this expression, we can solve for time this is nothing but, x 2 minus x 2 0 divided by U and hence, we can write that x 1 is nothing but, now time variable can be substituted, and we can write it that way. And also observe that, if U is plus or minus 1 by U is also plus or minus 1 actually, so whatever 1 by U is, there you can substitute with this value again.

So, we can we can write it this way now, so this is an expression between x 1 and x 2, including the initial condition for x 1 and x 2, and that do not forget actually. The moment I know initial condition for x 1 and x 2, this x 1 at any time and x 2 at any time, can be written that way basically.

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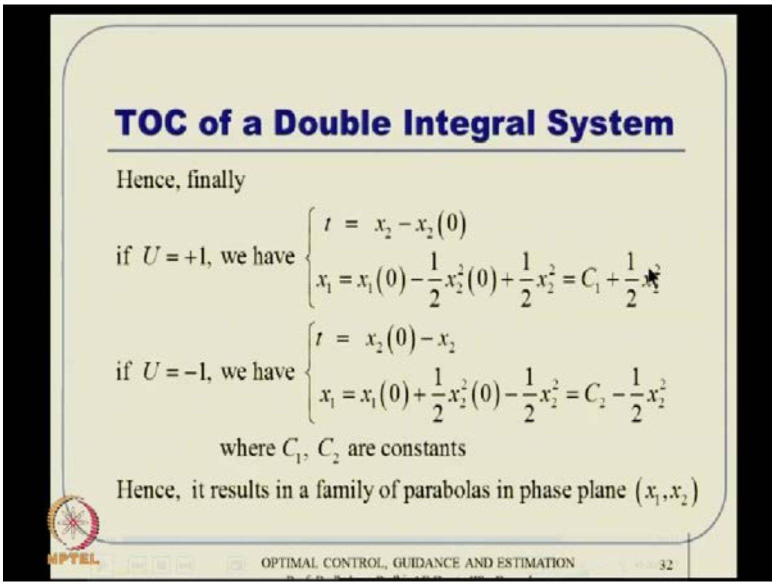

So, using that it is very good to expression, I mean we can write this expression quickly, that if U is positive or U is plus 1, then time happens to be this expression x 2 minus x 2 0 by 1, so it is x 2 minus x 2 0; and x 1 this relationship, you can just drop this U, because U is plus 1 actually. So, you can you can take club these quantities, that happens to be something like C 1 some, because it is a constant number which is C 1 plus half x 2 square. And in the other one, other case, that U is minus 1, we end up with fairly similar things with side adjacent, and all that. So, it happens t happens to be like that, where x 1 is nothing, but these quantities actually.

So, C 1 and C 2, are constants, and depending on the situations where U is positive or negative, we land up with these expressions. So, irrespective of, what you can see that, x 1 is equal to C 1 plus half x 2 square or x 1 is C 2 minus half x 2 square, if you see these relationships these are nothing but, parabolas or other familiar parabolas in the in the phase plane actually.

So, depends on the constants which control again, the constants depends on the initial condition values largely, so essentially depending on the initial condition, where you are on the phase plane. And phase plane is nothing but, a plot between x 1, and x 2 really, so in the phase plane depending on the initial condition, you can you can land up with this types of parabola or this kind of a parabola actually.

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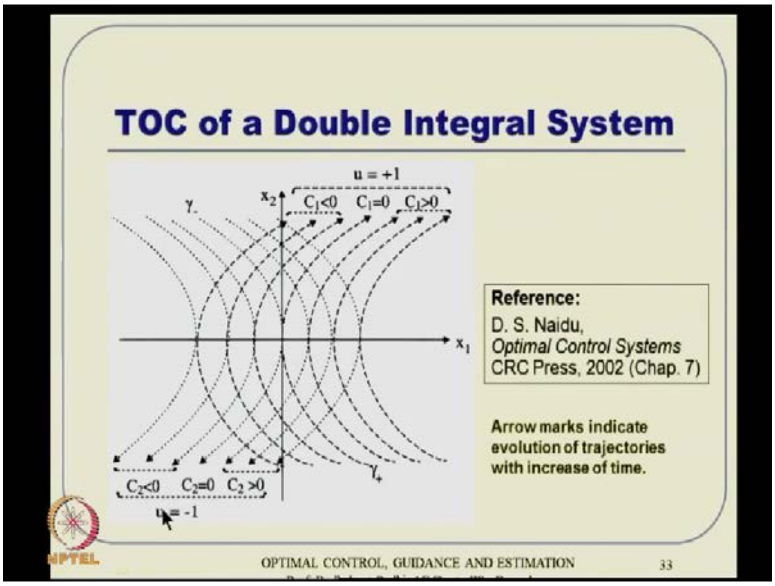

So, this is what is pictorially represented here, so here if U is positive that plus 1, then these are the parabolas; that will result, I mean depending on the initial condition actually. So, with time it will start from somewhere here, and it will start evolving that way, if you start from here it will start evolving that way actually like that.

But, if you start from somewhere here, it will evolve that way again depend on U is minus 1, if you if you apply control to be minus 1, it will it will happen that way. So, note that this arrow marks, indicate evolution of trajectories with increase of time, which is obvious actually, now our job is not yet done, we want to find good solution to that actually.

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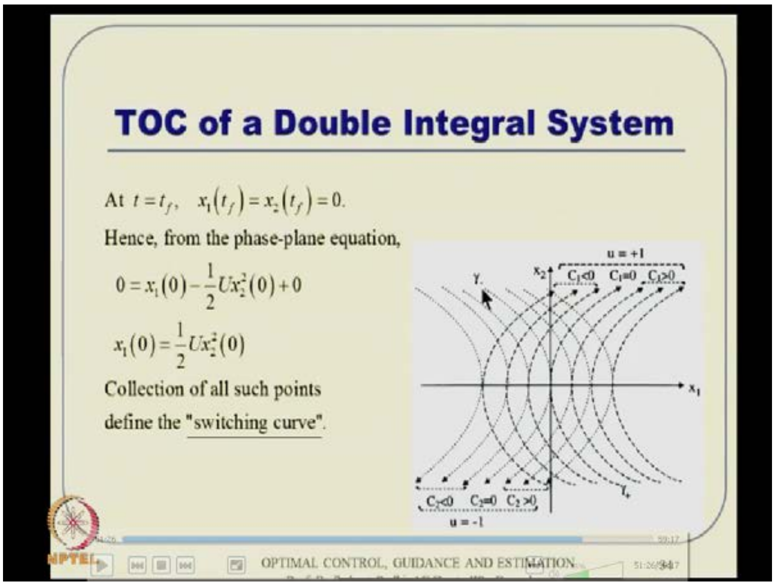

So, what happens here now, at t equal to t f our objective is to drive x 1 of t f, and x 2 of t f to equal to 0 basically, so we have to analyze that particular case and obviously, it will not happen everywhere the solution tells us, that is writes on this particular trajectory. So, here it has to comes 0 or if it is this side of the story, it has to write on this particular trajectory only basically, it will come to 0 any other thing, it will be this 0 and go back actually.

So, this particular situation you can, you depends again the initial conditions sort of relationship. So, it takes that relationship then put it 0 here and land up with some expression like this actually, so this expression what you see here each, we start with this expression; and then time where trying analyze actually. So, where putting this is, this is 0 and this is 0, and then analyzing this expression really, so this ends of this some equation like this (Refer Slide Time: 50:56)

So essentially, it is a collection of points, which defines the switching curve again depending on U, is plus 1 at result something U is minus, so result other family actually U is plus 1, this is what will the present, this is denoted as gamma plus and if it is U will minus 1, this will be result in that sense, this particular curve is defined as gamma minus.

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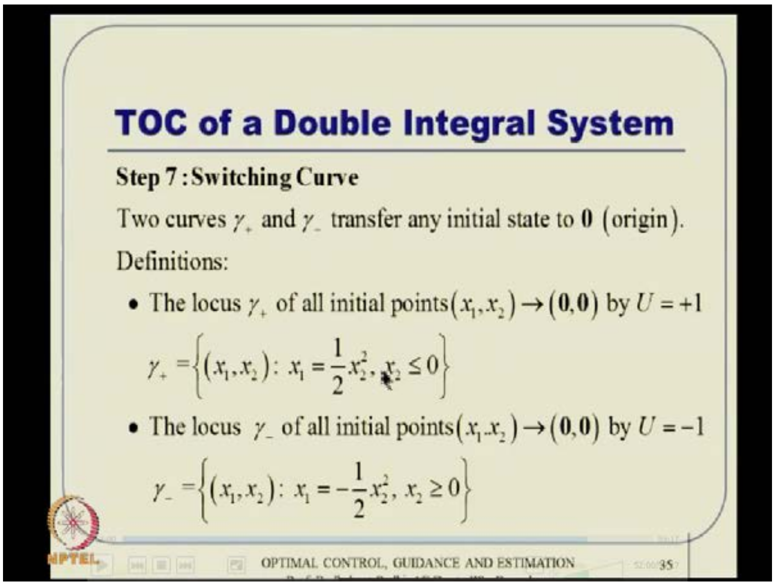

So, this two curves similar here, and one is called gamma plus, the other one is gamma minus; this will these are the on this curve, we are able to transfer any initial state to 0 actually. So, formally speaking the locus gamma plus, it defined something like these x 1, and x 2 such that, x 1 is equal to half x 2 square, and gamma minus is x 1, and x 2 such that, x 1 is minus of x 2 square.

This is valid in this, when x 2 is negative this segment, in this segment and this segment x 2 is negative, so this is this gamma plus is defined in this segment especially, these trajectory only and other case is especially better vector actually.

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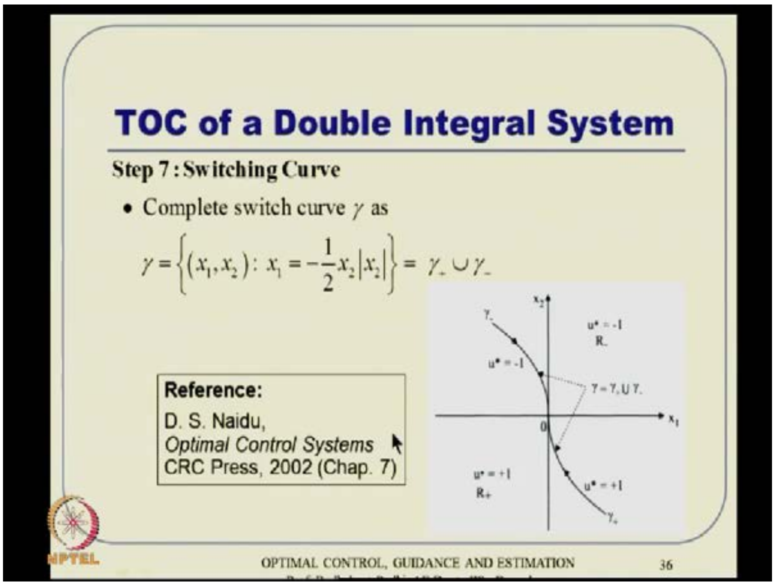

So, we can combine write them a something like these; we can timing from for form an union of this two curves, gamma plus union gamma minus, and combinely, it can be written something like that x 1 is nothing but, equal to minus half of x 2 times module of x of x 2, this will represent this side actually; so this is called switching curve actually.

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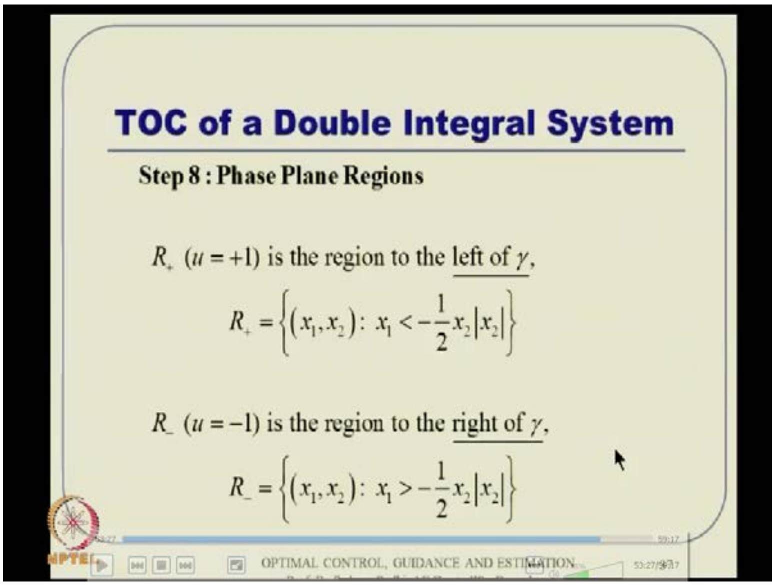

Now, accordingly after knowing this switching curve, we can define some phase plane regions, and accordingly will be define R plus and R minus, and R plus happens to be this all these points; where this condition is a satisfied x 1 is strictly less then this expression minus half of x 2 time x 2 modulus and R minus is the region, where this x 1 is strictly greater than this x same expression. In other words, you once you defined this gamma; which is it is curve, this entire the region towards the left side is nothing but, R plus and entire towards the right side is R minus actually, this is what is written here.

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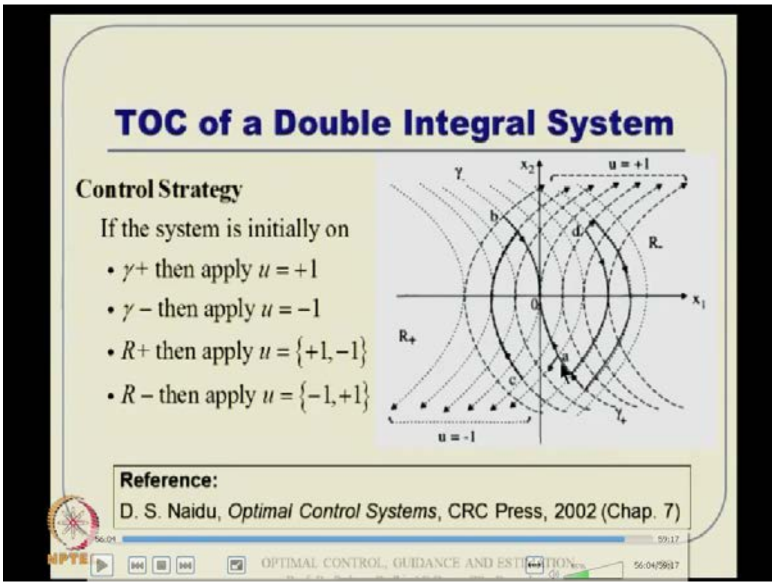

So, now what here telling here, the control strategy is something like these, if the if the system is initially on gamma plus, if some of the initial conditions happen to be on gamma plus, then simply apply, flow equal to plus 1, you have done actually it is some point of time, you will reach the final point trajectory. If it happens to be on gamma minus, then you simply apply minus 1, and it some point time it will take to do that actually again. But, if it if it is happen to be somewhere in the R plus or R minus what you do.

Now, this observation is very very interesting here, if it happens to be R plus let us say, so, what will do we have to apply plus one control, so the trajectory will where somewhere like these; I am here, it will actually is interest this is, this switching curve of some point of time. Once it is interest, you write on the switching curve, it will take you there similarly, if it happens to be R minus something like these; then you first apply minus 1, and it will it will intersect switching curve, and once it intersect the switching curve, then you change the this inter controller, it will take you there actually.

So, very very close to what is called as sliding mode and all that, something very similar happening here so now, somebody can argue what is second, I can I can actually do this way always also, how should I take this, I mean how should I take this negative minus negative controller, and go this way. And then way, then go to 0, I can actually also start with some positive control; then apply negative control, I will interest again this switching curve go the thing is it is not allowed.

First of all, you can clearly see that (()) feel that this path will take more time compare to this path, so that over that the (()) time situation, and also it valid from the theorem, that we told the maximum number of switches can at the most two here, sorry at the most one actually. You can go back to the theorem, at the most n minus 1 time, if original system is normal, and it as all eigen values being positive, and all that; then, it can switch at the most n minus 1 type n is 2 here, so n minus is 1. So, that kind of situation each is not allowed really, this will valid that condition also basically (Refer Slide Time: 55:43).

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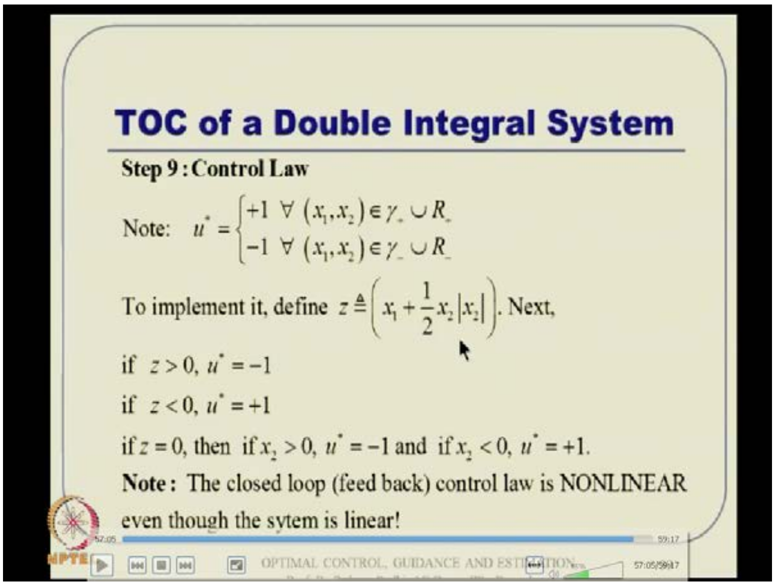

So, what is the final form of controller it turns out that it is positive 1 plus 1, if it happens to be in this region negative 1, if it happen to be in that region actually, but implement it. You can define this quantity, that is nothing but, this expression, and if z happens to be strictly negative we apply plus 1, and if a z happen to be exactly 0 and all that. Then depending on x 2 is positive u apply minus 1 and x 2 is negative it happens to be plus 1; that is on the switching curve really, and these are outside the switching curve actually.

So, essentially one can observe that, if you if you define a quantitative like these, and operative controller likes that, this is essentially a non-linear control even though the system is linear.

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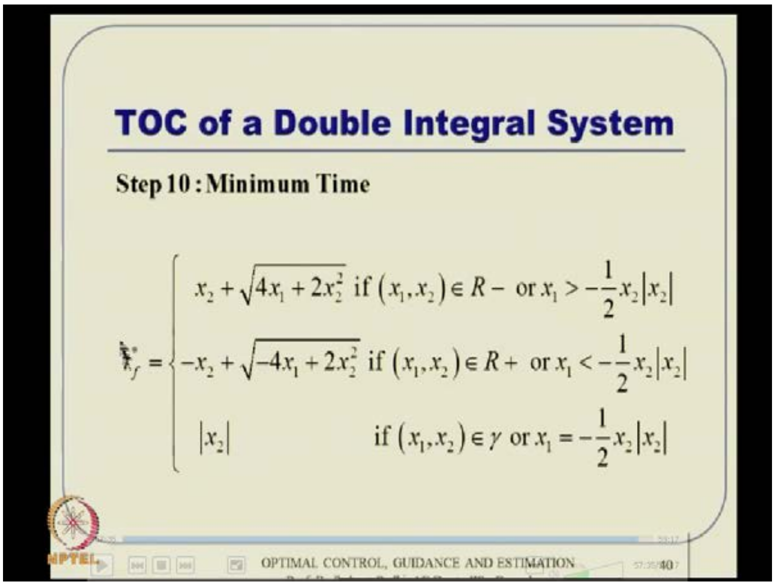

So, we land up with a bang-bang control for this minimum time optimal control problem, but the elegance is, thing is, we actually come up with a state feedback control law is basically. Then as a curiosity something body wants to compute how much time really, it takes to there to region and all, that expression can also we derive it depending on, where you are, and this expressions happens to be like this actually.

Some references you can all, and you can see or you can derive it yourself also basically, so this can also be computed, and to know how much time it takes, and you can actually implement this control in computed as, and then monitor that time take, and all it will it will much for that value actually.

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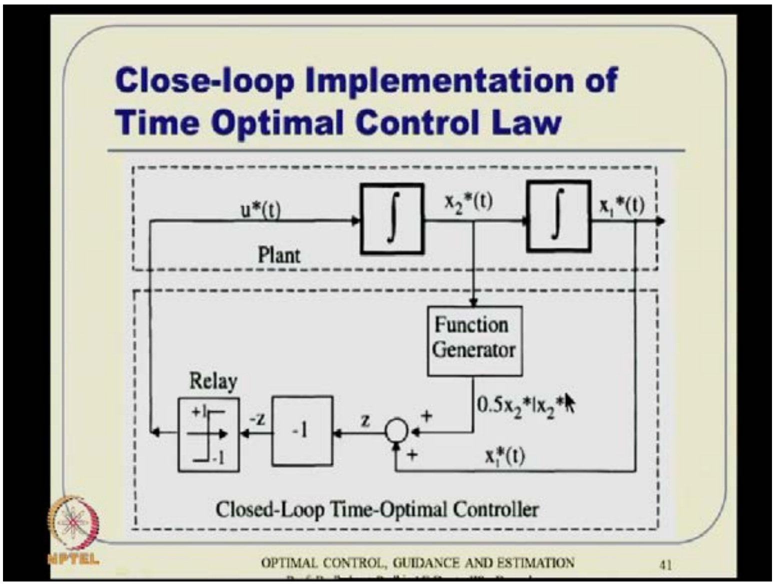

So, finally you can some of it; you want to implement this control, it is a other very easy all that you have to do is, this a double integrated system plan, so this is an integral here integral here u is coming here, so integral after integration, it is here it is x 2 dot, which is u after integration is x 2 of that is nothing but, x 1 dot, so after integration it is x 1. Now here, you tape x 2, and then generate this half x 2 times x 2 modules sort of thing, and then combine that is x 1, so that that will give us z, that is the expression, what we looking for that will give us z and you if the sign, and take to the relay function.

And then you get, you controller reduce completely in the in the state feedback form actually, which is very need to say that, and that is how we got this h of X and all that here, the closed loop expression that we are looking for actually. The close form expression results in a very need control structure actually. So, more on that, you can study many texts books and then, different problems you can see in the literature, and thing like that to have more knowledge actually.

So, that is where will stop this lecture and them the next lecture onwards, you will see some fuel optimal control, and then energy optimal control is time for and probably state constraint problem as well actually, so let us stop here in this class, thank you.